# AVERAGE GROWTH OF THE SPECTRAL FUNCTION ON A RIEMANNIAN MANIFOLD

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Abstract. We study average growth of the spectral function of the Laplacian on a Riemannian manifold. Two types of averaging are considered: with respect to the spectral parameter and with respect to a point on a manifold. We obtain as well related estimates of the growth of the pointwise $\zeta$-function along vertical lines in the complex plane. Some examples and open problems regarding almost periodic properties of the spectral function are also discussed.

## 1. Introduction

1.1. **Notation.** Throughout the paper, we use the following notation.

- $C$ denotes various positive constants whose precise values are not important for our purposes; dependence on various parameters is indicated by lower indices.
- $d_{x,y}$ is the Riemannian distance between $x$ and $y$.
- $\mathbb{R}_+ := (0, +\infty)$.
- $t, s, \lambda, \mu, \tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$.
- $\lambda_+ := \max\{\lambda, 0\}$, $\lceil\lambda\rceil = \min\{n \in \mathbb{Z} |\, \lambda \leqslant n\}$ is the ceiling function and $\lfloor\lambda\rfloor = \max\{n \in \mathbb{Z} |\, \lambda \geqslant n\}$ is the floor function.
- $\hat{f}(\xi) = \int f(x) e^{-ix\xi}\, \mathrm{d}x$ is the Fourier transform of $f$ and $(f)^\vee$ is the inverse Fourier transform, so that $(\hat{f})^\vee = f$.
- $*$ denotes the convolution on $\mathbb{R}$.
- $\Gamma$ is the gamma-function and $\binom{m}{k}$ are the binomial coefficients.
- $J_\alpha$ is the Bessel function of the first kind of order $\alpha$.

Let $f$ and $g$ be real-valued functions on $\mathbb{R}_+$, and let $g > 0$. We write

- $f(\lambda) = O(\lambda^p)$ if $\limsup_{\lambda\to+\infty} \lambda^{-p} f(\lambda) < \infty$;
- $f(\lambda) = o(\lambda^p)$ if $\lim_{\lambda\to+\infty} \lambda^{-p} f(\lambda) = 0$;
- $f(\lambda) = O(\lambda^{-\infty})$ if $\lambda^m f(\lambda) \to 0$ as $\lambda \to +\infty$ for all $m \in \mathbb{R}_+$;
- $f(\lambda) \neq o(g(\lambda))$ if there exists a sequence $\mu_n \to +\infty$ such that $\frac{|f(\mu_n)|}{g(\mu_n)} \geqslant C > 0$ for all $n$;
- $f(\lambda) \gg g(\lambda)$ if there exists $\lambda_0$ such that $f(\lambda) > C\, g(\lambda)$ for all $\lambda > \lambda_0$.

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1.2. **Spectral function.** Let $M$ be a compact Riemannian $n$-dimensional manifold and $\Delta$ be the Laplacian on $M$ with the eigenvalues $0 = \lambda_0 < \lambda_1^2 \leqslant \lambda_2^2 \leqslant \dots \lambda_j^2 \leqslant \dots$ and the corresponding orthonormal basis of eigenfunctions $\{\phi_j(x)\}$. Let $N_{x,y}(\lambda)$ be the *spectral function*,

$$N_{x,y}(\lambda) \;:=\; \sum_{0<\lambda_j<\lambda} \phi_j(x)\,\overline{\phi_j(y)}\,, \tag{1.2.1}$$

The spectral function $N_{x,y}(\lambda)$ is the integral kernel of the spectral projection of the operator $\sqrt{\Delta}$, corresponding to the interval $(0,\lambda)$. This way the spectral function could be defined in the non-compact case as well, see Example 1.3.1.

By the Weyl formulae,

$$N_{x,x}(\lambda) \;=\; \frac{\lambda^n}{(4\pi)^{\frac{n}{2}}\Gamma(\frac{n}{2}+1)} \;+\; O(\lambda^{n-1}) \tag{1.2.2}$$

and

$$N_{x,y}(\lambda) \;=\; O(\lambda^{n-1}) \tag{1.2.3}$$

for all $x \neq y \in M$ (see [Av, Le, Ho1]. The estimates (1.2.2) and (1.2.3) are sharp and attained on a round sphere. However, the bounds can be improved to $o(\lambda^{n-1})$ instead of $O(\lambda^{n-1})$, provided that the set of geodesic loops originating at $x$ and the set of geodesics joining $x$ and $y$ are of measure zero (see [S1] or [SV]).

The following condition is used to prove the lower bounds on the spectral function (see subsection 5.1). Let $\mathcal{L}_{x,y}$ be the set of lengths of all geodesic segments joining $x, y \in M$.

**Condition 1.2.4.** There exists $d \in \mathcal{L}_{x,y}$ such that $x$ and $y$ are not conjugate along any geodesic segment of length $d$.

It follows immediately from [Mil, Corollary 18.2] that Condition 1.2.4 is true for all $x \in M$ and almost all $y \in M$. If this condition is fulfilled then, in the same way as in the proof of [JP, Theorem 1.1.3], one can show that

$$N_{x,y}(\lambda) \;\neq\; o\left(\lambda^{\frac{n-1}{2}}\right)\,. \tag{1.2.5}$$

1.3. **Discussion.** Let $\sigma_{x,y}(t)$ be defined by $(\sigma_{x,y}(t))^\vee = \sum_j \phi_j(x)\,\phi_j(y)\,\delta(\lambda-\lambda_j)$. By the spectral theorem, $\sigma_{x,y}(t)$ coincides with the distributional kernel of the operator $\exp(-it\sqrt{\Delta})$ or, in other words, with the fundamental solution of the corresponding hyperbolic equation. Proofs of (1.2.2)–(1.2.5) are based on the study of singularities of the distribution $\sigma_{x,y}(t)$, which is usually done by means of Fourier integral operators (see, for example, [DG] or [SV]). It is well known that these singularities lie in $\mathcal{L}_{x,y}$. In particular, $\sigma_{x,y}(t)$ with $y \neq x$ is infinitely smooth in a neighbourhood of the origin, whereas $\sigma_{x,x}(t)$ has a strong singularity at $t = 0$ related to the main term in the Weyl formula (1.2.2). If Condition 1.2.4 is fulfilled then the singularity at $t = d$ can be explicitly described, which leads to the lower bound (1.2.5).

The purpose of this paper is to study the *average* growth of $N_{x,y}(\lambda)$, where averaging is considered either with respect to the spectral parameter $\lambda$ or with respect to a point $y$. The following examples indicate that, while both estimates (1.2.3) and (1.2.5) are sharp, the typical order of growth of the spectral function is more likely to be $\lambda^{\frac{n-1}{2}}$ rather than $\lambda^{n-1}$.

**Example 1.3.1.** (cf. [Pe, Example 4.1]) For the Laplacian in $\mathbb{R}^n$ the spectral function is given by

$$N_{x,y}(\lambda) \;=\; \frac{1}{(2\pi)^n}\int_{|\xi|\leqslant\lambda} e^{i\langle x-y,\xi\rangle}\,\mathrm{d}\xi \;=\; \frac{\lambda^n}{(2\pi)^n}\,\hat{\chi}_{B^n}(\lambda\,(x-y))$$

where $\chi_{B^n}$ is the characteristic function of the unit ball. Taking into account [Pe, formula (4.4)] and [GR, formula 3.771(8)], one gets $N_{x,y}(\lambda) = \mathcal{N}(\lambda, d_{x,y})$ where

$$\mathcal{N}(\lambda, d_{x,y}) \;:=\; (2\pi)^{-\frac{n}{2}}\, d_{x,y}^{-\frac{n}{2}}\,\lambda^{\frac{n}{2}}\, J_{\frac{n}{2}}\,(d_{x,y}\,\lambda) \tag{1.3.2}$$

and $d_{x,y} = |x-y|$. Using [GR, formula 8.451(1)] we obtain for all $x \neq y$:

$$\mathcal{N}(\lambda, d_{x,y}) \;=\; \frac{2\,\lambda^{\frac{n-1}{2}}}{(2\pi d_{x,y})^{\frac{n+1}{2}}}\,\sin\left(\lambda\, d_{x,y} - \frac{(n-1)\pi}{4}\right) + O\left(\frac{\lambda^{(n-3)/2}}{d_{x,y}^{(n+3)/2}}\right). \tag{1.3.3}$$

**Example 1.3.4.** Let $\mathbb{S}^n$ be a round sphere of dimension $n \geqslant 2$. Given $x \in \mathbb{S}^n$, denote by $-x$ the diametrically opposite point. Then $N_{x,y}(\lambda) = O\left(\lambda^{\frac{n-1}{2}}\right)$ for all $y \neq \pm x$ and $N_{x,-x}(\lambda) \neq o(\lambda^{n-1})$.

1.4. **Plan of the paper.** The paper is organized as follows. In the next section we present our main results. In Section 3 we discuss some examples and open problems regarding almost periodic properties of the spectral function. In Section 4 we give proofs of the upper bounds formulated in the subsections 2.1–2.4. Proofs of the lower bounds stated in the subsection 2.5 are given in Section 5. All examples are justified in Section 6. Finally, in Section 7 we establish the estimates on the $\zeta$-function presented in the subsection 2.6.

## 2. Main results

2.1. **Average over the manifold.** The principal object of study in the present paper is the *rescaled spectral function*

$$\tilde{N}_{x,y}(\lambda) \;:=\; \lambda^{\frac{1-n}{2}}\, N_{x,y}(\lambda) \tag{2.1.1}$$

on a compact $n$-dimensional Riemannian manifold $M$. Note that, by (1.3.3), in the Euclidean case the rescaled spectral function $\tilde{\mathcal{N}}(\lambda, d_{x,y}) := \lambda^{\frac{1-n}{2}}\,\mathcal{N}(\lambda, d_{x,y})$ is bounded for each fixed $x \neq y$. Our main result is

**Theorem 2.1.2.** *For any compact $n$-dimensional Riemannian manifold $M$ there exists a constant $C_M$ such that*

$$\int_M \left|\tilde N_{x,y}(\lambda) - \tilde{\mathcal{N}}(\lambda, d_{x,y})\right|^2 \mathrm{d}y \;\leqslant\; C_M\,, \qquad \forall\, x \in M\,,\ \forall\, \lambda \in \mathbb{R}_+\,, \tag{2.1.3}$$

*where $d_{x,y}$ is the Riemannian distance between $x, y \in M$.*

*Remark* 2.1.4. Note that $\tilde N_{x,x}(\lambda) \gg \lambda^{\frac{n+1}{2}}$. We subtract the term $\tilde{\mathcal{N}}(\lambda, d_{x,y})$ in (2.1.3) in order to "regularize" the rescaled spectral function in a neighbourhood of the diagonal. Geometrically this can be interpreted as follows. Consider a normal coordinate system centred at $x$. In these coordinates the Riemannian metric is Euclidean at the point $x$, and hence near $x$ the geometric Laplacian $\Delta$ can be viewed as a *perturbation* of the Euclidean Laplacian (a similar idea was used in [Pol, Section 3] and goes back to [AK, Theorem 6.1]). We justify the regularization using the Hadamard parametrix for the wave kernel, see the subsection 4.3.

2.2. **Spectral average.** Theorem 2.1.2 implies a number of results. The first one describes the growth of the spectral function on average with respect to $\lambda$.

**Theorem 2.2.1.** *For every finite measure $\nu$ on $\mathbb{R}_+$ and each fixed $x \in M$, there exists a subset $M_{x,\nu} \subset M$ of full measure such that*

$$\int_0^\infty |\tilde N_{x,y}(\lambda)|^2 \,\mathrm{d}\nu(\lambda) \;<\; \infty\,, \qquad \forall y \in M_{x,\nu}\,. \tag{2.2.2}$$

It should be emphasized that, generally speaking, the set $M_{x,\nu}$ depends on the choice of $\nu$. Otherwise (2.2.2) would mean that $\tilde N_{x,y}(\lambda)$ is bounded for almost all $y \in M$, which is not always the case (see the next subsection).

In particular, taking $\mathrm{d}\nu = (\lambda+1)^{-1}(\ln\lambda)^{-1-\varepsilon}\,\mathrm{d}\lambda$ and applying Theorem 2.2.1, we see that

$$\int_0^\infty \lambda^{-1}(\ln\lambda)^{-1-\varepsilon}\,|\tilde N_{x,y}(\lambda)|^2\,\mathrm{d}\lambda \;<\; \infty \tag{2.2.3}$$

for all $\varepsilon > 0$, all $x \in M$ and almost all $y \in M$.

*Remark* 2.2.4. This paper was inspired by [Ran], where the estimate (2.2.2) was proved for surfaces of constant negative curvature under the assumption that $\mathrm{d}\nu(\lambda) = (\lambda+1)^{-1-\varepsilon}\,\mathrm{d}\lambda$ with some $\varepsilon > 0$. Randol's proof is based on the estimate (2.6.4) for the pointwise $\zeta$-function of the Laplacian and uses some results of complex analysis obtained in [HIP]. We use a more direct approach that is applicable in higher generality and gives better estimates. It also allows us to improve the bounds for the $\zeta$-function. Even though these estimates are not needed in our proof, they seem to be of independent interest. We have included them in Section 2.6, where we also explain the relation between Theorem 2.2.1 and properties of the $\zeta$-function.

2.3. **Growth of the rescaled spectral function.** Theorem 2.2.1 and Example 1.3.4 suggest that $\tilde{N}_{x,y}(\lambda) = O(1)$ for almost all $x, y \in M$ on any manifold $M$. However, this is not true. In particular, for any negatively curved manifold $M$ there exists a constant $\alpha_M > 0$ depending on certain dynamical properties of the geodesic flow, such that $\tilde{N}_{x,y}(\lambda) \neq o\left((\ln\lambda)^{\alpha_M}\right)$ for all $x, y \in M$ (see [JP]). At the same time, for fixed $x$ and $y$ on a negatively curved manifold, the sequence of $\lambda$-s yielding the logarithmic growth of the rescaled spectral function is very scarce. This sequence is quite sensitive to the choice of the points $x$ and $y$. In particular, for fixed $x$ and $\lambda$, the set of points $y$, for which the function $\tilde{N}_{x,y}(\lambda)$ has a logarithmic "peak", is expected to be very small.

The theorem below shows that a similar effect takes place on any manifold, not necessarily negatively curved.

**Theorem 2.3.1.** *There exists a constant $C_0$ not depending on $M$, and a constant $C_M$ depending on the geometry of $M$, such that for any $C > C_0$, the measure of the set*

$$\Omega_x(\lambda,\mu) \ := \ \{\, y \in M \ : \ |\tilde{N}_{x,y}(\lambda)| \geqslant C\,(\mu + d_{x,y}^{-\frac{n+1}{2}})\,\} \tag{2.3.2}$$

*satisfies*

$$\operatorname{meas}(\Omega_x(\lambda,\mu)) \leqslant \frac{C_M}{C^2\mu^2}$$

*for any point $x \in M$ and any $\lambda \in \mathbb{R}_+$.*

**Corollary 2.3.3.** *Let $\{\mu_k\}$ and $\{\tau_k\}$ be positive increasing sequences converging to $+\infty$. Then, for each fixed $x \in M$, the measure of the set of points $y \in M$ such that*

$$|\tilde{N}_{x,y}(\tau_k)| \ \geq \ \mu_k\,, \qquad \forall k = 1, 2, \dots, \tag{2.3.4}$$

*is equal to zero. Moreover, if $\sum_k \mu_k^{-2} < \infty$ then*

$$\limsup_{k\to\infty} |\mu_k^{-1} \tilde{N}_{x,y}(\tau_k)| \ = \ 0 \tag{2.3.5}$$

*for almost all $y \in M$.*

2.4. **Weighted average over the manifold.** Let us consider the average of the rescaled spectral function over $M$ with a weight given by a power of the distance function $d_{x,y}$.

**Theorem 2.4.1.** *For all $\varkappa \geqslant 0$, $x \in M$ and $\lambda \in \mathbb{R}_+$, we have*

$$\int_M d_{x,y}^{\varkappa}\, |\tilde{N}_{x,y}(\lambda)|^2\, \mathrm{d}y \ \leqslant \ \begin{cases} C_{\varkappa,M}\,(1 + \lambda^{1-\varkappa}), & \varkappa \neq 1\,, \\ C_M\,(1 + |\ln\lambda|)\,, & \varkappa = 1. \end{cases} \tag{2.4.2}$$

Note that $\int_M |\tilde{N}_{x,y}(\lambda)|^2\, \mathrm{d}y = \lambda^{1-n}\, N_{x,x}(\lambda) = C\lambda + O(1)$. Therefore the estimate (2.4.2) with $\varkappa = 0$ is order sharp. We also remark that, as follows from the proof of Theorem 2.4.1, the constant $C_{\varkappa,M}$ blows up as $\frac{1}{|1-\varkappa|}$ when $\varkappa \to 1$.

**Corollary 2.4.3.** *Let $u(x,y)$ be a function on $M\times M$ such that $|u(x,y)|\leqslant C\,d_{x,y}^{\varkappa}$ with some nonnegative constants $\varkappa$ and $C$, and let $K_\lambda$ be the operator defined by the integral kernel $\mathcal{K}_\lambda(x,y):=u(x,y)\,|\tilde N_{x,y}(\lambda)|^2$. Then $K_\lambda$ maps $L^p(M)$ into $L^p(M)$ and*

$$\|K_\lambda\|_{L^p\to L^p}\ \leqslant\ \begin{cases} C_{\varkappa,M}\,(1+\lambda^{1-\varkappa}), & \varkappa\neq 1\,,\\ C_M\,(1+|\ln\lambda|)\,, & \varkappa=1. \end{cases}\qquad \forall p\in[1,\infty].$$

In particular, Corollary 2.4.3 implies that the commutator of the operator given by the integral kernel $|\tilde N_{x,y}(\lambda)|^2$ with the multiplication by a smooth function $f$ is bounded in $L^p(M)$ and its norm is $O\,(|\ln\lambda|)$. Indeed, the integral kernel of this commutator coincides with $(f(x)-f(y))\,|\tilde N_{x,y}(\lambda)|^2$, and $|f(x)-f(y)|\leqslant C\,d_{x,y}$ with some $C>0$.

2.5. **Lower bounds.** The following theorem shows that our upper estimates cannot be significantly improved.

**Theorem 2.5.1.** *If Condition* 1.2.4 *is fulfilled then*

$$\lambda^{-q-1}\int_0^\lambda \mu^q\,|\tilde N_{x,y}(\mu)|^p\,d\mu\ \gg\ 1\,,\qquad \forall q\geqslant 0\,,\quad p\geqslant 1\,. \tag{2.5.2}$$

In particular, $\lambda^{-1}\int_0^\lambda|\tilde N_{x,y}(\mu)|\,d\mu\gg 1$.

**Corollary 2.5.3.** *Let Condition* 1.2.4 *be fulfilled, and let $f$ be a positive function on $\mathbb{R}_+$. If there exists a constant $q\geqslant 0$ such that*

$$\limsup_{\mu\to+\infty}\left(\mu^{q+1}\inf_{\tau\leqslant\mu}\left(\tau^{-q}f(\tau)\right)\right)\ >\ 0 \tag{2.5.4}$$

*then $\int f(\mu)\,|\tilde N_{x,y}(\mu)|^p\,\mathrm{d}\mu=\infty$ for all $p\geqslant 1$. In particular, we have*

$$\int_0^\infty \mu^{-1}\,|\tilde N_{x,y}(\mu)|^p\,\mathrm{d}\mu\ =\ \infty\,,\qquad \forall p\geqslant 1\,, \tag{2.5.5}$$

*and*

$$\sum_k \mu_k^{-1}\int_{\mu_{k-1}}^{\mu_k}|\tilde N_{x,y}(\mu)|^p\,\mathrm{d}\mu\ =\ \infty\,,\qquad \forall p\geqslant 1\,, \tag{2.5.6}$$

*for every increasing sequence $\mu_k\to+\infty$.*

As follows from (2.5.5) with $p=2$, Theorem 2.2.1 fails for $\mathrm{d}\nu(\lambda)=(\lambda+1)^{-1}\mathrm{d}\lambda$ on any manifold $M$.

*Remark* 2.5.7. Theorem 2.5.1 improves upon [JP, Theorem 1.1.3]. It is quite possible that Condition 1.2.4 in this theorem can be removed: our proof works whenever $\sigma_{x,y}(t)$ has a sufficiently strong singularity. It is hard to imagine the situation where this does not happen; we are not aware of any counterexamples.

2.6. **The $\zeta$-function.** The function $Z_{x,y}$ of complex variable $z=t+is$ defined by

$$Z_{x,y}(z) \;:=\; \int_0^\infty \lambda^{-z}\,\mathrm{d}N_{x,y}(\lambda) \;=\; z\int_0^\infty \lambda^{-z-1}\,N_{x,y}(\lambda)\,\mathrm{d}\lambda \tag{2.6.1}$$

is said to be the *pointwise* $\zeta$*-function* of the Laplacian. It is the integral kernel of the pseudodifferential operator $\Delta^{-\frac{z}{2}}$. It is well known that $Z_{x,y}(z)$ is an entire function on $\mathbb{C}$ for all fixed $x\neq y$ (see, for example, [Sh, Theorem 12.1]).

*Remark* 2.6.2. Further on we call $Z_{x,y}(z)$ simply the $\zeta$*-function*. Note that $Z_{x,y}(z)$ should not be confused with the function $Z(z)=\operatorname{Tr}\Delta^{-\frac{z}{2}}$ that is usually referred to as the $\zeta$-function of the Laplacian.

The second equality in (2.6.1) implies that $Z_{x,y}(z)=z\,(\mathcal{M}\,N_{x,y})(-z)$ where $\mathcal{M}$ is the Mellin transform, $(\mathcal{M}f)(z):=\int_0^\infty \lambda^{z-1}\,f(\lambda)\,\mathrm{d}\lambda$. Recall that, for each $t\in\mathbb{R}$, the Mellin transform $(\mathcal{M}f)(t+is)$ of a distribution $f$ on $\mathbb{R}_+$ coincides with the inverse Fourier transform of the distribution $e^{t\mu}f(e^\mu)$ on $\mathbb{R}$ modulo the factor $(2\pi)^{-1}$. The inversion formula reads

$$f(\lambda)=(2\pi i)^{-1}\int_{t-i\infty}^{t+i\infty}\lambda^{-z}(\mathcal{M}f)(z)\,\mathrm{d}z\,,$$

where the integral is understood in the sense of distributions.

Obviously, $e^{t\mu}f(e^\mu)\in L^2(\mathbb{R})$ if and only if $\lambda^{t-\frac12}f(\lambda)\in L^2(\mathbb{R}_+)$. Therefore $(t+is)^{-1}\,Z_{x,y}(t+is)\in L^2(\mathbb{R})$ if and only if $\lambda^{-t-\frac12}N_{x,y}(\lambda)\in L^2(\mathbb{R}_+)$ for each fixed $t$, and

$$N_{x,y}(\lambda) \;=\; (2\pi i)^{-1}\int_{t-i\infty}^{t+i\infty}\lambda^{z}\,z^{-1}Z_{x,y}(z)\,\mathrm{d}z \tag{2.6.3}$$

in the sense of distributions. In particular, if $(t_0+is)^{-1}\,Z_{x,y}(t_0+is)\in L^2(\mathbb{R})$ for some $t_0\in\mathbb{R}$ then $(t+is)^{-1}\,Z_{x,y}(t+is)\in L^2(\mathbb{R})$ for all $t>t_0$.

In [Ran], for a surface of constant negative curvature, it was shown that

$$(t+i\,\cdot)^{-1}Z_{x,y}(t+i\,\cdot)\in L^2(\mathbb{R})\,,\qquad \forall t>\frac{n-1}{2}\,, \tag{2.6.4}$$

almost everywhere. By the above, this inclusion is equivalent to Theorem 2.2.1 with $\mathrm{d}\nu=(\lambda+1)^{-1-\varepsilon}\,\mathrm{d}\lambda$.

Let $\langle s\rangle := (1+|s|^2)^{1/2}$. In the last section we shall prove the following two theorems.

**Theorem 2.6.5.** $|\,Z_{x,y}(t+is)\,| \;\leqslant\; \begin{cases} C_t\,, & \textit{if}\;\; n<t\,,\\ C_t\left(|s|^{n-t}+d_{x,y}^{t-n}\right)\,, & \textit{if}\;\; \frac{n}{2}\leqslant t<n\,,\\ C_t\left(|s|^{n-t}+d_{x,y}^{t-n}\langle s\rangle^{\frac{n}{2}-t}\right)\,, & \textit{if}\;\; t<\frac{n}{2}\,.\end{cases}$

**Theorem 2.6.6.** *For all* $x \in M$ *and all* $\varepsilon > 0$ *, we have*

$$\int_M \frac{|Z_{x,y}(t+is)|^2}{d_{x,y}^{2t-n-\varepsilon}+1}\,\mathrm{d}y \;\leqslant\; C_{t,\varepsilon}\left(\langle s\rangle^{n-2t}+1\right), \qquad \forall t \neq \frac{n}{2}\,. \tag{2.6.7}$$

In view of Fubini's theorem, (2.6.7) immediately implies (2.6.4) for all $x$ and almost all $y$. However, even the improved estimate (2.6.7) does not seem to be sufficient to obtain our upper bounds for the spectral function.

*Remark* 2.6.8. It is quite possible that Theorems 2.6.5 and 2.6.6 remain valid for $t = n$ and $t = \frac{n}{2}$, but our proof does not work in these cases.

2.7. **Possible generalizations.** All the above results can easily be extended to an elliptic self-adjoint pseudodifferential operator $A$ acting on a compact manifold without boundary. For such an operator, one has to consider trajectories of the Hamiltonian flow generated by its principal symbol instead of geodesics, and to use Fourier integral operators or the global parametrix constructed in [SV] instead of the Hadamard representation (4.3.1) for the study of singularities of $\sigma_{x,y}(t)$ .

Note that some of our estimates do not require the spectrum to be discrete. In particular, it may well be possible to extend the results that do not involve integration over $M$ to the case of noncompact manifolds. For instance, the functions $\int \rho_1(\lambda-\mu)\,\mathrm{d}N_{x,y}(\mu)$ and $\int |\rho_1(\lambda-\mu)|^2\,\mathrm{d}N_{x,y}(\mu)$ are the integral kernels of the operators $\rho_1(\lambda - A)$ and $|\rho_1(\lambda-A)|^2$ , where $A$ is the restriction of $\sqrt{\Delta}$ to the subspace spanned by the eigenfunctions $\phi_1, \phi_2, \ldots$. Therefore the key equality (4.2.1) is easily obtained by rewriting the obvious operator identity $(\rho_1(\lambda-A))^*\,\rho_1(\lambda-A) = |\rho_1(\lambda-A)|^2$ in terms of integral kernels.

It would be also interesting to extend our results to an elliptic self-adjoint differential operator on a manifold with boundary, subject to suitable boundary conditions. In this case the role of geodesics is played by Hamiltonian billiards. One has to consider interior points $x$ and $y$ and to make appropriate assumptions to avoid problems with the so-called grazing and dead-end trajectories (see [SV]).

## 3. Almost periodic properties of the spectral function

3.1. **Besicovitch almost periodic functions.** Let $p \geqslant 1$ . Recall that, for a measurable function $f$ on $\mathbb{R}_+$, its *Besicovitch seminorm* $||f||_{\mathcal{B}^p}$ is defined by

$$||f||_{\mathcal{B}^p} \;:=\; \limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T |f(\lambda)|^p\,\mathrm{d}\lambda\right)^{1/p} \tag{3.1.1}$$

The space $B^p$ of *Besicovitch almost periodic functions* is defined as the completion of the linear space of all finite trigonometric sums $\sum_{k=1}^N a_k e^{i\omega_k x}$ with $a_k \in \mathbb{C}$ and $\omega_k \in \mathbb{R}$ with respect to the Besicovitch seminorm. Clearly, $B^{p_1} \subset B^{p_2}$ and $\|f\|_{B^{p_2}} \leqslant \|f\|_{B^{p_1}}$ for all $p_1 > p_2$ .

For each $f \in B^p$, there exists a sequence of real numbers $\omega_k$ called the *frequencies* of the function $f$, such that

$$\lim_{N\to\infty} ||f - \sum_{k=1}^{N} a_k\, e^{i\omega_k x}||_{\mathcal{B}^p} \;=\; 0$$

where $a_k$ are some complex constants (see, for example, [Bes]). If the above identity holds, we shall write $f \sim \sum a_k e^{i\omega_k x}$.

Theorem 2.2.1 motivates the following

**Conjecture 3.1.2.** *On any compact Riemannian manifold $M$, $\|\tilde{N}_{x,y}(\lambda)\|_{\mathcal{B}^2} < \infty$ for each fixed $x \in M$ and almost all $y \in M$.*

Conjecture 3.1.2 holds for round spheres and flat 2-tori. In fact, in Examples 3.2.4 and 3.2.7 we present a stronger result: the rescaled spectral function on these manifolds is $B^2$–almost periodic for each fixed $x$ and almost all $y$. In both cases, the set of frequencies coincides with the set $\mathcal{L}_{x,y}$ of lengths of all geodesic segments joining $x$ and $y$.

*Remark* 3.1.3. It was proved in [Bl1] and [KMS] that the rescaled error term in Weyl's law has an almost periodic expansion in $B^2$ on surfaces of revolution and in $B^1$ on Liouville tori. On Zoll manifolds, the rescaled Weyl remainder (albeit with a different order of rescaling) has an almost periodic expansion in $B^2$ [Sch]. The frequencies of these expansions are the lengths of closed geodesics. The spectral function, similarly to the Weyl remainder, has oscillatory behaviour, and hence it is natural to study it in the context of almost periodic functions. Moreover, as indicated by Theorem 2.2.1, the order of rescaling in this case could be chosen *universally* for all manifolds of a given dimension. As was mentioned in Section 1.3, the lengths of geodesic segments joining $x$ and $y$ are the singularities of the distribution $\sigma_{x,y}(t)$, and hence play the same role for the spectral function as the lengths of closed geodesics for the Weyl remainder. The link between the spectral function and the set of lengths $\mathcal{L}_{x,y}$ has a natural interpretation from the viewpoint of the quantum–classical correspondence.

3.2. **Spectral function on spheres and tori.** Let us start with a toy example — the spectral function on the unit circle $\mathbb{S}^1$:

$$\tilde{N}_{x,y}(\lambda) = N_{x,y}(\lambda) = \frac{1}{\pi}\sum_{1\leqslant n<\lambda}\cos(n\, d_{x,y}) = -\frac{1}{2\pi} + \frac{1}{2\pi}\frac{\sin((\lceil\lambda\rceil - \frac{1}{2})d_{x,y})}{\sin(\frac{d_{x,y}}{2})} \tag{3.2.1}$$

Note that in dimension one no rescaling occurs, and the constant term $(2\pi)^{-1}$ is subtracted, because the eigenfunction corresponding to the zero eigenvalue is excluded in the definition (1.2.1). In higher dimensions the contribution of the constant term to $\tilde{N}_{x,y}(\lambda)$ is negligible in $B^p$ due to the rescaling.

**Example 3.2.2.** If $M$ is the unit circle then the spectral function $N_{x,y}(\lambda)$ is $B^2$–almost periodic for all $x \neq y$ with the set of frequencies $\mathcal{L}_{x,y} = \{|d_{x,y} + 2\pi k|\}_{k\in\mathbb{Z}}$, and

$$N_{x,y}(\lambda) \sim -\frac{1}{2\pi} + \sum_{-\infty}^{+\infty} \frac{1}{\pi|d_{x,y} + 2\pi k|} \sin(\lambda|d_{x,y} + 2\pi k|). \tag{3.2.3}$$

The next two examples generalize Example 3.2.2 to round spheres and flat two dimensional tori.

**Example 3.2.4.** On a flat square 2-torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, the rescaled spectral function is $B^2$–almost periodic for all $x \neq y$ and

$$\tilde{N}_{x,y}(\lambda) \sim \sum_{\eta\in\mathbb{Z}^2} \frac{2\sin\left(\lambda\,|x-y+2\pi\eta| - \frac{\pi}{4}\right)}{(2\pi)^{3/2}\,|x-y+2\pi\eta|^{3/2}}. \tag{3.2.5}$$

Here $|\cdot|$ denotes the length of a vector in $\mathbb{R}^2$. A similar formula holds on flat tori corresponding to arbitrary lattices.

*Remark* 3.2.6. In dimensions $n \geqslant 3$ the situation is significantly more complicated. For example, following the argument of [Pet, Corollary 1.4] one could show that $||\tilde{N}_{x,y}(\lambda)||_{\mathcal{B}^2} = \infty$ on a flat $n$-torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ if the vector $\pi^{-1}(x-y)$ is *rational*. This happens due to unbounded multiplicities in the set $\mathcal{L}_{x,y}$. However, it does not contradict Conjecture 3.1.2 because for each fixed $x \in \mathbb{T}^n$, $\pi^{-1}(x-y) \notin \mathbb{Q}^n$ for almost all $y \in \mathbb{T}^n$.

Recall that the Morse index of a geodesic segment joining $x$ and $y$ is the number of points on the segment that are conjugate to $x$, counted with multiplicities (see [Mil, Theorem 15.1]). Let $H(x)$ be the "reversed" Heaviside function: $H(x) = 0$ if $x \geqslant 0$ and $H(x) = 1$ if $x < 0$.

**Example 3.2.7.** Let $M$ be the unit round sphere $\mathbb{S}^n$ of dimension $n \geqslant 2$, and let $x \neq y \in \mathbb{S}^n$ be any two non-opposite points. Then $\tilde{N}_{x,y}(\lambda) \in B^2$, and

$$\tilde{N}_{x,y}(\lambda) \sim \sum_{k=-\infty}^{+\infty} \frac{2\sin\left(\lambda\,|d_{x,y} + 2\pi k| - \frac{(n-1+2\omega_k)\pi}{4}\right)}{(2\pi)^{\frac{n+1}{2}}(\sin d_{x,y})^{\frac{n-1}{2}}\,|d_{x,y} + 2\pi k|} \tag{3.2.8}$$

where $\omega_k = (n-1)\,(2\,|k| - H(k))$ is the Morse index of the geodesic segment of length $|d_{x,y} + 2\pi k|$.

Note that, unlike (3.2.5), the phase shifts in the expansion (3.2.8) depend on the number of conjugate points on the geodesic segments.

3.3. **An open problem.** In this section we describe a possible route for generalizing Examples 3.2.4 and 3.2.7. Let $M$ be an arbitrary compact $n$-dimensional Riemannian manifold. Let $\Gamma_{x,y}$ be the set of all geodesic segments joining $x$ and $y$. For every $\gamma \in \Gamma_{x,y}$, let $l(\gamma)$ be its length and $\omega(\gamma)$ be its Morse index. As before, set $\mathcal{L}_{x,y} = \{l(\gamma)|\,\gamma \in \Gamma_{x,y}\}$. Along each geodesic segment $\gamma \in \Gamma_{x,y}$, consider the matrix Jacobi equation $A'' + RA = 0$, where the coefficient $R$ is defined in terms of the Riemann curvature tensor and the parallel transport along $\gamma$ (see [Ch1, p. 104]). We assume that $\gamma$ is naturally parametrized, and $A$ satisfies the initial conditions $A(0, \xi_\gamma) = 0$, $A'(0, \xi_\gamma) = 1$, where $\xi_\gamma$ is the unit tangent vector to $\gamma$ at the point $x$. Set $a(\gamma) = |\det A(l(\gamma), \xi_\gamma)|$. For example, $a(\gamma) = l(\gamma)^{n-1}$ on a flat square $n$-torus and $a(\gamma) = |\sin l(\gamma)|^{n-1}$ on a round $n$-sphere. In dimension two, $a(\gamma) = |J(l(\gamma))|$, where $J(t)$ is the orthogonal Jacobi field along $\gamma$ with the initial conditions $J(0) = 0$ and $J'(0) = 1$.

**Problem 3.3.1.** *Let $M$ be a compact $n$-dimensional Riemannian manifold. Is it true that for all $x \in M$ and almost all $y \in M$ the rescaled spectral function $\tilde{N}_{x,y}(\lambda)$ has an almost periodic expansion*

$$\tilde{N}_{x,y}(\lambda) \sim \frac{2}{(2\pi)^{\frac{n+1}{2}}} \sum_{\gamma \in \Gamma_{x,y}} \frac{\sin(\lambda l(\gamma) - \frac{(n-1)\pi}{4} - \omega(\gamma)\frac{\pi}{2})}{l(\gamma)\sqrt{a(\gamma)}} \tag{3.3.2}$$

*in $B^p$ for some $p \geqslant 1$?*

Let us show that the expansion (3.3.2) is well-defined if the points $x, y \in M$ are not conjugate along any geodesic joining them (by [Mil, Corollary 18.2], this condition is satisfied for any fixed $x \in M$ and almost all $y \in M$). Indeed, if the points $x, y$ are not conjugate along any geodesic, $a(\gamma) \neq 0$ for any $\gamma$, the set of lengths $\mathcal{L}_{x,y}$ is discrete, and each element has finite mulitplicity, i.e. appears in $\mathcal{L}_{x,y}$ at most a finite number of times [Mil, Theorem 16.3]. This implies, in particular, that the set $\mathcal{L}_{x,y}$ is infinite, because for any two points on a compact manifold there exists an infinite number of geodesic segments joining them [Ser]. If $M$ has no conjugate points, it is easy to check that (3.3.2) agrees with [JP, formula (5.1.3)].

*Remark* 3.3.3. If (3.3.2) does hold for some $p \geqslant 1$ on a given manifold, it would be interesting to determine the maximal possible value of $p$. For instance, it is quite likely that for round spheres one can take any $p \geqslant 1$. Note that if the almost periodic expansion is valid for some $p > 1$, then the Fourier coefficients of (3.3.2) lie in $l_q$ for $q = \max(2, \frac{p}{p-1})$ ([ABI, section 4]). This can be viewed as a dynamical condition on the manifold (see [Pat, Section 3.1] for some related results), and it is not clear whether it always holds. At the same time, even for $p = 1$, a positive answer to Problem 3.3.1 provides a lot of useful information about the spectral function; in particular, it implies that the rescaled spectral function has a *limit distribution* (see [KMS, Appendix II]). Understanding the properties of this distribution on a given manifold is a problem of independent interest.

## 4. Proofs of the upper bounds

4.1. **Auxiliary functions.** Let us fix a real-valued even rapidly decreasing function $\rho \in C^\infty(\mathbb{R})$ satisfying the following condition.

**Condition 4.1.1.** $\hat\rho \in C_0^\infty(\mathbb{R})$, $\hat\rho \equiv 1$ in a neighbourhood of the origin and $\operatorname{supp}\hat\rho \subset (-\delta,\delta)$ for some $\delta > 0$.

Condition 4.1.1 implies that $\int_0^\infty \rho(\tau)\,\mathrm{d}\tau = \frac12$ and $\int_0^\infty \tau^k\rho(\tau)\,\mathrm{d}\tau = 0$ for all $k = 1,2,\dots$ Let $\rho_1(\tau) := \operatorname{sign}\tau \int_{|\tau|}^\infty \rho(\mu)\,\mathrm{d}\mu$ if $\tau \neq 0$, and $\rho_1(0) := -\frac12$. The function $\rho_1$ is rapidly decreasing, odd and infinitely differentiable outside the origin $\tau = 0$, so that $\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_1(\tau) = -\rho(\tau)$ for all $\tau \neq 0$. It has a jump at the origin; in view of Condition 4.1.1, $\rho_1(0) = \rho_1(-0) = -\rho_1(+0) = -\frac12$.

Let

$$N_{x,y;0} := \rho * N_{x,y}\,, \qquad N_{x,y;1} := N_{x,y} - N_{x,y;0}\,. \tag{4.1.2}$$

The following elementary lemma is a slight variation of [S2, Lemma 1.2].

**Lemma 4.1.3.** *$N_{x,y;1}(\lambda) = \int \rho_1(\lambda-\mu)\,\mathrm{d}N_{x,y}(\mu)$ for all $\lambda \in \mathbb{R}$.*

*Proof.* If $\lambda$ is not an eigenvalue then, integrating by parts, we immediately obtain

$$\begin{aligned}\int \rho_1(\lambda-\mu)\,\mathrm{d}N_{x,y}(\mu) &= \int_{-\infty}^{\lambda} \rho_1(\lambda-\mu)\,\mathrm{d}N_{x,y}(\mu) + \int_{\lambda}^{\infty} \rho_1(\lambda-\mu)\,\mathrm{d}N_{x,y}(\mu)\\ &= (\rho_1(+0)-\rho_1(-0))\,N_{x,y}(\lambda) - \int \rho(\lambda-\mu)\,N_{x,y}(\mu)\,\mathrm{d}\mu\,,\end{aligned}$$

where the right hand side coincides with $N_{x,y;1}(\lambda)$. If $\lambda = \lambda_j$ then the same equality holds for the function

$$N_{x,y}^{(j)}(\lambda) \;:=\; N_{x,y}(\lambda) - (N_{x,y}(\lambda_j+0) - N_{x,y}(\lambda_j-0))\;\chi_j(\lambda)\,,$$

where $\chi_j(\lambda)$ is the characteristic function of the interval $(\lambda_j,\infty)$. Since

$$\begin{aligned}\int \rho_1(\lambda_j-\mu)\,\mathrm{d}\chi_j(\mu) \;&=\; \rho_1(0) \;=\; \rho_1(-0) \;=\; -\int_0^\infty \rho(\tau)\,\mathrm{d}\tau\\ &=\; -\int \rho(\lambda_j-\mu)\,\chi_j(\mu)\,\mathrm{d}\mu \;=\; \chi(\lambda_j) - \int \rho(\lambda_j-\mu)\,\chi_j(\mu)\,\mathrm{d}\mu\,,\end{aligned}$$

the lemma remains valid when $\lambda$ is an eigenvalue. □

*Remark* 4.1.4. The above lemma turns out to be very useful for obtaining estimates of the spectral and counting functions. Usually, the singularities of $\sigma_{x,y}(t)$ for small values of $t$ can be described explicitly. Then, taking the inverse Fourier transform, one obtains full asymptotic expansion of $N_{x,y;0}(\lambda)$. The asymptotic behaviour of $N_{x,y;1}(\lambda)$ is determined by nonzero singularities of $\sigma_{x,y}(t)$, which are much more difficult to study. According to Lemma 4.1.3, the function $N_{x,y;1}(\lambda)$ can also be written as a convolution. The main technical problem is that the function $\rho_1$ has a jump, and therefore the straightforward integration by parts does not yield any new results. However, as we shall see in the next subsection, this jump disappears when we square $N_{x,y;1}(\lambda)$ and integrate over $y$. Note also that $|\rho_1|$ is estimated by a smooth function whose Fourier transform has a compact support. This observation allows one to simplify and refine the well known Fourier Tauberian Theorems (see [S2]).

4.2. **Upper bounds for** $N_{x,y;1}(\lambda)$**.** Since the eigenfunction $\phi_j$ are orthogonal in $L^2(M)$, Lemma 4.1.3 implies that

$$(4.2.1)\qquad \int_M |N_{x,y;1}(\lambda)|^2\,\mathrm{d}y \;=\; \int_M |\int \rho_1(\lambda-\mu)\,\mathrm{d}N_{x,y}(\mu)|^2\,\mathrm{d}y$$

$$=\; \int_M \iint \rho_1(\lambda-\mu)\,\rho_1(\lambda-\tau)\,\mathrm{d}N_{x,y}(\mu)\,\mathrm{d}N_{x,y}(\tau)\,\mathrm{d}y \;=\; \int |\rho_1(\lambda-\mu)|^2\,\mathrm{d}N_{x,x}(\mu)$$

(an operator interpretation of this equality has been given in the subsection 2.7). The function $|\rho_1(\tau)|^2$ is continuous and infinitely differentiable outside the origin. Integrating by parts and changing variables in the right hand side, we obtain

$$\int_M |N_{x,y;1}(\lambda)|^2\,\mathrm{d}y = \int |\rho_1(\lambda-\mu)|^2\,\mathrm{d}N_{x,x}(\mu) = \int \tilde\rho_1(\mu)\,N_{x,x}(\lambda-\mu)\,\mathrm{d}\mu\,,$$

where $\tilde\rho_1(\tau) := \frac{\mathrm{d}}{\mathrm{d}\tau}\,|\rho_1(\tau)|^2$ is a rapidly decreasing odd function. Since $\tilde\rho_1$ is odd,

$$\int \tilde\rho_1(\mu)\,N_{x,x}(\lambda-\mu)\,\mathrm{d}\mu = \int_0^\infty \tilde\rho_1(\mu)\,(N_{x,x}(\lambda-\mu)-N_{x,x}(\lambda+\mu))\,\mathrm{d}\mu\,.$$

Substituting (1.2.2) in the right hand side and estimating

$$|(\lambda-\mu)^n-(\lambda+\mu)^n| \;\leqslant\; C\,(|\mu|\,\lambda^{n-1}+|\mu|^n)\,,\qquad |\lambda\pm\mu|^{n-1} \;\leqslant\; C\,(\lambda^{n-1}+|\mu|^{n-1})\,,$$

we see that

$$(4.2.2)\qquad \int_M |N_{x,y;1}(\lambda)|^2\,\mathrm{d}y \;\leqslant\; C\,(\lambda^{n-1}+1)\,,\qquad \forall\lambda>0\,,$$

where the constant $C$ depends only on the dimension, the remainder term in the Weyl formula (1.2.2) and the auxiliary function $\rho$.

4.3. **Upper bounds for** $N_{x,y;0}(\lambda)$**.** Since $\frac{\mathrm{d}}{\mathrm{d}\lambda}N_{x,y;0}(\lambda) = \int\rho(\lambda-\mu)\,\mathrm{d}N_{x,y}(\mu)$ and

$$\int \rho(\lambda-\mu)\,\mathrm{d}N_{x,y}(\mu) = \rho * N'_{x,y}(\lambda) = (\hat\rho\,\sigma_{x,y})^\vee\,(\lambda)\,,$$

Condition 4.1.1 implies that the asymptotic behaviour of $N_{x,y;0}(\lambda)$ for large $\lambda$ is determined by the singularities of $\sigma_{x,y}$ on $\operatorname{supp}\hat\rho$. For second order differential operators, it is slightly more convenient to deal with the cosine Fourier transform $e_{x,y}(t) := \int\cos(t\lambda)\,\mathrm{d}N_{x,y}(\lambda)$. The distribution $e_{x,y}$ coincides with the fundamental solution of the wave equation. For all sufficiently small $t$, it admits the following Hadamard representation

$$(4.3.1)\qquad e_{x,y}(t) \;=\; |t|\sum_{j=0}^\infty u_j(x,y)\,\frac{(t^2-d_{x,y}^2)_+^{j-\frac{n+1}{2}}}{\Gamma(j-\frac{n+1}{2}+1)} \quad \bmod C^\infty$$

where $u_j(x,y)$ are some smooth functions (see [Ba] or [Ber, Proposition 27]). Let us assume that $\delta$ in Condition 4.1.1 is small enough so that (4.3.1) holds on the interval $(-\delta,\delta)$ (and, consequently, on $\operatorname{supp}\hat\rho$).

Denote $\Psi_{x,y}(\lambda) := N_{x,y}(\lambda) - N_{x,y}(-\lambda)$. Changing variables and taking into account that the function $\rho$ is even, we see that

$$\rho * \Psi_{x,y}(\lambda) = N_{x,y;0}(\lambda) - N_{x,y;0}(-\lambda)\,.$$

Since $N_{x,y}(\lambda)=0$ for $\lambda<0$ and $\rho$ is a rapidly decreasing function, $N_{x,y;0}(-\lambda)$ vanishes faster than any power of $\lambda$ as $\lambda\to+\infty$. Thus we have

$$N_{x,y;0}(\lambda) \;=\; \rho*\Psi_{x,y}(\lambda) \;+\; o(\lambda^{-m})\,, \qquad \forall m\in\mathbb{R}_+\,. \tag{4.3.2}$$

Integrating by parts, we obtain

$$e_{x,y}(t) \;=\; t\int \sin(t\lambda)\, N_{x,y}(\lambda)\,\mathrm{d}\lambda \;=\; -(2i)^{-1}\, t\,\hat\Psi_{x,y}(t)\,. \tag{4.3.3}$$

In view of (4.3.1) and (4.3.3),

$$\hat\Psi_{x,y}(t) \;=\; -2i\ \operatorname{sign} t\sum_{j=0}^{\infty} u_j(x,y)\,\frac{(t^2-d_{x,y}^2)_+^{j-\frac{n+1}{2}}}{\Gamma(j-\frac{n+1}{2}+1)} \quad \operatorname{mod} C^\infty \tag{4.3.4}$$

on $\operatorname{supp}\hat\rho$. We have

$$\begin{aligned}\int_{|t|>d_{x,y}} e^{it\lambda}\operatorname{sign} t\,\frac{(t^2-d_{x,y}^2)^{p-\frac12}}{\Gamma\left(p+\frac12\right)}\,\mathrm{d}t \;&=\; 2i\int_{d_{x,y}}^{\infty}\sin(t\lambda)\,\frac{(t^2-d_{x,y}^2)^{p-\frac12}}{\Gamma\left(p+\frac12\right)}\,\mathrm{d}t\\ &=\; i\sqrt{\pi}\left(\frac{2\,d_{x,y}}{\lambda}\right)^{p} J_{-p}(d_{x,y}\lambda)\,,\end{aligned} \tag{4.3.5}$$

(see, for example, [GR, formula 3.771(7)] or [GS, Chapter II, Section 2.5]). Therefore, taking the inverse Fourier transform of each term in the right-hand side of (4.3.4), we obtain an asymptotic series

$$\sum_{j=0}^{\infty} w_j(x,y)\,d_{x,y}^{\,j-\frac n2}\,\lambda^{\frac n2-j}\,J_{\frac n2-j}\left(d_{x,y}\,\lambda\right) \tag{4.3.6}$$

with some bounded functions $w_j(x,y)$. Let $\Psi_{0;x,y}(\lambda)$ be an arbitrary function such that $\Psi_{0;x,y}(\lambda)\sim\sum_{j=0}^\infty w_j(x,y)\,d_{x,y}^{\,j-\frac n2}\,\lambda^{\frac n2-j}\,J_{\frac n2-j}\left(d_{x,y}\,\lambda\right)$ as $\lambda\to+\infty$ uniformly with respect to $x,y\in M$. By the above,

$$\rho*\Psi_{x,y}(\lambda)-\rho*\Psi_{0;x,y}(\lambda) \;=\; O(\lambda^{-\infty})\,. \tag{4.3.7}$$

Recall that the Bessel function $J_p(\tau)$ is estimated by $C\,\tau^p$ in a neighbourhood of the origin and does not exceed $C\,\tau^{-\frac12}$ for large $\tau$ (see [GR, 8.440 and 8.451(1)]). Therefore

$$|d_{x,y}^{-p}\,\lambda^{\frac12}\,J_p\left(d_{x,y}\,\lambda\right)| \;\leqslant\; \begin{cases} C\,\lambda^{p+\frac12}\,, & \lambda\leqslant d_{x,y}^{-1}\,,\\ C\,d_{x,y}^{-p-\frac12}\,, & \lambda> d_{x,y}^{-1}\,,\end{cases} \qquad \forall p\in\mathbb{R}\,. \tag{4.3.8}$$

The inequalities (4.3.8) imply that the terms in the expansion (4.3.6) are bounded by $C_j\, d_{x,y}^{j-\frac{n+1}{2}}\, \lambda^{\frac{n-1}{2}-j}$ for $\lambda \geqslant d_{x,y}^{-1}$ and are estimated by $C_j\, \lambda^{n-2j}$ for $\lambda \in (\varepsilon, d_{x,y}^{-1})$, where $\varepsilon$ is an arbitrary positive constant. Thus, if $\lambda > \varepsilon$ then

$$\left|w_j(x,y)\, d_{x,y}^{j-\frac{n}{2}}\, \lambda^{\frac{n}{2}-j}\, J_{\frac{n}{2}-j}\,(d_{x,y}\,\lambda)\right| \ \leqslant\ C_j \left(d_{x,y}^{j-\frac{n+1}{2}} + 1\right) \lambda^{\frac{n-1}{2}-j}$$

for all $j = 0, 1, 2, \ldots$ and, consequently,

$$\left|\Psi_{0;x,y}(\lambda) - w_0(x,y)\, d_{x,y}^{-\frac{n}{2}}\, \lambda^{\frac{n}{2}}\, J_{\frac{n}{2}}\,(d_{x,y}\,\lambda)\right| \ \leqslant\ C \left(d_{x,y}^{\frac{1-n}{2}} + 1\right) \lambda^{\frac{n-3}{2}}. \tag{4.3.9}$$

By direct calculation, $w_0(x,y) = 2^{-\frac{n}{2}}\pi^{-\frac{1}{2}} u_0(x,y)$. We have

$$u_0(x,y) \ =\ \pi^{\frac{1-n}{2}}\, (\det\{g_{jk}(x,y)\})^{-1/4}\ ,$$

where $\{g_{jk}(x,y)\}$ is the metric tensor in geodesic normal coordinates with origin $x$ ([JP, formula (3.1.1)]), so that $w_0(x,y) = (2\pi)^{-\frac{n}{2}} + O(d_{x,y}^2)$ (see [Ros, p. 101]). From this estimate and (4.3.9) it follows that

$$\int_M \left|\Psi_{0;x,y}(\lambda) - (2\pi)^{-\frac{n}{2}}\, d_{x,y}^{-\frac{n}{2}}\, \lambda^{\frac{n}{2}}\, J_{\frac{n}{2}}\,(d_{x,y}\,\lambda)\right|^2 \mathrm{d}y \ \leqslant\ C_\varepsilon\, \lambda^{n-1} \int_M \left(d_{x,y}^{1-n} + 1\right) \mathrm{d}y$$

for all $\lambda > \varepsilon$, where $\varepsilon$ is an arbitrary positive number and $C_\varepsilon$ is a constant depending only on $\varepsilon$, the auxiliary function $\rho$ and the geometry of $M$. Finally, since the function $\rho$ is rapidly decreasing, the above inequality, (4.3.2) and (4.3.7) imply that

$$\begin{aligned} &\int_M \left|N_{x,y;0}(\lambda) - (2\pi)^{-\frac{n}{2}}\, d_{x,y}^{-\frac{n}{2}}\, \lambda^{\frac{n}{2}}\, J_{\frac{n}{2}}\,(d_{x,y}\,\lambda)\right|^2 \mathrm{d}y \\ &\qquad\leqslant\ C_\varepsilon\, \lambda^{n-1} \int_M \left(d_{x,y}^{1-n} + 1\right) \mathrm{d}y\,, \qquad \forall \lambda \geqslant \varepsilon > 0\,, \end{aligned} \tag{4.3.10}$$

where $C_\varepsilon$ is another constant depending only on $\varepsilon$, $\rho$ and the geometry of $M$.

*Remark* 4.3.11. The above proof is a slight modification of arguments in [Ba].

4.4. **Proof of Theorem 2.1.2.** The theorem follows from (4.3.10), (4.2.2) and (1.3.2).

4.5. **Proof of Theorem 2.2.1.** If $\varkappa > 1$ then, in view of (2.4.2),

$$\int_0^\infty \int_M d_{x,y}^{\varkappa}\, |\tilde N_{x,y}(\lambda)|^2\, \mathrm{d}y\, \mathrm{d}\nu(\lambda) \ <\ \infty\,.$$

Therefore, by Fubini's theorem, the integral $\int_0^\infty |\tilde N_{x,y}(\lambda)|^2\, \mathrm{d}\nu(\lambda)$ is finite for almost all $y \in M$.

4.6. **Proof of Theorem 2.3.1.** By (4.3.8), there exists a constant $C_1$ such that

$$\left| d_{x,y}^{-\frac{n}{2}}\,\mu^{\frac12}\,J_{\frac{n}{2}}\,(d_{x,y}\,\mu)\right| \;\leqslant\; C_1\,d_{x,y}^{-\frac{n+1}{2}}\,.$$

If $C > \sqrt{2}C_1$ then the above inequality and (2.1.3) imply that

$$\begin{aligned}
2\,C_M \;&\geqslant\; 2\int_{\Omega_x(\lambda,\mu)} \left|\tilde N_{x,y}(\lambda) - d_{x,y}^{-\frac{n}{2}}\,\mu^{\frac12}\,J_{\frac{n}{2}}\,(d_{x,y}\,\mu)\right|^2 \mathrm{d}y\\
&\geqslant\; \int_{\Omega_x(\lambda,\mu)} \left(|\tilde N_{x,y}(\lambda)|^2 - 2\,C_1^2\,d_{x,y}^{-n-1}\right)\mathrm{d}y\\
&\geqslant\; \int_{\Omega_x(\lambda,\mu)} \left(C^2\,\mu^2 + (C^2-2\,C_1^2)\,d_{x,y}^{-n-1}\right)\mathrm{d}y \;\geqslant\; C^2\,\mu^2\int_{\Omega_x(\lambda,\mu)}\mathrm{d}y\,.
\end{aligned}$$

This proves the corollary.

4.7. **Proof of Corollary 2.3.3.** If $y \neq x$ then the inequalities (2.3.4) imply that $y \in \bigcap_{k\geqslant N}\Omega_x(\tau_k,(2C)^{-1}\mu_k)$ for all sufficiently large $N$. By Theorem 2.3.1, the measure of this intersection is zero.

If $\sum_k \mu_k^{-2} < \infty$ then, according to Theorem 2.3.1, the sum of measures of the sets $\Omega_x(\tau_k,t\mu_k)$ is finite for all $t>0$. By the Borel–Cantelli lemma, in this case almost every point $y\in M$ belongs only to finitely many sets $\Omega_x(\tau_k,t\mu_k)$. This implies that $\limsup_{k\to\infty}\left(\mu_k^{-1}\,\tilde N_{x,y}(\tau_k)\right) < t$ almost everywhere. Letting $t\to 0$, we obtain (2.3.5).

4.8. **Proof of Theorem 2.4.1.** Let $d_M := \operatorname{diam} M$. We have

$$\begin{aligned}
\int_M d_{x,y}^{\varkappa}\left|d_{x,y}^{-\frac{n}{2}}\,\lambda^{\frac12}\,J_{\frac{n}{2}}\,(d_{x,y}\,\lambda)\right|^2\mathrm{d}y \;&\leqslant\; C\,\lambda\int_M d_{x,y}^{\varkappa-n}\,J_{\frac{n}{2}}^2\,(d_{x,y}\,\lambda)\,\mathrm{d}y\\
&\leqslant\; C\,\lambda\int_0^{d_M} r^{\,\varkappa-1}\,J_{\frac{n}{2}}^2(r\,\lambda)\,\mathrm{d}r \;=\; C\,\lambda^{1-\varkappa}\int_0^{\lambda\,d_M} r^{\,\varkappa-1}\,J_{\frac{n}{2}}^2(r)\,\mathrm{d}r\,.
\end{aligned}$$

Even if spherical coordinates centred at $x$ do not exist globally on $M$, it is still possible to use the pull-back of the metric under the exponential map to change the integration variable to $r$. Since the Bessel function $J_{\frac{n}{2}}$ is bounded by $C\,r^{\frac{n}{2}}$ in a neighbourhood of the origin and does not exceed $C\,r^{-\frac12}$ for large $r$, the right hand side is estimated by $C\,\lambda^{1-\varkappa}\left((n+\varkappa)^{-1} + \int_1^{\lambda\,d_M} r^{\,\varkappa-2}\,\mathrm{d}r\right)$ if $\lambda\,d_M > 1$. Thus we obtain

$$\int_M d_{x,y}^{\varkappa}\left|w_0(x,y)\,d_{x,y}^{-\frac{n}{2}}\,\lambda^{\frac12}\,J_{\frac{n}{2}}\,(d_{x,y}\,\lambda)\right|^2\mathrm{d}y \;\leqslant\; \begin{cases} C_\varkappa\,(\lambda^{1-\varkappa} + d_M^{\varkappa-1})\,, & \varkappa\neq 1\,,\\ C\,|\ln\lambda|\,, & \varkappa = 1\,.\end{cases}$$

with $C_\varkappa = C|\varkappa-1|^{-1}$. This estimate together with (2.1.3) (multiplied by $d_M^{\varkappa}$ on both sides) imply (2.4.2) by triangle inequality.

4.9. **Proof of Corollary 2.4.3.** Let $g_\varkappa(\lambda) := C_{\varkappa,M}\,(1+\lambda^{1-\varkappa})$ if $\varkappa \neq 1$ and $g_1(\lambda) := C_M(1+|\ln\lambda|)$. By (2.4.2),

$$|\int_M \mathcal{K}_\lambda(x,y)\,u(y)\,\mathrm{d}y\,| \;\leqslant\; g_\varkappa(\lambda)\,\|u\|_{L^\infty}$$

for all $u \in L^\infty(M)$ and

$$\int_M \left|\int_M \mathcal{K}_\lambda(x,y)\,v(y)\,\mathrm{d}y\right|\,\mathrm{d}x \leqslant \int_M \left(\int_M \mathcal{K}_\lambda(x,y)\,\mathrm{d}x\right)|v(y)|\,\mathrm{d}y \leqslant g_\varkappa(\lambda)\,\|v\|_{L^1}$$

for all $v \in L^1(M)$. These estimates imply the required result for $p=\infty$ and $p=1$ respectively. For other values of $p$ the result follows from the Riesz interpolation theorem (see [So, Theorem 0.1.13]).

## 5. Proofs of the lower bounds

5.1. **Proof of Theorem 2.5.1.** Let $x,y \in M$ be not conjugate along any geodesic segment of length $d \in \mathcal{L}_{x,y}$. Then, as follows from [Mil, Theorem 16.2], the number $m$ of geodesic segments of length $d$ joining $x$ and $y$ is finite, and there exists $\varepsilon > 0$ such that $(d-\varepsilon, d+\varepsilon) \cap \mathcal{L}_{x,y} = d$. Set $\psi(\lambda) = \exp(i\lambda\,d)\,\rho(\lambda)$, where $\rho$ satisfies Condition 4.1.1. Then $\hat\psi(d) = 1$, and one could choose $\rho$ in such a way that $\operatorname{supp}\hat\psi \subset (d-\varepsilon, d+\varepsilon)$. By [LSV, Theorem 4.2] (see also [JP, Proposition 3.3.6]) we have:

$$\left(\hat\psi\,\sigma_{x,y}\right)^\vee(\mu) \;=\; \int \psi(\mu-\tau)\,\mathrm{d}N_{x,y}(\tau) \;=\; P(\mu)\,\mu^{\frac{n-1}{2}} + O\left(\mu^{\frac{n-3}{2}}\right) \tag{5.1.1}$$

with a periodic function $P(\mu)$ of the form $P(\mu) = \sum_{k=1}^m C_k\,e^{-id\mu+\theta_k}$, where $C_k$ are positive constants and $\theta_k$ are phase shifts along the geodesics. Since

$$\int |\psi'(\mu-\tau)|\,|N_{x,y}(\tau)|\,\mathrm{d}\tau \;\geqslant\; |\int \psi'(\mu-\tau)\,N_{x,y}(\tau)\,\mathrm{d}\tau| \;=|\int \psi(\mu-\tau)\,\mathrm{d}N_{x,y}(\tau)|\,,$$

(5.1.1) implies that

$$\int \mu^q\,|\psi'(\mu-\tau)|\,|N_{x,y}(\tau)|\,\mathrm{d}\tau \;\geqslant\; |P(\mu)|\,\mu^{\frac{n-1}{2}+q} + O\left(\mu^{\frac{n-3}{2}+q}\right)\,, \qquad \forall q \geq 0\,.$$

The rest of the proof is similar to that of [Sar, Lemma 5.1] Integrating the above inequality and taking into account the estimate

$$\int_0^{\frac{\lambda}{2}} |P(\mu)|\,\mu^{\frac{n-1}{2}+q}\,\mathrm{d}\mu \;\gg\; \lambda^{\frac{n+1}{2}+q}\,, \tag{5.1.2}$$

we obtain

$$\int \tilde\psi(\lambda,\tau)\,|N_{x,y}(\tau)|\,\mathrm{d}\tau \;\geqslant\; C\,\lambda^{\frac{n+1}{2}+q} + O\left(\lambda^{\frac{n-1}{2}+q}\right), \tag{5.1.3}$$

where $\tilde\psi(\lambda,\tau) := \int_0^{\frac{\lambda}{2}} \mu^q\,|\psi'(\mu-\tau)|\mathrm{d}\mu$ . Since $\psi'$ is a rapidly decreasing function, we have

$$\tilde\psi(\lambda,\tau) \;\leqslant\; C\int_0^{\frac{\lambda}{2}} (\tau^q+|\mu-\tau|^q)\,|\psi'(\mu-\tau)|\mathrm{d}\mu \;\leqslant\; C\,(\tau^q+1)\,, \qquad \forall\tau\in\mathbb{R}_+\,,$$

and $\tilde\psi(\lambda,\tau) \;\leqslant\; C_j\,(\lambda\,|\lambda-\tau|)^{-j}$ for all $\tau\geqslant\lambda$ and $j=1,2,\dots$ These estimates imply, respectively, that

$$\int_0^{\lambda}\tilde\psi(\lambda,\tau)\,|N_{x,y}(\tau)|\,\mathrm{d}\tau \leqslant\; C\int_0^{\lambda}\tau^q\,|N_{x,y}(\tau)|\,\mathrm{d}\tau \tag{5.1.4}$$

and

$$\int_\lambda^{\infty}\tilde\psi(\lambda,\tau)\,|N_{x,y}(\tau)|\,\mathrm{d}\tau \;\leqslant\; C\int_\lambda^{\infty}\tilde\psi(\lambda,\tau)\,\tau^n\,\mathrm{d}\tau \;=\; O(1)\,. \tag{5.1.5}$$

Putting together (5.1.3)–(5.1.5), we see that

$$\lambda^{-q-1}\int_0^{\lambda}\tau^q\,|\tilde N_{x,y}(\tau)|\,\mathrm{d}\tau \;\gg\; \lambda^{-\frac{n+1}{2}-q}\int_0^{\lambda}\tau^q\,|N_{x,y}(\tau)|\,\mathrm{d}\tau \;\gg\; 1\,.$$

Now (2.5.2) with $p>1$ follows from Jensen's inequality. □

*Remark* 5.1.6. The inequality (5.1.1) holds for $N_{x,x}(\lambda)$ under the assumption that $x$ is not conjugate to itself along any geodesic loop of some fixed length $d$. Therefore, similar lower bounds can be proved for the oscillatory error term $R_x^{osc}(\lambda)$ introduced in [JP, section 1.2]. Note that in dimension two this term coincides with the usual pointwise remainder in Weyl's law.

5.2. **Proof of Corollary 2.5.3.** Condition (2.5.4) implies that there is a sequence of points $\mu_k\to+\infty$ and a positive constant $C$ such that

$$\mu_k^{q+1}\mu^{-q}f(\mu) \;\geqslant\; C\,, \qquad \forall\mu\leqslant\mu_k\,.$$

Let $\chi_k(\mu)$ be the characteristic functions of the intervals $(0,\mu_k]$ , $k=1,2,\dots$ By the above, the functions $g_k(\mu) := \mu_k^{-q-1}\chi_k(\mu)\,\mu^q\,(f(\mu))^{-1}$ are uniformly bounded by a constant. Obviously, $g_k(\mu)\to0$ as $k\to\infty$ for each fixed $\mu$. If the integral $\int f(\mu)\,|N_{x,y}(\mu)|^p\,\mathrm{d}\mu$ were finite then, by the Lebesgue dominated convergence theorem, we would have

$$\mu_k^{-q-1}\int_0^{\mu_k}\mu^q\,|N_{x,y}(\mu)|^p\,\mathrm{d}\mu = \int g_k(\mu)\,f(\mu)\,|N_{x,y}(\mu)|^p\,\mathrm{d}\mu \;\to\; 0\,, \qquad k\to\infty\,.$$

However, this contradicts to (2.5.2).

The estimates (2.5.5) and (2.5.6) are obtained by taking $f(\mu):=\mu^{-1}$ and $f(\mu):=\mu_k^{-1}\,(\chi_k(\mu)-\chi_{k-1}(\mu))$ respectively.

## 6. Examples

6.1. **Circle and 2-tori.** The purpose of this subsection is to justify Examples 3.2.2 and 3.2.4. Let $n = 1, 2$ and let $\Gamma$ be a lattice in $\mathbb{R}^n$. Consider the flat torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$. If $n = 1$ one can assume that $\Gamma = 2\pi\mathbb{Z}$, and thus $\mathbb{T}^1 \cong \mathbb{S}^1$. We have

$$N_{x,y}(\lambda) \;=\; \frac{1}{\mathrm{Vol}(\Gamma)} \sum_{\xi\in\Gamma^*} e^{2\pi i\langle x-y,\xi\rangle}\, \chi_{B^n}(2\pi\xi/\lambda)$$

where $\chi_{B^n}$ is the characteristic function of the unit ball, $\mathrm{Vol}(\Gamma)$ is the volume of the torus and $\Gamma^* = \{\xi \in \mathbb{R}^n \,:\, (\xi,\eta) \in \mathbb{Z}\,,\, \forall \eta \in \Gamma\}$ is the dual lattice (see [Ch2, p. 29]. We are slightly abusing notation here, since we added to the spectral function the constant term corresponding to $\xi = 0$. Note that $N_{x,y} = N_{x-y,0}$, hence we can assume without loss of generality that $y = 0$.

Let $w(\xi) \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function depending only on $|\xi|$, such that $\int w(\xi)\,\mathrm{d}\xi = 1$. Denote $w_\varepsilon(\xi) := \varepsilon^{-n}\, w(\varepsilon^{-1}\xi)$ for $\varepsilon > 0$. By the Poisson summation formula,

$$\begin{aligned} N_{x,0}^{(\varepsilon)}(\lambda) \;:=\; & \frac{1}{\mathrm{Vol}(\Gamma)} \sum_{\xi\in\Gamma^*} e^{2\pi i\langle x,\xi\rangle} (w_\varepsilon * \chi_{B^n})(2\pi\xi/\lambda) \\ & = \sum_{\eta\in\Gamma} \int_{\mathbb{R}^n} e^{2\pi i\langle x+\eta,\xi\rangle} (w_\varepsilon * \chi_{B^n})(2\pi\xi/\lambda)\,\mathrm{d}\xi \\ & = \frac{\lambda^n}{(2\pi)^n} \sum_{\eta\in\Gamma} \hat{w}(\varepsilon\lambda(x+\eta))\, \hat{\chi}_{B^n}(\lambda(x+\eta))\,. \end{aligned}$$

In what follows we set $\varepsilon = (\lambda T)^{-1}$. Applying the asymptotic formula (1.3.3) for $\hat{\chi}_{B^n}$, one gets

$$\frac{\lambda^n}{(2\pi)^n}\, \hat{\chi}_{B^n}(\lambda(x+\eta)) = \frac{2\,\lambda^{(n-1)/2}\,\sin\left(\lambda\,|x+\eta| - \frac{(n-1)\pi}{4}\right)}{(2\pi)^{(n+1)/2}\,|x+\eta|^{(n+1)/2}} + O\left(\frac{\lambda^{(n-3)/2}}{|x+\eta|^{(n+3)/2}}\right).$$

Using the formulae above, let us show that $\tilde{N}_{x,0}(\lambda) \in B^2$ and

$$\tilde{N}_{x,0}(\lambda) \;\sim\; \sum_{\eta\in\Gamma} \frac{2\,\sin\left(\lambda\,|x+\eta| - \frac{(n-1)\pi}{4}\right)}{(2\pi)^{(n+1)/2}\,|x+\eta|^{(n+1)/2}}\,. \tag{6.1.1}$$

The proof of (6.1.1) follows closely [Bl2, section 3] and is split into four lemmas.

**Lemma 6.1.2.**

$$\int_1^T \left| \sum_{\xi\in\Gamma^*} e^{2\pi i\langle x,\xi\rangle} \left(\chi_{B^n}(2\pi\xi/\lambda) - w_{(\lambda T)^{-1}} * \chi_{B^n}(2\pi\xi/\lambda)\right)\right|^2 \frac{\mathrm{d}\lambda}{\lambda^{n-1}} \;=\; O(1).$$

*Proof.* Set

$$f_\xi(\lambda) = \chi_{B^n}(2\pi\xi/\lambda) - w_{(\lambda T)^{-1}} * \chi_{B^n}(2\pi\xi/\lambda) = \chi_{\lambda B^n}(2\pi\xi) - w_{T^{-1}} * \chi_{\lambda B^n}(2\pi\xi),$$

where $\lambda B^n$ is a ball of radius $\lambda$. Since by definition $\int w_{T^{-1}}(\xi) d\xi = 1$ and $w_{T^{-1}} \in C_0^\infty$, it follows that $f_\xi(\lambda)$ is supported on an interval of size $O(T^{-1})$ centred at $\lambda = 2\pi|\xi|$. By a result of [Col], we have

$$\#\{\xi \in \Gamma^*, |\xi| < R\} \;=\; \frac{C_n R^n}{\mathrm{Vol}(\Gamma)} + O(R^{n-2+\frac{2}{n+1}})\,, \qquad \forall n = 1, 2, \dots$$

Therefore, counting lattice points in an annulus of exterior radius $|\xi| + O(T^{-1})$ and interior radius $\xi - O(T^{-1})$ we get

$$\#\{\xi \in \Gamma^*, \operatorname{supp} f_\xi(\lambda) \cap \operatorname{supp} f_\xi(\lambda) \neq \varnothing\} = O(T^{-1}|\xi|^{n-1} + |\xi|^{n-2+\frac{2}{n+1}}). \tag{6.1.3}$$

This gives us an estimate on the number of non-vanishing cross-terms in the expression under the integral in Lemma 6.1.2. Since all the terms in this expression are bounded by a constant, taking into account (6.1.3) and the size of the support of $f_\xi(\lambda)$, we obtain

$$\int_1^T \left| \sum_{\xi \in \Gamma^*} e^{2\pi i \langle x, \xi\rangle} \left(\chi_{B^n}(2\pi\xi/\lambda) - w_{(\lambda T)^{-1}} * \chi_{B^n}(2\pi\xi/\lambda)\right)\right|^2 \frac{\mathrm{d}\lambda}{\lambda^{n-1}}$$
$$\leqslant\; C T^{-1} \sum_{2\pi|\xi| \leqslant T} (T^{-1} + |\xi|^{-3+2/(n+1)}) \;=\; O(1)\,.$$

Note that it is essential for the final equality that $n < 3$. ☐

**Lemma 6.1.4.**

$$\int_1^T \left| \sum_{\eta \in \Gamma} \frac{|\hat{w}(T^{-1}\,(x+\eta))|}{\lambda\,|x+\eta|^{(n+3)/2}} \right|^2 \mathrm{d}\lambda \;=\; O(1)\,.$$

*Proof.* Indeed,

$$\int_1^T \left| \sum_{\eta \in \Gamma} \frac{|\hat{w}(T^{-1}\,(x+\eta))|}{\lambda\,|x+\eta|^{(n+3)/2}} \right|^2 \mathrm{d}\lambda \;\leqslant\; \left( \sum_{\eta \in \Gamma} \frac{|\hat{w}(T^{-1}(x+\eta))|}{|x+\eta|^{(n+3)/2}} \right)^2$$
$$\leqslant\; \left( \sum_{\eta \in \Gamma} \frac{1}{|x+\eta|^{(n+3)/2}} \right)^2 \;=\; O(1).$$

Here the first inequality holds since $\int_1^T \lambda^{-2} d\lambda \leqslant 1$, the second inequality is true because $|\hat{w}(x)| = |\int e^{-i\langle x,\xi\rangle} w(\xi)\,\mathrm{d}\xi| \leqslant \int w(\xi)\,\mathrm{d}\xi = 1$, and in the last step we use that $n < 3$. ☐

In the lemmas below all the summations are also taken over elements $\eta \in \Gamma$.

**Lemma 6.1.5.**

$$\lim_{Q\to+\infty}\lim_{T\to+\infty}\frac{1}{T}\int_1^T\left|\sum_{|\eta|>Q}\frac{\hat{w}(T^{-1}(x+\eta))}{|x+\eta|^{(n+1)/2}}\ \sin\left(\lambda\,|x+\eta|-\frac{(n-1)\pi}{4}\right)\right|^2 \mathrm{d}\lambda\ =0$$

The proof of this lemma is a modification of the proof of [Bl2, Lemma 3.3].

*Proof.* One can check that

$$\left|\int_1^T \sin\left(\lambda\,|x+\eta|-\frac{(n-1)\pi}{4}\right)\sin\left(\lambda\,|x+\xi|-\frac{(n-1)\pi}{4}\right)\mathrm{d}\lambda\ \right| \leqslant\ C\min\{T,||x+\eta|-|x+\xi||^{-1}\}\,.$$

Expanding the squared sum in the integral below and taking into account that $|\hat{w}(x)|<C(1+|x|)^{-2n}$ as $\hat{w}$ is rapidly decreasing, we obtain

$$(6.1.6)\quad \frac{1}{T}\int_1^T\left|\sum_{|\eta|>Q}\frac{\hat{w}(T^{-1}(x+\eta))}{|x+\eta|^{\frac{n+1}{2}}}\ \sin\left(\lambda\,|x+\eta|-\frac{(n-1)\pi}{4}\right)\right|^2\mathrm{d}\lambda$$
$$\leqslant\ C\sum_{|\eta|>Q}\frac{T^{-1}|\eta|^{n-1}+|\eta|^{n-2+\frac{2}{n+1}}}{|\eta|^{(n+1)}(1+T^{-1}|\eta|)^{4n}}$$
$$+\ \frac{C}{T}\sum_{|\eta|>Q}\ \sum_{k=\lceil T^{-1}|\eta|^{1-\frac{2}{n+1}}\rceil}^{\lfloor|\eta|^{1-\frac{2}{n+1}}\rfloor}\frac{|\eta|^{1-\frac{2}{n+1}}}{k}\,\frac{|\eta|^{n-2+\frac{2}{n+1}}}{|\eta|^{(n+1)}(1+T^{-1}|\eta|)^{4n}}$$
$$+\ \frac{C}{T}\sum_{k=1}^{+\infty}\frac{1}{k}\sum_{j>Q}\frac{j^{\frac{n-3}{2}}(j+k)^{\frac{n-3}{2}}}{(1+T^{-1}j)^{2n}(1+T^{-1}(j+k))^{2n}}\,.$$

In the right hand side of (6.1.6), the first sum bounds the contribution of the terms corresponding to pairs $\eta,\xi$ such that $||x+\eta|-|x+\xi||\leqslant T^{-1}$. The second sum estimates the contribution of the terms such that $T^{-1}<||x+\eta|-|x+\xi||\leqslant 1$. Here we consider each subinterval of length $|x+\eta|^{-1+\frac{2}{n+1}}$ separately. The last sum takes care of pairs $\eta,\xi$ such that $k<||x+\eta|-|x+\xi||\leqslant k+1$, with $k\geqslant 1$.

For $T>Q$, the first sum in (6.1.6) is bounded by

$$\sum_{|\eta|>Q}\frac{T^{-1}|\eta|^{n-1}+|\eta|^{n-2+\frac{2}{n+1}}}{|\eta|^{n+1}(1+T^{-1}|\eta|)^{4n}}\ \leqslant\ \begin{cases}C\,(Q^{-1}+T^{-1}Q^{-1})\,, & n=1\,,\\ C\,(Q^{-1/3}+T^{-1}|\ln T-\ln Q|)\,, & n=2\,.\end{cases}$$

In order to estimate the second sum in (6.1.6), we first note that

$$\sum_{k=\lceil T^{-1}|\eta|^{1-\frac{2}{n+1}}\rceil}^{\lfloor|\eta|^{1-\frac{2}{n+1}}\rfloor}\frac{|\eta|^{1-\frac{2}{n+1}}}{k}\ <\ C|\eta|^{1-\frac{2}{n+1}}\ \ln T.$$

for all sufficiently large $T$. Estimating the sum over $\eta$ by an integral we get

$$\frac{1}{T}\sum_{|\eta|>Q}\sum_{k=\lceil T^{-1}|\eta|^{1-\frac{2}{n+1}}\rceil}^{\lfloor|\eta|^{1-\frac{2}{n+1}}\rfloor}\frac{|\eta|^{1-\frac{2}{n+1}}}{k}\frac{|\eta|^{n-2+\frac{2}{n+1}}}{|\eta|^{n+1}\,(1+T^{-1}|\eta|)^{4n}}$$
$$<\frac{C\ln T}{T}\int_Q^{+\infty}\frac{r^{n-3}\mathrm{d}r}{(1+T^{-1}r)^{4n}}\leqslant\begin{cases}C_Q\,T^{-1}\ln T\,, & n=1\,,\\ C_Q\,T^{-1}\ln^2 T\,, & n=2\,.\end{cases}$$

To find a bound for the third sum in (6.1.6), we use the inequality $2a^{\frac{n-3}{2}}b^{\frac{n-3}{2}}\leqslant a^{n-3}+b^{n-3}$ and once again estimate the sum by an integral,

$$\sum_{j>Q}\frac{j^{\frac{n-3}{2}}\,(j+k)^{\frac{n-3}{2}}}{(1+T^{-1}j)^{2n}(1+T^{-1}(j+k))^{2n}}$$
$$<\frac{C}{(1+T^{-1}k)^{2n}}\int_Q^{+\infty}\frac{r^{n-3}\mathrm{d}r}{(1+T^{-1}r)^{2n}}+C\int_{Q+k}^{+\infty}\frac{r^{n-3}\mathrm{d}r}{(1+T^{-1}r)^{2n}}\,.$$

Hence we have

$$\frac{C}{T}\sum_{k=1}^{+\infty}\frac{1}{k}\sum_{j>Q}\frac{j^{\frac{n-3}{2}}\,(j+k)^{\frac{n-3}{2}}}{(1+T^{-1}j)^{2n}(1+T^{-1}(j+k))^{2n}}\leqslant\begin{cases}C_Q\,T^{-1}\ln T\,, & n=1\,,\\ C_Q\,T^{-1}\ln^2 T\,, & n=2\,.\end{cases}$$

This completes the proof of the lemma. □

Note that the proof of Lemma 6.1.5 uses the fact that the limit with respect to $T$ is taken first.

**Lemma 6.1.7.**

$$\int_1^T\left|\sum_{|\eta|\leqslant Q}\frac{1-\hat{w}(T^{-1}(x+\eta))}{|x+\eta|^{(n+1)/2}}\;\sin\left(\lambda\,|x+\eta|-\frac{(n-1)\pi}{4}\right)\right|^2\mathrm{d}\lambda\;=\;o(T)\,.$$

*Proof.* This estimate holds because the sum involves a finite number of terms and $\lim_{T\to+\infty}\hat{w}(T^{-1}x)=1$ for all $x$. □

By Lemma 6.1.2 we get

$$\lim_{T\to+\infty}\frac{1}{T}\int_1^T\left|N_{x,0}(\lambda)-N_{x,0}^{((\lambda T)^{-1})}(\lambda)\right|^2\frac{\mathrm{d}\lambda}{\lambda^{n-1}}=0\,.$$

Lemmas 6.1.4, 6.1.5 and 6.1.7 imply

$$\lim_{Q\to+\infty}\lim_{T\to+\infty}\frac{1}{T}\int_1^T\left|\lambda^{\frac{1-n}{2}}N_{x,0}^{((\lambda T)^{-1})}(\lambda)-\sum_{|\eta|\leqslant Q}\frac{2\,\sin\left(\lambda\,|x+\eta|-\frac{(n-1)\pi}{4}\right)}{(2\pi)^{(n+1)/2}\,|x+\eta|^{(n+1)/2}}\right|^2\mathrm{d}\lambda\;=\;0\,,$$

so that

$$\lim_{Q\to+\infty}\lim_{T\to+\infty}\frac{1}{T}\int_1^T\left|\tilde{N}_{x,0}(\lambda)-\sum_{|\eta|\leqslant Q}\frac{2\,\sin\left(\lambda\,|x+\eta|-\frac{(n-1)\pi}{4}\right)}{(2\pi)^{(n+1)/2}\,|x+\eta|^{(n+1)/2}}\right|^2\mathrm{d}\lambda \;=\; 0\,.$$

This proves (6.1.1) and therefore implies Examples 3.2.2 and 3.2.4.

*Remark* 6.1.8. As was indicated in Remark 3.2.6, this approach can not work for $n\geqslant 3$. Indeed, in higher dimensions we can no longer bound the sums under the integral in Lemmas 6.1.2, 6.1.4 and 6.1.5 by taking the absolute value of each term. A more delicate analysis is required in this case (see [BB] for some related results).

We also note that there is a simpler and more direct proof of Example 3.2.2 based on the identity [GR, formula 1.422(4)]

$$\frac{1}{\sin^2(\frac{s}{2})}=\sum_{k=-\infty}^{+\infty}\frac{4}{(s+2\pi k)^2}. \tag{6.1.9}$$

It works in dimension $n=1$ only, and we leave the details to the interested reader.

6.2. **Spheres.** In this subsection we are going to justify Examples 1.3.4 and 3.2.7. The spectral function on $\mathbb{S}^n$ depends only on the distance or, equivalently, on the angle $s$ between the points $x$ and $y$. For any $x\in\mathbb{S}^n$, the eigenfunctions orthogonal to the Legendre polynomials $P_m(n,t)$, with $t=\cos s$ and $m\geqslant 0$, vanish at $x$. Hence, $P_m(n,t)$ are the only eigenfunctions that matter to compute the spectral function. They satisfy the differential equation

$$((1-t^2)\partial_t^2-nt\partial_t)P_m(n,t)=-m(m+n-1)P_m(n,t)\,,$$

which is equivalent to the eigenvalue problem $\Delta f=m(m+n-1)f$ on $\mathbb{S}^n$ with $f$ depending only on the parameter $t\in[-1,1]$. Note also that

$$\int_{-1}^1 P_m^2(n,t)(1-t^2)^{\frac{n-2}{2}}dt=\frac{D_n}{D_{n-1}}\frac{1}{N(n,m)},\qquad P_m(n,1)=1,$$

where

$$D_n=\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\quad\text{and}\quad N(n,m)=\frac{(2m+n-1)\Gamma(m+n-1)}{\Gamma(m+1)\Gamma(n)}$$

(see [Mu, Lemma 10]). We assume that $x$ and $y$ are non-conjugate points, so that $0<s<\pi$. The rescaled spectral function on $\mathbb{S}^n$, for $n\geqslant 2$, is given by

$$\tilde{N}_s(\mu)=\mu^{\frac{1-n}{2}}N_s(\mu)=\mu^{\frac{1-n}{2}}\sum_{0\leqslant m(m+n-1)\leqslant\mu^2}\frac{N(n,m)P_m(n,\cos s)}{D_n}$$

Here we are abusing notation slightly, since according to (1.2.1) the inequalities in the summation limits should be strict. However, because of the rescaling this makes no difference in $B^2$.

Consider the following generating function ([Mu], Lemma 17):

$$\sum_{m=0}^{+\infty} N(n,m)P_m(n,t)z^m \;=\; \frac{1-z^2}{(1+z^2-2zt)^{\frac{n+1}{2}}}\,. \tag{6.2.1}$$

It is easy to see that $\sum_{k=0}^{+\infty} a_k z^k = f(z)$ implies $\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{k} a_j\right) z^k = \frac{f(z)}{1-z}$. Therefore (6.2.1) implies

$$\sum_{m=0}^{+\infty} N_s(\sqrt{m(m+n-1)})z^m \;=\; \frac{1+z}{D_n(1+z^2-2zt)^{\frac{n+1}{2}}}\,. \tag{6.2.2}$$

**Lemma 6.2.3.** *For $m \in \mathbb{N}$, the spectral function on $\mathbb{S}^n$ satisfies the following asymptotic formula*

$$\begin{aligned} &N_s(\sqrt{m(m+n-1)}) \\ &= \frac{2\cos(\frac{s}{2})\, m^{\frac{n-1}{2}}}{(2\pi\sin s)^{\frac{n+1}{2}}}\cos\left(\left(\frac{n}{2}+m\right)s - \frac{(n+1)}{4}\pi\right) + O(m^{\frac{n-3}{2}})\,, \qquad m\to\infty\,. \end{aligned}$$

*Proof.* Following the approach of [CoH, Chapter VII, Sections 6.6 and 6.7], we prove Lemma 6.2.3 using the Darboux method applied to the generating function (6.2.2). The Darboux method relies on the following fact: the coefficients $a_k$ of the Taylor expansion at the origin of the function $f(z) = \sum_{k=0}^{+\infty} a_k z^k$, holomorphic in the open disc $\mathbb{D}$, decay as $O(k^{-r})$ if $f(e^{ix}) \in C^r(\mathbb{R})$. The first step to obtain the asymptotic formula is to approximate the generating function (6.2.2), taking into account the singularities of the highest order.

Near the singular point $x = e^{\pm is}$, we can write

$$\begin{aligned} \frac{1}{(1+z^2-2zt)^{\frac{n+1}{2}}} &= \frac{((e^{\pm is}-e^{\mp is})+(z-e^{\pm is}))^{-\frac{n+1}{2}}}{(z-e^{\pm is})^{\frac{n+1}{2}}} \\ &= \frac{e^{\pm i\frac{3(n+1)\pi}{4}}}{(2\sin s)^{\frac{n+1}{2}}(z-e^{\pm is})^{\frac{n+1}{2}}}\sum_{k=0}^{+\infty}\binom{-\frac{n+1}{2}}{k}\left(\frac{z-e^{\pm is}}{\pm 2i\sin s}\right)^k, \end{aligned}$$

where

$$\binom{\alpha}{k} = \begin{cases} 1, & k=0,\\ \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}, & k>0\,. \end{cases}$$

For $p > \frac{n+1}{2}$, we have:

(6.2.4)

$$\frac{1}{(2\sin s)^{\frac{n+1}{2}}}\sum_{k=0}^{p}\binom{-\frac{n+1}{2}}{k}\left(e^{i\frac{3(n+1)\pi}{4}}\frac{(z-e^{is})^{k-\frac{n+1}{2}}}{(2i\sin s)^k}+e^{-i\frac{3(n+1)\pi}{4}}\frac{(z-e^{-is})^{k-\frac{n+1}{2}}}{(-2i\sin s)^k}\right) - \frac{1}{(1+z^2-2zt)^{\frac{n+1}{2}}} \in C^1(\bar{\mathbb{D}})\,.$$

Expanding the powers of $z - e^{\pm is}$ by Taylor's formula at $z = 0$, we see that Taylor's expansion at $z = 0$ of the difference (6.2.4) is given by

$$\frac{2}{(2\sin s)^{\frac{n+1}{2}}}\sum_{k=0}^{p}\sum_{l=0}^{+\infty}\binom{-\frac{n+1}{2}}{k}\binom{k-\frac{n+1}{2}}{l}\frac{\cos\left((\frac{n+1}{2}+l-k)s+(l-\frac{n+1}{4}-\frac{k}{2})\pi\right)}{(2\sin s)^k}z^l.$$

Using (6.2.2) and comparing the coefficients in front of the same powers of $z$, we deduce that

$$\begin{aligned}
&D_nN_s(\sqrt{m(m+n-1)})\\
&=\frac{2}{(2\sin s)^{\frac{n+1}{2}}}\sum_{k=0}^{p}\binom{-\frac{n+1}{2}}{k}\left(\frac{\cos\left((\frac{n+1}{2}+m-k)s+(m-\frac{n+1}{4}-\frac{k}{2})\pi\right)}{(2\sin s)^k}\binom{k-\frac{n+1}{2}}{m}\right)\\
&+\frac{2}{(2\sin s)^{\frac{n+1}{2}}}\sum_{k=0}^{p}\binom{-\frac{n+1}{2}}{k}\left(\frac{\cos\left((\frac{n-1}{2}+m-k)s+(m-\frac{n+5}{4}-\frac{k}{2})\pi\right)}{(2\sin s)^k}\binom{k-\frac{n+1}{2}}{m-1}\right)\\
&\qquad+O(m^{-1}).
\end{aligned}$$

as $m \to \infty$. Here the error estimate follows from (6.2.4) in view of the remark in the beginning of the proof. Taking into account the estimate

$$\frac{\Gamma(m+\frac{n+1}{2})}{\Gamma(m+1)} \;=\; m^{\frac{n-1}{2}}+O(m^{\frac{n-3}{2}})\,,$$

we see that the main term of the expansion above corresponds to $k = 0$. Hence we obtain

$$\begin{aligned}
&D_nN_s(\sqrt{m(m+n-1)})\\
&\quad=\frac{4\cos(\frac{s}{2})\,\Gamma(m+\frac{n+1}{2})}{(2\sin s)^{\frac{n+1}{2}}\,\Gamma(m+1)\,\Gamma(\frac{n+1}{2})}\;\cos\left(\left(\frac{n}{2}+m\right)s-\frac{(n+1)}{4}\pi\right)+O(m^{\frac{n-3}{2}})\,.
\end{aligned}$$

This completes the proof of the lemma. □

Note that on $\mathbb{S}^n$ there are simple expressions for $N_s(\mu)$ at conjugate points:

$$N_0(\sqrt{m(m+n-1)})=\frac{(n+2m)\,(n+m-1)!}{D_n\,m!\,n!}=\frac{2\pi^{n/2}}{(2\pi)^n\Gamma(n/2)n}\;m^n+O(m^{n-1})\,,$$

$$N_\pi(\sqrt{m(m+n-1)}) = \frac{(-1)^m(n+m-1)!}{D_n\, m!\,(n-1)!} \neq o(m^{n-1})$$

These formulae and Lemma 6.2.3 prove Example 1.3.4.

Let us now complete the proof of Example 3.2.7. Lemma 6.2.3 implies that

$$\int_1^T \left| \tilde{N}_s(\lambda) - \frac{2\cos(\frac{s}{2})}{(2\pi\sin s)^{\frac{n+1}{2}}} \cos\left(\left(\lfloor \lambda - \frac{n-1}{2} \rfloor + \frac{n}{2}\right) s - \frac{(n+1)}{4}\pi\right)\right|^2 d\lambda$$
$$\leqslant C \sum_{m=1}^{T} \frac{1}{m} = O(\ln T)\,.$$

Here we have used that the interval

$$\left[\sqrt{m(m+n-1)}, \sqrt{(m+1)(m+n)}\right) \bigcap \{\lambda \,:\, \lfloor \lambda - (n-1)/2 \rfloor \neq m\},$$

where the formula above does not agree with Lemma 6.2.3, is of size $O(m^{-1})$. This follows from the simple asymptotic formula $\sqrt{m(m+n-1)} = m + \frac{n-1}{2} + O(m^{-1})$.

Therefore, $\tilde{N}_s(\lambda)$ is $B^2$-equivalent to

$$\frac{2\cos(\frac{s}{2})}{(2\pi\sin s)^{\frac{n+1}{2}}} \cos\left(\left(\lfloor \lambda - \frac{n-1}{2} \rfloor + \frac{n}{2}\right) s - \frac{(n+1)}{4}\pi\right)$$

which can be rewritten as

$$\frac{1}{(2\pi)^{\frac{n+1}{2}} \sin^{\frac{n-1}{2}}(s)} \frac{1}{\sin(\frac{s}{2})} \sin\left(\left(\lfloor \lambda - \frac{n-1}{2} \rfloor + \frac{n}{2}\right) s - \frac{(n-1)}{4}\pi\right) \tag{6.2.5}$$

**Lemma 6.2.6.** *The following expansions hold in $B^2$ for $0 < s < \pi$:*

$$\frac{\sin((\lfloor\lambda\rfloor + \frac{1}{2})s)}{2\sin(\frac{s}{2})} \sim \sum_{k=-\infty}^{+\infty} \frac{1}{|s+2\pi k|} \sin(\lambda|s+2\pi k|)\,,$$

$$\frac{\sin(\lfloor\lambda + \frac{1}{2}\rfloor s)}{2\sin(\frac{s}{2})} \sim \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{|s+2\pi k|} \sin(\lambda|s+2\pi k|)\,,$$

$$\frac{\cos((\lfloor\lambda\rfloor + \frac{1}{2})s)}{2\sin(\frac{s}{2})} \sim \sum_{k=-\infty}^{+\infty} \frac{(-1)^{H(k)}}{|s+2\pi k|} \cos(\lambda|s+2\pi k|)\,,$$

$$\frac{\cos(\lfloor\lambda + \frac{1}{2}\rfloor s)}{2\sin(\frac{s}{2})} \sim \sum_{k=-\infty}^{+\infty} \frac{(-1)^{k+H(k)}}{|s+2\pi k|} \cos(\lambda|s+2\pi k|)\,.$$

*Here $H(x)$ is the "reversed" Heaviside function: $H(x) = 0$ if $x \geqslant 0$, and $H(x) = 1$ if $x < 0$.*

*Proof.* Each of these expansions can be obtained in a similar way as (6.1.1) in the case $n = 1$. Alternatively, they could be proved directly using the identity (6.1.9). $\square$

Applying Lemma 6.2.6 to the formula (6.2.5), we deduce that

$$\tilde N_s(\lambda) \sim \frac{1}{\pi\,(2\pi\sin s)^{\frac{n-1}{2}}} \sum_{j=0}^{\infty} \frac{\sin\left(\lambda\, l(\gamma_j) - \frac{(n-1)\pi}{4} - \omega(\gamma_j)\frac{\pi}{2}\right)}{l(\gamma_j)}\,,$$

where the sum runs over all geodesic segments $\gamma_j$ starting at $x$ and ending at $y$. Here $l(\gamma_j)$ is the length of $\gamma_j$ and $\omega(\gamma_j)$ is the Morse index of $\gamma_j$. Counting the points conjugate to $x$ on the geodesic $\gamma$ of length $|s+2\pi k|$, one gets $\omega(\gamma) = (n-1)\,(2|k| - H(k))$. This completes the proof of Example 3.2.7.

## 7. The $\zeta$-function: proofs

7.1. **Auxiliary functions.** Throughout the section we denote

- $t := \operatorname{Re} z\,,\ s := \operatorname{Im} z$ and
- $\langle s\rangle := (1+|s|^2)^{1/2}$.

Let $\rho$ be a real-valued even function satisfying Condition 4.1.1 with a sufficiently small $\delta$. Let us fix an arbitrary $c>0$ such that $2\,c<\lambda_1$ and $c<d_{x,y}^{-1}$ for all $x,y\in M$, and denote

$$Z_{x,y;j}(z) := z\int_c^\infty \lambda^{-z-1}\,N_{x,y;j}(\lambda)\,\mathrm{d}\lambda\,,\qquad j=0,1, \tag{7.1.1}$$

where $N_{x,y;j}(\lambda)$ are the functions defined by (4.1.2). Then

$$Z_{x,y} \;=\; Z_{x,y;0}(z) \;+\; Z_{x,y;1}(z)$$

and, due to the finite speed of propagation, $|Z_{x,y}(z) - Z_{x,y;1}(z)| \leqslant C_t$ whenever $\delta < d_{x,y}$.

By Lemma 4.1.3, we have

$$Z_{x,y;1}(z) \;:=\; \int h_1(z,\mu)\,\mathrm{d}N_{x,y}(\mu)\,, \tag{7.1.2}$$

where $h_1(z,\mu) := z\int_c^\infty \lambda^{-z-1}\,\rho_1(\lambda-\mu)\,\mathrm{d}\lambda$. Substituting $z\lambda^{-z-1} = -\frac{\mathrm{d}}{\mathrm{d}\lambda}\,(\lambda^{-z})$, integrating by parts and then changing variables, we obtain

$$h_1(z,\mu) \;=\; \mu^{-z} + c^{-z}\,\rho_1(c-\mu) - \int_{c-\mu}^\infty (\lambda+\mu)^{-z}\,\rho(\lambda)\,\mathrm{d}\lambda\,,\quad \forall\mu>c\,. \tag{7.1.3}$$

Since one can differentiate under the integral sign, $h_1(z,\mu)$ is an entire function of the variable $z\in\mathbb{C}$ smoothly depending on the parameter $\mu\in(2c,+\infty)$, such that

$$\frac{\partial^m}{\partial\mu^m}\,h_1(z,\mu) = (-1)^m\,\frac{\Gamma(z+m)}{\Gamma(z)}\,h_1(z+m,\mu)\,. \tag{7.1.4}$$

**Lemma 7.1.5.** *For all $\mu\in(2c,+\infty)$, we have $|h_1(z,\mu)|\leqslant C_t\,\mu^{-t}$ and*

$$|h_1(z,\mu)| \;\leqslant\; C_{t,\varkappa}\,\mu^{-t-\varkappa}\,\langle s\rangle^{\varkappa}\,(1+|\ln\langle s\rangle - \ln\mu|)^{-r}\,,\qquad \forall\varkappa,r>0\,. \tag{7.1.6}$$

*Proof.* The identities (7.1.3) and $\int\rho(\lambda)\,\mathrm{d}\lambda=1$ imply that

$$h_1(z,\mu)=c^{-z}\rho_1(c-\mu)+\mu^{-z}\int_{-\infty}^{c-\mu}\rho(\lambda)\,\mathrm{d}\lambda+\int_{c-\mu}^{-\frac{\mu}{2}}\left(\mu^{-z}-(\mu+\lambda)^{-z}\right)\rho(\lambda)\,\mathrm{d}\lambda$$
$$+\int_{\frac{\mu}{2}}^{\infty}\left(\mu^{-z}-(\mu+\lambda)^{-z}\right)\rho(\lambda)\,\mathrm{d}\lambda+\int_{-\frac{\mu}{2}}^{\frac{\mu}{2}}\left(\mu^{-z}-(\mu+\lambda)^{-z}\right)\rho(\lambda)\,\mathrm{d}\lambda\,.$$

Since $\rho$ is rapidly decreasing, the first four terms in the right hand side are bounded by $C_{t,m}\,\mu^{-m}$ for all $m>0$. Therefore we only need to estimate $\int_{-\frac{\mu}{2}}^{\frac{\mu}{2}}\left((\mu+\lambda)^{-z}-\mu^{-z}\right)\rho(\lambda)\,\mathrm{d}\lambda$.

By Taylor's formula,

$$(7.1.7)\quad (\mu+\lambda)^{-z}-\mu^{-z}=\sum_{k=1}^{m-1}(-1)^k\,\frac{\Gamma(z+k)}{k!\,\Gamma(z)}\,\lambda^k\mu^{-z-k}$$
$$+(-1)^m\,\frac{\Gamma(z+m)}{(m-1)!\,\Gamma(z)}\,\lambda^m\int_0^1(1-t)^{m-1}\,(\mu+t\lambda)^{-z-m}\,\mathrm{d}t\,,\qquad m=1,2,\dots$$

(unless $-z$ is a nonnegative integer, in which case the terms with negative exponents $-z-k$ are absent). Substituting (7.1.7) into the integral, we see that

$$(7.1.8)\quad \int_{-\frac{\mu}{2}}^{\frac{\mu}{2}}\left((\mu+\lambda)^{-z}-\mu^{-z}\right)\rho(\lambda)\,\mathrm{d}\lambda$$
$$=\sum_{k=1}^{m-1}\frac{(-1)^k\,\Gamma(z+k)}{k!\,\Gamma(z)}\,\mu^{-z-k}\int_{-\frac{\mu}{2}}^{\frac{\mu}{2}}\lambda^k\,\rho(\lambda)\,\mathrm{d}\lambda$$
$$+(-1)^m\,\frac{\Gamma(z+m)}{(m-1)!\,\Gamma(z)}\int_{-\frac{\mu}{2}}^{\frac{\mu}{2}}\int_0^1(1-\tau)^{m-1}\,(\mu+\tau\lambda)^{-z-m}\,\lambda^m\,\rho(\lambda)\,\mathrm{d}\tau\,\mathrm{d}\lambda\,.$$

Since $\rho$ is rapidly decreasing, Condition 4.1.1 implies that the sum in the right hand side of (7.1.8) is estimated by $C_m\,\mu^{-t-m}\,\langle s\rangle^{m-1}$ for all $m\geqslant 1$. On the other hand, for all $j,m\geqslant 0$,

$$(7.1.9)\qquad \int_{-\frac{\mu}{2}}^{\frac{\mu}{2}}|(\mu+\tau\lambda)^{-z-m}\,\lambda^j\,\rho(\lambda)|\,\mathrm{d}\lambda\ \leqslant\ C_{t,m}\,\mu^{-t-m}\,,\qquad \forall\tau\in[0,1]\,.$$

Therefore the last term in (7.1.8) does not exceed $C_{t,m}\,\mu^{-t-m}\,\langle s\rangle^m$. Thus we have

$$(7.1.10)\quad \left|\int_{-\frac{\mu}{2}}^{\frac{\mu}{2}}\left(\mu^{-z}-(\mu+\lambda)^{-z}\right)\rho(\lambda)\,\mathrm{d}\lambda\right|\ \leqslant\ C_{t,m}\,\mu^{-t}\,b_{s,\mu}^m\,,\qquad \forall m=0,1,\dots,$$

where $b_{s,\mu}:=\mu^{-1}\langle s\rangle$ (the estimate with $m=0$ is a particular case of (7.1.9) ). Since $b_{s,\mu}^{\varkappa}\,(1+|\ln b_{s,\mu}|)^{-r}\geqslant C_{r,\varkappa}\min\{b_{s,\mu}^{3\varkappa/2},b_{s,\mu}^{\varkappa/2}\}$ whenever $\varkappa>0$ and

$$\mu^{-\varkappa}\,\langle s\rangle^{\varkappa}\,(1+|\ln\langle s\rangle-\ln\mu|)^{-r}=b_{s,\mu}^{\varkappa}\,(1+|\ln b_{s,\mu}|)^{-r}\,,$$

interpolating between the estimates (7.1.10), we arrive at (7.1.6). $\square$

7.2. **Estimates for $Z_{x,y;1}(z)$.** Lemma 7.1.5 implies the following corollaries.

**Corollary 7.2.1.** *$Z_{x,y;1}(z)$ is an entire function of $z$ for each $(x,y) \in M \times M$, such that*

$$|Z_{x,y;1}(t+is)| \;\leqslant\; C_t \left(\langle s\rangle^{n-t}+1\right)\,, \qquad \forall t \neq n\,. \tag{7.2.2}$$

*Proof.* In view of (7.1.4) and (7.1.6), one can differentiate under the integral sign in the definition of $Z_{x,y;1}(z)$. Therefore the function $Z_{x,y;1}(z)$ is analytic.

Let $\tilde N_{x,y}(\lambda;a_1,a_2) := \sum_{\lambda_j<\lambda} |a_1\,\varphi_j(x) + a_2\,\varphi_j(y))|^2$ where $a_1, a_2$ are complex constants such that $|a_1|^2+|a_2|^2 = 2$. The function $N_{x,y}(\lambda;a_1,a_2)$ is nondecreasing and, in view of the Weyl formulae (1.2.2), is estimated by $C\,\lambda^n$. We have

$$2\;\mathrm{Re}\,N_{x,y}(\lambda) = N_{x,y}(\lambda;1,1) - N_{x,x}(\lambda) - N_{y,y}(\lambda)\,,$$
$$2i\;\mathrm{Im}\,N_{x,y}(\lambda) = N_{x,y}(\lambda;i,1) - N_{x,x}(\lambda) - N_{y,y}(\lambda)\,.$$

Therefore it is sufficient to prove that the estimate (7.2.2) holds for the integral $\int_{2c}^\infty h_1(z,\lambda)\,\mathrm{d}G(\lambda)$, where $G(\lambda)$ is a nondecreasing function bounded by $C\,\lambda^n$.

If $t<n$ then (7.1.6) with $\varkappa = n-t$ and $r=2$ implies that

$$\begin{aligned}\int_{2c}^\infty h_1(z,\lambda)\,\mathrm{d}G(\lambda) \;&\leqslant\; C_t\,\langle s\rangle^{n-t}\int_{2c}^\infty \lambda^{-n}\,(1+|\ln\langle s\rangle-\ln\lambda|)^{-2}\,\mathrm{d}G(\lambda)\\ &\leqslant\; C_t\,\langle s\rangle^{n-t}\int_{2c}^\infty \lambda^{-n-1}\,(1+|\ln\langle s\rangle-\ln\lambda|)^{-2}\,G(\lambda)\,\mathrm{d}\lambda\\ &\leqslant\; C_t\,\langle s\rangle^{n-t}\int_{2c}^\infty (1+|\ln\langle s\rangle-\ln\lambda|)^{-2}\,\lambda^{-1}\,\mathrm{d}\lambda \;\leqslant\; C_t\,\langle s\rangle^{n-t}\,.\end{aligned}$$

If $t>n$ then the required estimate is obtained in a similar way, with the use of the inequality $|h_1(z,\lambda)| \leqslant C_t\,\lambda^{-t}$. $\square$

*Remark* 7.2.3. Note that (7.2.2) holds for $x=y$.

**Corollary 7.2.4.**

$$\int_M |Z_{x,y;1}(t+is)|^2\,\mathrm{d}y \;\leqslant\; C_t\left(\langle s\rangle^{n-2t}+1\right)\,, \qquad \forall t\neq \frac{n}{2}\,, \quad \forall x \in M\,. \tag{7.2.5}$$

*Proof.* In the same way as in (4.2.1), we obtain

$$\int_M |Z_{x,y;1}(z)|^2\,\mathrm{d}y \;=\; \int |h_1(z,\lambda)|^2\,\mathrm{d}N_{x,x}(\lambda)\,.$$

If $t<n/2$ then (7.1.6) with $\varkappa = \frac{n}{2}-t$ and $r=2$ implies that

$$\begin{aligned}\int |h_1(z,\lambda)|^2\,\mathrm{d}N_{x,x}(\lambda) \;&\leqslant\; C_{t,\varkappa}\,\langle s\rangle^{n-2t}\int_\varepsilon^\infty \lambda^{-n}\,(1+|\ln\langle s\rangle-\ln\lambda|)^{-2}\,\mathrm{d}N_{x,x}(\lambda)\\ &\leqslant\; C_t\,\langle s\rangle^{n-2t}\int_\varepsilon^\infty \lambda^{-n-1}\,(1+|\ln\langle s\rangle-\ln\lambda|)^{-2}\,N_{x,x}(\lambda)\,\mathrm{d}\lambda\,.\end{aligned}$$

Using the Weyl formula and estimating integrals as in the proof of the previous corollary, we see that the right hand side is not greater than $C_t \langle s\rangle^{n-2t}$.

If $t > \frac{n}{2}$ then the required inequality is obtained in the same way, with the use of the estimate $|h_1(z,\lambda)|^2 \leqslant C_t\,\lambda^{-2t}$. □

7.3. **Estimates for** $Z_{x,y;0}(z)$**.** If $x \neq y$ then $Z_{x,y;0}(z)$ is an entire function of $z$. Our goal is to estimate $Z_{x,y;0}(z)$ uniformly with respect to $s$ and $x, y \in M$. Further on

- $O_t(d_{x,y}^p)$ denotes a function which is estimated by $C_t(d_{x,y}^p + 1)$.

Obviously, if $|f(\lambda)| \leqslant C\,\langle\lambda\rangle^{-m}$ for all $\lambda \in \mathbb{R}_+$ then $\int_c^\infty \lambda^{-z} f(\lambda)\,\mathrm{d}\lambda$ is an analytic function in the half-plane $\{z : t > 1-m\}$ and $\left|\int_c^\infty \lambda^{-z} f(\lambda)\,\mathrm{d}\lambda\right| \leqslant C_t$ for all $t > 1-m$. In particular, if $\Psi_{x,y}(\lambda) = N_{x,y}(\lambda) - N_{x,y}(-\lambda)$ then, in view of (4.3.2), we have

$$Z_{x,y;0}(z) = z\int_c^\infty \lambda^{-z-1}\,N_{x,y;0}(\lambda)\,\mathrm{d}\lambda = \int_c^\infty \lambda^{-z}\,(\rho * \Psi_{x,y})'(\lambda)\,\mathrm{d}\lambda \;+\; O_t(1)\,.$$

Let us assume that $\delta$ in Condition 4.1.1 is small enough, so that the Hadamard representation (4.3.1) holds on $\operatorname{supp}\hat\rho$. Then the Fourier transform of the derivative $\Psi'_{x,y}$ admits an asymptotic expansion of the form (4.3.1) on $\operatorname{supp}\hat\rho$. This implies that

$$(\rho * \Psi_{x,y})'(\lambda) \;-\; \int_c^\infty \rho(\lambda-\mu)\,\Psi'_{0;x,y}(\mu)\,\mathrm{d}\mu \;=\; O(\lambda^{-\infty})\,,$$

uniformly with respect to $x, y \in M$, where $\Psi'_{0;x,y}(\mu)$ is a function whose asymptotic expansion for $\mu \to \infty$ is obtained from (4.3.6) by differentiating each term with respect to $\lambda$ and putting $\lambda = \mu$. Thus we have

$$Z_{x,y;0}(z) \;=\; \int_c^\infty\int_c^\infty \lambda^{-z}\rho(\lambda-\mu)\,\Psi'_{0;x,y}(\mu)\,\mathrm{d}\mu\,\mathrm{d}\lambda \;+\; O_t(1)\,. \tag{7.3.1}$$

For all $m = 0, 1, 2\ldots$, we have $\frac{\mathrm{d}^m}{\mathrm{d}\tau^m} J_p(\tau) \leqslant C\,\tau^{p-m}$ for small values of $\tau$ and

$$\frac{\mathrm{d}^m}{\mathrm{d}\tau^m} J_p(\tau) \;\sim\; \tau^{-\frac12}\,\cos\tau\sum_{k=0}^\infty a_{m,k}\,\tau^{-k} + \tau^{-\frac12}\,\sin\tau\sum_{k=0}^\infty b_{m,k}\,\tau^{-k}\,,\qquad \tau \to \infty\,,$$

where $a_{m,k}$ and $b_{m,k}$ are some real coefficients (see, for example, [GR, 8.440 and 8.451(1)]). Therefore the asymptotic formula for $\Psi'_{0;x,y}$ implies that $|\Psi'_{0;x,y}(\mu)| \leqslant C\,\langle\mu\rangle^{n-1}$ for all $\mu \in (c, d_{x,y}^{-1})$ and

$$\begin{aligned}\Psi'_{0;x,y}(\mu) \;=\; &\sum_{j+k<N} u_{j,k}(x,y)\,d_{x,y}^{2j+1-n}\,(d_{x,y}\,\mu)^{\frac{n-1}{2}-j-k}\,\exp(i\,d_{x,y}\,\mu)\\ &+\sum_{j+k<N} v_{j,k}(x,y)\,d_{x,y}^{2j+1-n}\,(d_{x,y}\,\mu)^{\frac{n-1}{2}-j-k}\,\exp(-i\,d_{x,y}\,\mu) \;+\; R_N(x,y,\mu)\end{aligned} \tag{7.3.2}$$

for all $\mu \in (d_{x,y}^{-1}, \infty)$ and $N = 1, 2, \dots$, where $u_{j,k}$, $v_{j,k}$ are bounded functions and

$$|R_N(x,y,\mu)| \leqslant C\, d_{x,y}^{1-n}\, (d_{x,y}\,\mu)^{\frac{n-1}{2}-N}, \qquad \forall \mu \in (d_{x,y}^{-1}, \infty)\,.$$

In particular, $|\Psi'_{0;x,y}(\mu)| \leqslant C\, d_{x,y}^{1-n}\, (d_{x,y}\,\mu)^{\frac{n-1}{2}} \leqslant C\,\mu^{n-1}$ whenever $\mu \geqslant d_{x,y}^{-1}$ and, consequently, $|\Psi'_{0;x,y}(\mu)| \leqslant C\,\langle\mu\rangle^{n-1}$ for all $\mu \geqslant c$.

The above estimates and the obvious inequalities

$$\langle\lambda\rangle^{-1}\langle\mu\rangle \leqslant 2\langle\lambda-\mu\rangle\,, \qquad \langle\mu\rangle^{-1}\langle\lambda\rangle \leqslant 2\langle\lambda-\mu\rangle \tag{7.3.3}$$

imply that

$$\begin{aligned} \left|\lambda^{-z}\rho(\lambda-\mu)\,\Psi'_{0;x,y}(\mu)\right| &\leqslant C_t\,\langle\lambda\rangle^{n-t-1}\langle\lambda-\mu\rangle^{n-1}\,|\rho(\lambda-\mu)|\,,\\ \left|\lambda^{-z}\rho(\lambda-\mu)\,\Psi'_{0;x,y}(\mu)\right| &\leqslant C_t\,\langle\mu\rangle^{n-t-1}\langle\lambda-\mu\rangle^{|t|}\,|\rho(\lambda-\mu)| \end{aligned}$$

and

$$\begin{aligned} \left|\lambda^{-z}\rho(\lambda-\mu)\,R_N(x,y,\mu)\right| &\leqslant C\,d_{x,y}^{t-n+1}\left|(d_{x,y}\lambda)^{-t}\,\rho(\lambda-\mu)\,(d_{x,y}\mu)^{\frac{n-1}{2}-N}\right|\\ &\leqslant C_t\,d_{x,y}^{t-n+1}\langle d_{x,y}\lambda\rangle^{\frac{n-1}{2}-N-t}\langle d_{x,y}\lambda-d_{x,y}\mu\rangle^{\frac{n-1}{2}+N}|\rho(\lambda-\mu)| \end{aligned}$$

for all $\lambda,\mu > d_{x,y}^{-1}$. Since the function $\rho$ is rapidly decreasing, integrating the above estimates over $\lambda$ and $\mu$, we obtain

$$\left|\int_c^{d_{x,y}^{-1}}\int_c^\infty \lambda^{-z}\rho(\lambda-\mu)\,\Psi'_{0;x,y}(\mu)\,\mathrm{d}\mu\,\mathrm{d}\lambda\right| \leqslant \begin{cases} C_t\,(d_{x,y}^{t-n}+1)\,, & t\neq n\,,\\ C\,|\ln d_{x,y}|\,, & t=n\,, \end{cases} \tag{7.3.4}$$

$$\left|\int_{d_{x,y}^{-1}}^\infty\int_c^{d_{x,y}^{-1}} \lambda^{-z}\rho(\lambda-\mu)\,\Psi'_{0;x,y}(\mu)\,\mathrm{d}\mu\,\mathrm{d}\lambda\right| \leqslant \begin{cases} C_t\,(d_{x,y}^{t-n}+1)\,, & t\neq n\,,\\ C\,|\ln d_{x,y}|\,, & t=n\,, \end{cases} \tag{7.3.5}$$

and

$$\begin{aligned} &\left|\int_{d_{x,y}^{-1}}^\infty\int_{d_{x,y}^{-1}}^\infty \lambda^{-z}\rho(\lambda-\mu)\,R_N(x,y,\mu)\,\mathrm{d}\mu\,\mathrm{d}\lambda\right| \qquad (7.3.6)\\ &\quad\leqslant C_t\,d_{x,y}^{t-n-1}\int_1^\infty\int_1^\infty \langle\lambda\rangle^{\frac{n-1}{2}-N-t}\langle\lambda-\mu\rangle^{\frac{n-1}{2}+N}|\rho\left(d_{x,y}^{-1}(\lambda-\mu)\right)|\,\mathrm{d}\mu\,\mathrm{d}\lambda\\ &\quad\leqslant C_t\,d_{x,y}^{t-n-1}\left(\int_1^\infty\langle\lambda\rangle^{\frac{n-1}{2}-N-t}\,\mathrm{d}\lambda\right)\left(\int\langle\tau\rangle^{\frac{n-1}{2}+N}|\rho\left(d_{x,y}^{-1}\tau\right)|\,\mathrm{d}\tau\right) = O_t(d_{x,y}^{t-n}) \end{aligned}$$

for all $N > \frac{n+1}{2} - t$.

It remains to estimate the integrals

$$\begin{aligned} &\int_{d_{x,y}^{-1}}^\infty\int_{d_{x,y}^{-1}}^\infty \lambda^{-z}\rho(\lambda-\mu)\,d_{x,y}^{2j+1-n}\,(d_{x,y}\,\mu)^{\frac{n-1}{2}-j-k}\,e^{\pm i\,d_{x,y}\,\mu}\,\mathrm{d}\mu\,\mathrm{d}\lambda \qquad (7.3.7)\\ &\qquad= d_{x,y}^{2j+t-n-1+is}\int_1^\infty\int_1^\infty \lambda^{-z}\rho\left(d_{x,y}^{-1}(\lambda-\mu)\right)\,\mu^{\frac{n-1}{2}-j-k}\,e^{\pm i\,\mu}\,\mathrm{d}\mu\,\mathrm{d}\lambda \end{aligned}$$

generated by the main asymptotic terms in (7.3.2). Expanding $\mu^{\frac{n-1}{2}-j-k}$ by Taylor's formula at $\mu=\lambda$, we see that the right hand side of (7.3.7) coincides with

$$d_{x,y}^{2j+t-n-1+is}\int_1^\infty \psi(z,\lambda,\mu)\left(\int_1^\infty \rho\left(d_{x,y}^{-1}(\lambda-\mu)\right)\,e^{\pm i\,\mu}\,\mathrm{d}\mu\right)\mathrm{d}\lambda\;+\;O_t(d_{x,y}^{2j+t-n})\,,$$

where

$$\psi(z,\lambda,\mu)\;=\;\sum_{m=0}^{l}\binom{\frac{n-1}{2}-j-k}{m}\lambda^{\frac{n-1}{2}-j-k-m-z}\,(\mu-\lambda)^m$$

with an arbitrary $l\geqslant\frac{n+1}{2}-t-j-k$, and the remainder estimate follows from the inequalities

$$|\psi(z,\lambda,\mu)-\lambda^{-z}\mu^{\frac{n-1}{2}-j-k}|\leqslant C\lambda^{-t}(\mu-\lambda)^{l+1}(\mu^{\frac{n-1}{2}-j-k-l-1}+\lambda^{\frac{n-1}{2}-j-k-l-1})$$

and

$$\mu^{\frac{n-1}{2}-j-k-l-1}\leqslant C\langle\mu-\lambda\rangle^{\frac{n-1}{2}+j+k+l+1}\lambda^{\frac{n-1}{2}-j-k-l-1}.$$

Note that the last inequality is a consequence of (7.3.3).

Obviously,

$$\begin{aligned}&d_{x,y}^{-1}\int_1^\infty\int_{-\infty}^1 \lambda^p\,(\mu-\lambda)^m\left|\rho\left(d_{x,y}^{-1}(\lambda-\mu)\right)\right|\mathrm{d}\mu\,\mathrm{d}\lambda\\ \leqslant\;& d_{x,y}^m\int_1^\infty\int_{d_M^{-1}(\lambda-1)}^\infty \lambda^p\,|\mu^m\rho\,(\mu)|\,\mathrm{d}\mu\,\mathrm{d}\lambda\;\leqslant\;C_{p,m}\,,\qquad \forall p\in\mathbb{R}\,,\ \forall m=0,1,2,\dots,\end{aligned}$$

where $d_M$ is the diameter of $M$. This estimate and (7.3.3) imply that the integral (7.3.7) is equal to

$$\begin{aligned}&d_{x,y}^{2j+t-n-1+is}\int_1^\infty \psi(z,\lambda,\mu)\left(\int_{-\infty}^\infty \rho\left(d_{x,y}^{-1}(\lambda-\mu)\right)\,e^{\pm i\,\mu}\,\mathrm{d}\mu\right)\mathrm{d}\lambda\\ =\;&\sum_{m=0}^{l}\binom{\frac{n-1}{2}-j-k}{m}d_{x,y}^{2j+t-n+m+is}(-i)^m\,\hat\rho^{(m)}(d_{x,y})\int_1^\infty \lambda^{\frac{n-1}{2}-j-k-m-z}\,e^{\pm i\lambda}\,\mathrm{d}\lambda\end{aligned}\tag{7.3.8}$$

modulo $O_t(d_{x,y}^{2j+t-n})$, where $\hat\rho^{(m)}$ is the $m$th derivative of the Fourier transform.

For each $p\in\mathbb{R}$, the integral $\int_1^\infty\lambda^{p-z}\,e^{\pm i\lambda}\,\mathrm{d}\lambda$ defines an analytic function on the half-plane $\{z\,:\,t>p+1\}$, where it is bounded on each vertical line by a constant $C_t$. This function admits an analytic continuation to the whole complex plane, obtained by replacing $e^{\pm i\lambda}$ with $(\mp i)^m\,\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\,e^{\pm i\lambda}$ and integrating by parts. This continuation coincides with the difference between the meromorphic continuations of the integrals $\int_0^\infty\lambda^{p-z}\,e^{\pm i\lambda}\,\mathrm{d}\lambda$ and $\int_0^1\lambda^{p-z}\,e^{\pm i\lambda}\,\mathrm{d}\lambda$. According to [GR, 3.381] and [GR, 8.328(1)],

$$\int_0^\infty \lambda^{p-z}\,e^{\pm i\lambda}\,\mathrm{d}\lambda\;=\;\pm\,i\,e^{\pm\,i\pi(p-z)/2}\,\Gamma(p+1-z)$$

and

$$\left|e^{\pm i\pi(p-z)/2}\,\Gamma(p+1-z)\right| = \left|e^{\pi|s|/2}\,\Gamma(p+1-t+is)\right| \leqslant C_{p-t}\,|s|^{p-t+1/2}$$

whenever $|s|>1$. Replacing $\lambda^{p-z}$ with $(m+p-z)^{-1}\dots(1+p-z)^{-1}\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\lambda^{m+p-z}$ and integrating by parts, we see that $\left|\int_0^1 \lambda^{p-z}\,e^{\pm i\lambda}\,\mathrm{d}\lambda\right| \leqslant C_{p-t}$ whenever $|s|>1$. Therefore

$$\left|\int_1^\infty \lambda^{p-z}\,e^{\pm i\lambda}\,\mathrm{d}\lambda\right| \;\leqslant\; C_{p-t}\,(1+\langle s\rangle^{p-t+1/2})\,,\qquad \forall p\in\mathbb{R}\,,\ \forall z\in\mathbb{C}\,.$$

The above inequality implies that the integral in the left hand side of (7.3.8) is estimated by $C_t\,d_{x,y}^{2j+t-n}\,(1+\langle s\rangle^{\frac{n}{2}-t-j-k})$ and, consequently,

$$\left|\int_{d_{x,y}^{-1}}^\infty\int_{\mu>d_{x,y}^{-1}} \lambda^{-z}\rho(\lambda-\mu)\,d_{x,y}^{2j+1-n}\;(d_{x,y}\,\mu)^{\frac{n-1}{2}-j-k}\;e^{\pm i\,d_{x,y}\,\mu}\,\mathrm{d}\mu\,\mathrm{d}\lambda\right| \leqslant\; C_t\,d_{x,y}^{2j+t-n}\,(1+\langle s\rangle^{\frac{n}{2}-t-j-k})\,. \tag{7.3.9}$$

Now, putting together (7.3.4)–(7.3.6) and (7.3.9), we obtain

$$|Z_{x,y;0}(t+is)| \;\leqslant\; \begin{cases} C_t\left(d_{x,y}^{t-n}\,\langle s\rangle^{\frac{n}{2}-t} + d_{x,y}^{t-n}+1\right)\,, & t\neq n\,,\\ C\left(\langle s\rangle^{\frac{n}{2}-t} + |\ln d_{x,y}|\right)\,, & t=n\,. \end{cases} \tag{7.3.10}$$

7.4. **Proof of Theorems 2.6.5 and 2.6.6.** Theorem 2.6.5 is an immediate consequence of (7.2.2) and (7.3.10). Since the function $\left(d_{x,y}^{2t-n-\varepsilon}+1\right)^{-1}$ is bounded and is estimated by $d_{x,y}^{n-2t+\varepsilon}$ for small values of $d_{x,y}$, Theorem 2.6.6 follows from (7.2.5) and (7.3.10).

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