PSEUDODIFFERENTIAL OPERATORS AND LINEAR CONNECTIONS

Yu. Safarov

King's College London

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Introduction

The aim of the paper is to construct a calculus of pseudodifferential operators (ψDOs) on a smooth manifold M without using local coordinate systems. Instead we deal with linear connections Γ of M.

The fact that a linear connection Γ is a global object enables one to associate with a ψDO its full symbol, which is a function on the cotangent bundle T^*M (depending on the choice of Γ). This idea was put forward by H. Widom. He has suggested a method of defining full symbols of ψDOs on a manifold with a linear connection and constructed a

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version of symbolic calculus [W1], [W2]. More advanced results in this direction were obtained later in [FK3]. However, in these papers the classes of pseudodifferential operators are defined in local coordinates, with use of standard local phase functions.

On the contrary, we define ψDOs in a coordinate-free way, using invariant oscillatory integrals over T^*M. The phase functions in these oscillatory integrals are linear with respect to the phase variables and are determined by Γ, and the symbols are functions on T^*M. (In [FK3], [Re] it is mentioned that similar phase functions have been introduced by L. Drager [D].) We also include in the oscillatory integrals a weight factor which allows us to consider ψDOs in the spaces of $\varkappa$-densities for any $\varkappa \in \mathbf{R}^1$. If the connection Γ is (locally) flat then the phase functions take the usual form $(x-y)\cdot\theta$ and the weight factor is identically equal to 1 in the "flat" coordinates. So our construction is a generalization of the standard one, which corresponds from our point of view to locally flat connections.

The invariant approach allows us to define τ-symbols of ψDOs acting on a manifold; in particular, we define the Weyl symbols. In fact, it is a kind of quantization on a manifold provided with a linear connection. This problem is of interest for physicists (see, for example, [FK1], [FK2], [LQ]). But we discuss only local results, and this subject is left out of the paper.

We deal with Hörmander's classes of symbols $S^m_{\rho,\delta}$ assuming $1 \leqslant \delta < \rho \leqslant 1$. The standard definition of these classes is given in local coordinates, and when $\rho \geqslant 1-\delta$ (and, consequently, $\rho > 1/2$) they turn out to be invariant under change of coordinates. However, in order to introduce these classes it is sufficient only to define the horizontal and vertical derivatives. This can be done in terms of a linear connection Γ. We associate with Γ some classes of symbols $\mathrm{S}^m_{\rho,\delta}(\Gamma)$. If $1-\rho \leqslant \delta$ then these classes are independent of Γ and the corresponding classes of ψDOs coincide with the usual ones. In this case our construction leads only to a version of invariant symbolic calculus.

When $\delta < 1-\rho$ we obtain new classes of pseudodifferential operators. In local coordinates their amplitudes belong to $\mathrm{S}^m_{\rho,1-\rho}$. If $\rho \leqslant 1/2$, one can neither define symbols of these ψDOs nor obtain most of the other standard results in the usual way. But it can be done by means of invariant oscillatory integrals. We prove that all the basic results of the classical theory of ψDOs ([H], [Sh], [T], [Tr]) remain valid for these new classes when $\rho \geqslant 1/3$ and the connection Γ is symmetric. Moreover, some of these results are valid without any additional restrictions.

To illustrate the main idea of the paper let us consider the class of differential (or pseudodifferential) operators with constant coefficients in $\mathbf{R}^n$. This class is invariant with respect to linear transformations of coordinates, and in any "linear" coordinates we can easily obtain various results (in particular, concerning functional calculus) in terms of the full symbols. If we chose other ("non-linear") coordinates on $\mathbf{R}^n$, then the operators would have variable coefficients and most of the results (even local ones) relying upon full symbols would fail. Thus, studying the operators with constant coefficients we single out a class of "preferred" coordinates on $\mathbf{R}^n$ which is invariant with respect to linear transformations. From the geometric point of view this means that we fix a linear connection of $\mathbf{R}^n$.

Analogously, given a class of ψDOs or a particular ψDO acting on a manifold M, we can try to choose a "preferred" linear connection of M which allows us to obtain more advanced results in terms of the corresponding symbolic calculus. Note that the "preferred" linear connection may well be non-flat even when $M = \mathbf{R}^n$ or M is a domain in $\mathbf{R}^n$.

In particular, in Section 10 we consider (pseudo)differential operators which are semi-elliptic with respect to the linear connection generated by a system of (non-commuting) linearly independent vector fields. Under some additional restrictions on the vector fields we prove that such operators are hypoelliptic and admit pseudodifferential parametrices.

In Section 11 we deal with functions of the Laplace operator Δ on a closed Riemannian manifold. It appears that for some natural class $\mathrm{S}^m_\rho(\mathbf{R}^1)$ of functions ω the operator $\omega(\Delta)$ is a ψDO corresponding to the Levi–Civita connection. The same is valid for operators of the form $\omega(\Delta+\nu)$ where ν is a lower order differential operator. This allows one to study fine properties of the operators like $\omega(\Delta+\nu)$ or $\omega(\Delta+\nu)-\omega(\Delta)$. For instance, we immediately obtain that the operator $\omega(\Delta+\nu)$ is pseudolocal as far as $\omega \in \mathrm{S}^m_\rho(\mathbf{R}^1)$. These results have already been known for $\rho \in (1/2,1]$ (see [T, Ch.12.1]). Our technique works in the case $0<\rho\leqslant 1/2$ as well.

There are many papers devoted to generalizations of the classical ψDOs calculus where much more general classes of symbols (than $\mathrm{S}^m_{\rho,\delta}$) have been introduced. However, the known generalizations are connected with other kinds of estimates for the derivatives of symbols and have nothing to do with the phase functions. On the contrary, we use very standard estimates but non-standard phase functions. Therefore our results are not contained in any other known results but can be combined with them (see Remark 3.5).

Throughout the paper we shall use the following standard notations:

- $[\cdot,\cdot]$ denotes the commutator of two operators or vector fields;
- $a \asymp b$ means that $C^{-1}\,a \leqslant b \leqslant C\,a$ with some positive constant C;
- the sign $\sim$ stands for an asymptotic expansion which is uniform with respect to all the parameters when they run over a compact set, and which can be differentiated infinitely many times.

Indices are denoted by Latin letters $i,j,\dots$, multi-indices — by Greek letters $\alpha,\beta,\dots$. As usual, we set $D^\alpha_\theta = (-i)^{|\alpha|}\partial_\theta$.

We shall use some elementary notions and results from differential geometry. We recall them in Sections 1 and 2; for more details see, for example, [KN].

1. Linear connections

1. Let M be a smooth n-dimensional manifold. We denote the points of M by x, y or z, and the covectors from T^*_xM, T^*_yM, T^*_zM by ξ, η, ζ respectively. The same letters denote also coordinates on M and the corresponding dual coordinates in the fibres of T^*M.

We are going to consider operators acting in the spaces of $\varkappa$-densities on M, $\varkappa\in\mathbf{R}^1$. Recall that a complex "function" u on M is said to be a $\varkappa$-density if it behaves under change of coordinates in the following way

$$u(y) \;=\; |\det\{\partial x^i/\partial y^j\}|^{\varkappa}\, u\big(x(y)\big)\,.$$

The usual functions on M are 0-densities. The $\varkappa$-densities are sections of some complex linear bundle $\Omega^\varkappa$ over M. We denote by $C^\infty(M;\Omega^\varkappa)$ and $C^\infty_0(M;\Omega^\varkappa)$ the spaces of smooth $\varkappa$-densities and smooth $\varkappa$-densities with compact supports respectively. If $u \in C_0^\infty(M;\Omega^\varkappa)$ and $v \in C^\infty(M;\Omega^{1-\varkappa})$ then the product $u\,v$ is a density and the integral $\int_M u\,v\,dx$ is independent of the choice of coordinates. This allows us to define the inner product $(u,v)=\int_M u\,\bar v\,dx$ on the space of half-densities $C_0^\infty(M;\Omega^{1/2})$ and to introduce the Hilbert space $L_2(M;\Omega^{1/2})$ in the standard way.

We shall deal with the (p,q)-tensor bundles over M and with the induced bundles over T^*M. For the sake of simplicity the sections of the induced bundles are also called (p,q)-tensors. The components of symmetric tensors in local coordinates are often numbered by multi-indices instead of sets of indices. For example, $\{F^\alpha\}_{|\alpha|=q}$ denotes the $(0,q)$-tensor whose components $F_{i_1,i_2,\dots,i_q}$ coincide with F^α if the set of indices $\{i_1,i_2,\dots,i_q\}$ corresponds to the multi-index α.

We shall always identify vector fields with the corresponding first order differential operators.

2. We assume the manifold M to be provided with a linear connection Γ (which may be non-complete). This means that in any local coordinate system we have defined a set of smooth "functions" $\Gamma^{\,i}_{jk}(x)$, $i,j,k=1,\dots,n$, which are called the *Christoffel symbols*. Under change of coordinates the $\Gamma^{\,i}_{jk}$ behave as follows

$$\sum_l \frac{\partial y^i}{\partial x^l}\,\Gamma^{\,l}_{pq}(x) \;=\; \sum_{p,q} \frac{\partial y^j}{\partial x^p}\,\Gamma^{\,i}_{jk}\big(y(x)\big)\,\frac{\partial y^k}{\partial x^q} \;+\; \frac{\partial^2 y^i}{\partial x^p\,\partial x^q}\,. \tag{1.1}$$

The Christoffel symbols can be chosen arbitrarily (assuming that (1.1) is fulfilled), and each set of Christoffel symbols determines some linear connection of M.

A curve $y(t)\subset M$, $t\in\mathbf{R}^1$, is said to be a geodesic if

$$\ddot y^k(t) \;+\; \sum_{i,j}\Gamma^{\,k}_{ij}\big(y(t)\big)\,\dot y^i(t)\,\dot y^j(t) \;=\; 0\,. \tag{1.2}$$

For any point $x\in M$ and any $\theta\in T_xM$ there exists a unique geodesic $y(t)$ starting at x such that $\dot y(0)=\theta$.

3. Let $x\in M$ be a fixed point and U_x be a sufficiently small neighbourhood of x. One can choose local coordinates $y=\{y^k\}$ on U_x in such a way that all the geodesics starting at x have the form $y(t)=t\theta+y(x)$, where $y(x)$ are the coordinates of the points x and $\theta=\dot y(0)$. Such coordinate systems on U_x are called normal coordinate systems (n.c.s.) with origin x. Normal coordinates with a fixed origin are invariant with respect to linear (and only linear) transformations.

Let $\{x^k\}$ be some local coordinates on U_x. We say that the n.c.s. $\{y^k\}$ with origin x is associated with the coordinates $\{x^k\}$ if $y^k(x)=x^k$ and the Jacobi matrix $\{\partial y^j/\partial x^k\}$ is equal to I at the point x. Obviously, a change of the coordinates $\{x^k\}\to\{\tilde x^k\}$ leads to the linear transformation of the associated normal coordinates $\{y^k\}\to\{\tilde y^k\}=\{\sum_j(\partial\tilde x^k/\partial x^j)\,y^j\}$. From (1.2) it follows that in the n.c.s. $y=\{y^k\}$ associated with coordinates $\{x^k\}$ we have

$$\sum_{i,j}\Gamma^{\,k}_{ij}(y)\,(y^i-x^i)\,(y^j-x^j) \;\equiv\; 0\,. \tag{1.3}$$

For $y \in U_x$ we denote by $\gamma_{y,x}(t)$ the "shortest" (lying in U_x) geodesic joining x and y such that $t \in [0,1]$, $\gamma_{y,x}(0) = x$, $\gamma_{y,x}(1) = y$. This geodesic exists and is uniquely defined. We denote by $\dot\gamma_{y,x}(t)$ its tangent vector at the point $\gamma_{y,x}(t)$.

Let $\gamma^k_{y,x}(t)$ be the y-coordinates of the point $\gamma_{y,x}(t)$, $\dot\gamma^k_{y,x}(t)$ be the corresponding components of the vector $\dot\gamma_{y,x}(t)$, and

$$\dot\gamma^\alpha_{y,x}(t) \;=\; (\dot\gamma^1_{y,x})^{\alpha_1}\dots(\dot\gamma^n_{y,x})^{\alpha_n}$$

(where $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multi-index). If $\{y^k\}$ is the n.c.s. with origin x associated with some coordinates $\{x^k\}$ then

$$\gamma^k_{y,x}(t) \;=\; x^k \;+\; t\,(y^k - x^k)\,, \quad \dot\gamma^k_{y,x}(t) \;=\; (y^k - x^k)\,, \quad \dot\gamma^\alpha_{y,x}(t) \;=\; (y-x)^\alpha\,, \tag{1.4}$$

where x^k and y^k are the coordinates of the points x and y respectively.

Let now $\{y^k\}$ be the same coordinate system as $\{x^k\}$. Then (1.2) implies

$$\gamma^k_{y,x}(t) \;\sim\; x^k \;+\; t\,\dot\gamma^k_{y,x}(0) \;-\; \frac{t^2}{2}\sum_{i,j}\Gamma^k_{ij}(x)\,\dot\gamma^i_{y,x}(0)\,\dot\gamma^j_{y,x}(0) \\ +\; \sum_{|\alpha|\geqslant 3}\frac{t^{|\alpha|}}{\alpha!}\,\Gamma^k_\alpha(x)\,\dot\gamma^\alpha_{y,x}(0)\,, \qquad t\to 0\,,$$

where Γ^k_α are some polynomials in the Christoffel symbols and their derivatives. Obviously, $t\,\dot\gamma_{y,x}(0) = \dot\gamma_{z_t,x}(0)$ where $z_t = \gamma_{y,x}(t)$. Therefore the same asymptotic expansion holds as $t = 1$ and $y \to x$. From here it follows that

$$\dot\gamma^k_{y,x}(0) \;\sim\; (y^k - x^k) \;+\; \frac{1}{2}\sum_{i,j}\Gamma^k_{ij}(x)\,(y^i - x^i)\,(y^j - x^j) \\ +\; \sum_{|\alpha|\geqslant 3}\frac{1}{\alpha!}\,\tilde\Gamma^k_\alpha(x)\,(y-x)^\alpha\,, \qquad y\to x\,, \tag{1.5}$$

where $\tilde\Gamma^k_\alpha$ are also some polynomials in the Christoffel symbols and their derivatives.

4. Let $F = \{F^{i_1,\dots,i_p}_{j_1,\dots,j_q}\}$ be a (p,q)-tensor and $\nu = \sum \nu^k(y)\,\partial_{y^k}$ be a vector field. Then the "functions"

$$\mathbf{D}_\nu F^{i_1,\dots,i_p}_{j_1,\dots,j_q}(y) \;=\; \sum_k \nu^k(y)\,\partial_{y^k}\,F^{i_1,\dots,i_p}_{j_1,\dots,j_q}(y) \\ +\sum_{k,i'_1}\Gamma^{i_1}_{ki'_1}(y)\,\nu^k(y)\,F^{i'_1,\dots,i_p}_{j_1,\dots,j_q}(y) \;+\;\dots\;+\; \sum_{k,i'_p}\Gamma^{i_p}_{ki'_p}(y)\,\nu^k(y)\,F^{i_1,i_2,\dots,i'_p}_{j_1,j_2\dots,j_q}(y) \\ -\;\sum_{k,j'_1}\Gamma^{j'_1}_{kj_1}(y)\,\nu^k(y)\,F^{i_1,\dots,i_p}_{j'_1,\dots,j_q}(y) \;-\;\dots\;-\;\sum_{k,j'_q}\Gamma^{j'_q}_{kj_q}(y)\,\nu^k(y)\,F^{i_1,\dots,i_p}_{j_1,\dots,j'_q}(y)$$

are components of a (p,q)-tensor which is called the covariant derivative of F with respect to ν. Using this notion one can reformulate the definition of a geodesic: $y(t)$ is a geodesic if the covariant derivative of the vector field $\dot y(t)$ with respect to itself is equal to zero.

Let $\mathbf{D}_k$ be the covariant differentiation with respect to the coordinate vector fields ∂_{y^k}. A $(p,q+l)$-tensor with the components

$$\mathbf{D}_{k_1}\dots\mathbf{D}_{k_l}F^{i_1,\dots,i_p}_{j_1,\dots,j_q}(y)\,,\qquad 1\leqslant k_1,\dots,k_l\leqslant n\,,$$

is called the l-th *covariant differential* of F. Generally speaking, the covariant differential is not symmetric with respect to the indices $k_1,\dots,k_l$. We denote its symmetrization with respect to $k_1,\dots,k_l$ by $\{\mathbf{D}^\alpha F\}_{|\alpha|=l}$. This tensor is said to be the l-th *symmetric covariant differential* of F. A tensor is said to be *parallel* if all its covariant differentials are identically equal to zero.

5. For arbitrary vector fields $\nu_l=\sum_k\nu_l^k(y)\partial_{y^k}$ we have

$$\mathbf{D}_{\nu_1}\nu_2\;-\;\mathbf{D}_{\nu_2}\nu_1\;-\;[\nu_1,\nu_2]\;=\;\sum_{i,j,k}T^i_{jk}(y)\,\nu_1^j(y)\,\nu_2^k(y)\,\partial_{y^i}\,,\tag{1.6}$$

$$\mathbf{D}_{\nu_1}\mathbf{D}_{\nu_2}\nu_3\;-\;\mathbf{D}_{\nu_2}\mathbf{D}_{\nu_1}\nu_3\;-\;\mathbf{D}_{[\nu_1,\nu_2]}\nu_3\;=\;\sum_{i,j,k,l}R^i{}_{jkl}(y)\,\nu_1^k(y)\,\nu_2^l(y)\,\nu_3^j(y)\,\partial_{y^i}\,,\tag{1.7}$$

where $T=\{T^i_{jk}\}$ and $R=\{R^i{}_{jkl}\}$ are some tensors. The tensors T and R are said to be the *torsion tensor* and the *curvature tensor* of the connection Γ. The tensor with the components $R_{kl}=\sum_i R^i{}_{kil}$ is called the *Ricci tensor*.

Let $T(\nu_1,\nu_2)$ and $R(\nu_1,\nu_2)\,\nu_3$ be the vector fields (1.6) and (1.7) respectively. Then

$$\mathfrak{S}\,\{\,R(\nu_1,\nu_2)\,\nu_3\,\}\;=\;\mathfrak{S}\,\{\,T\,(T(\nu_1,\nu_2),\nu_3)+\mathbf{D}_{\nu_1}T(\nu_2,\nu_3)\,\}\,,$$

$$\mathfrak{S}\,\{\,\mathbf{D}_{\nu_1}R(\nu_2,\nu_3)+R\,(T(\nu_1,\nu_2),\nu_3)\,\}\;=\;0\,,$$

where $\mathfrak{S}\,\{\,\cdot\,\}$ denotes the cyclic sum with respect to ν_1, ν_2 and ν_3. These formulae are called Bianchi's identities.

In an arbitrary coordinate system

$$T^i_{jk}(y)\;=\;\Gamma^i_{jk}(y)\;-\;\Gamma^i_{kj}(y)\,,$$

$$R^i{}_{jkl}(y)\;=\;\partial_{y^k}\Gamma^i_{lj}(y)\;-\;\partial_{y^l}\Gamma^i_{kj}(y)\;+\;\sum_p\Gamma^i_{kp}(y)\Gamma^p_{lj}(y)\;-\;\sum_p\Gamma^i_{lp}(y)\,\Gamma^p_{kj}(y)\,.$$

Obviously, $T^i_{jk}=-T^i_{kj}$ and $R^i{}_{jkl}=-R^i{}_{jlk}$. If $T\equiv 0$ then, by the first Bianchi's identity, we also have $R^i{}_{jkl}+R^i{}_{ljk}+R^i{}_{klj}=0$.

When $T\equiv 0$ and $R\equiv 0$ the linear connection is flat, i.e., $\Gamma^i_{jk}\equiv 0$ in any n.c.s. When $T\equiv 0$ the connection is said to be symmetric. If the connection is symmetric, then for any fixed point $x\in M$ there exists a coordinate system such that $\Gamma^i_{jk}(x)=0$. It is valid, for example, in any n.c.s. with origin x. In the general case the identity (1.3) implies that $\Gamma^i_{jk}=T^i_{jk}/2$ at the origin of a n.c.s.

In (1.2) one can replace Γ^i_{jk} by $(\Gamma^i_{jk}+\Gamma^i_{kj})/2$. Thus, for an arbitrary given connection Γ there exists a symmetric connection generating the same geodesics and normal coordinate systems; in a sense these objects are independent of the torsion tensor.

6. We say that a tensor F is a polynomial of tensors $F_{(j)}$ if $F=\sum_k c_k\,G^{(k)}$, where c_k are constants and $G^{(k)}$ are tensor products of $F_{(j)}$ or the traces of tensor products of $F_{(j)}$ with respect to a part of indices. Differentiating the identity (1.3) with respect to y one can prove the following simple lemma.

Lemma 1.1. *Let $y=\{y^k\}$ be a n.c.s. with origin x associated with coordinates $\{x^k\}$. Then $\{\partial_y^\alpha\Gamma^i_{jk}\big|_{y=x}\}_{|\alpha|=q}$ is a $(1,q+2)$-tensor in the coordinates $\{x^k\}$ which coincides with a polynomial of the torsion and curvature tensors and their symmetric covariant differentials.*

In particular, if Γ is symmetric then (1.3) implies

$$\partial_{y^l}\Gamma^i_{jk}(x)\ +\ \partial_{y^j}\Gamma^i_{kl}(x)\ +\ \partial_{y^k}\Gamma^i_{lj}(x)\ =\ 0\,,$$

and therefore

$$\partial_{y^l}\Gamma^i_{jk}(x)\ =\ \frac{1}{3}\left(R^{\,i}{}_{jlk}(x)+R^{\,i}{}_{klj}(x)\right). \tag{1.8}$$

7. When M is a Riemannian (or, more generally, pseudo-Riemannian) manifold, we denote the corresponding metric tensor by $\{g_{ij}\}$, and the inverse metric tensor — by $\{g^{ij}\}$. Then $|\xi|_x:=\left(\sum_{i,j}g^{ij}(x)\,\xi_i\,\xi_j\right)^{1/2}$ is the length of the covector $\xi\in T^*_yM$.

The *Laplace operator* Δ on M is defined in local coordinates by the formula

$$\Delta u(x)\ =\ g^{-1}(x)\sum_{i,j}\partial_{x^i}\left(g(x)\,g^{ij}(x)\,\partial_{x^j}u(x)\right),$$

where $g:=|\det\{g_{ij}\}|^{1/2}$. The "function" g is a smooth density on M, which is called the *canonical Riemannian density*. The operator $g^{\varkappa}\Delta\,g^{-\varkappa}$ is said to be the Laplace operator in the space of $\varkappa$-densities; we shall also denote it by Δ.

A linear connection of a Riemannian manifold is said to be *metric* if the metric tensor is parallel. There exists a unique symmetric metric connection which is called the *Levi–Civita connection*. When Γ is the Levi–Civita connection, the function $S=\sum_{j,k}g^{jk}R_{jk}$ is said to be the *scalar curvature* of M.

The curvature tensor of the Levi–Civita connection possesses some extra properties. In particular,

(1) the corresponding Ricci tensor is symmetric;
(2) $R_{ij,kl}=R_{kl,ij}=-R_{ji,kl}=-R_{ij,lk}$, where $R_{ij,kl}:=\sum_p g_{ip}R^{\,p}{}_{jkl}$.

2. Horizontal distribution and horizontal differentials

1. Let $\nu=\sum\nu^k(y)\,\partial_{y^k}$ be a vector field on M. Then by (1.1)

$$\nabla_\nu\ =\ \sum_k\nu^k(y)\,\partial_{y^k}\ +\ \sum_{i,j,k}\Gamma^i_{kj}(y)\,\nu^k(y)\eta_i\,\partial_{\eta_j} \tag{2.1}$$

is a vector field on T^*M. The vector field (2.1) is said to be the *horizontal lift* of ν. The horizontal lifts generate a n-dimensional subbundle $HT^*M\subset TT^*M$ which is called the *horizontal distribution.* The vertical vector fields $\partial_{\eta_1},\dots,\partial_{\eta_n}$ generate another n-dimensional subbundle $VT^*M\subset TT^*M$ which is called the *vertical distribution.*

Since $HT^*M\,\cap\,VT^*M=\{0\}$, we have $TT^*M=HT^*M\,+\,VT^*M$. Obviously, the horizontal distribution depends on the choice of Γ whereas the vertical distribution does not.

Lemma 2.1. *The horizontal distribution HT^*M is involutive if and only if the connection Γ is curvature free, and it is Lagrangian if and only if Γ is symmetric.*

Proof. In an arbitrary coordinate system we have

$$[\nabla_k,\nabla_l]\;=\;\sum_{i,j}R^{\,i}_{\;jkl}(y)\,\eta_i\,\partial_{\eta_j}\,,\qquad\langle\,dy\wedge d\eta\,,\,\nabla_k\wedge\nabla_l\,\rangle\;=\;\sum_i T^i_{kl}(y)\,\eta_i\,.$$

The second identity is equivalent to the second statement of the lemma. The first identity implies that the commutator of any horizontal vector fields is also a horizontal vector field if and only if $R\equiv 0$. It is equivalent to the first statement. $\square$

A curve in the cotangent bundle T^*M is said to be horizontal (or vertical) if its tangent vectors belong to HT^*M (or VT^*M). For any given curve $y(t)\subset M$ and covector $\eta_0\in T^*_{y(0)}M$ there exists just one horizontal curve $\big(y(t),\eta(t)\big)\subset T^*M$ starting at the point $(y(0),\eta_0)$. It is defined in local coordinates y by the equations

$$\frac{d}{dt}\,\eta_j(t)\;-\;\sum_{i,k}\Gamma^i_{kj}\big(y(t)\big)\,\dot y^k(t)\,\eta_i(t)\;=\;0\,,\qquad j=1,\dots,n\,.$$

The curve $\big(y(t),\eta(t)\big)$ is said to be the *horizontal lift* of $y(t)$. The corresponding linear transformation $\eta_0\to\eta(t)$ is called the *parallel displacement* along the curve $y(t)$. By duality horizontal curves and parallel displacements are defined in the tangent bundle TM, and then in all the tensor bundles over M.

Let F be a tensor and ν be a vector field on M. Then

$$\mathbf{D}_\nu F\;=\;\lim{}_{t\to 0}(F-F_t)/t\,, \tag{2.2}$$

where F_t is the tensor obtained from F by the parallel displacement along the integral curves of ν. If F is a $(p,0)$-tensor and

$$\mathcal{F}(y,\eta)\;=\;\sum_{i_1,\dots,i_p}F^{i_1,\dots,i_p}(y)\,\eta_{i_1}\dots\eta_{i_p}$$

is the corresponding polynomial on T^*M, then

$$\nabla_\nu\,\mathcal{F}(y,\eta)\;=\;\sum_{i_1,\dots,i_p}\mathbf{D}_\nu F^{i_1,\dots,i_p}(y)\,\eta_{i_1}\dots\eta_{i_p}\,. \tag{2.3}$$

Remark 2.2. The equality (2.2) implies that a curve $y(t)$ is a geodesic if and only if the curve $\big(y(t),\dot y(t)\big)\in TM$ is horizontal.

2. Let $\Phi_{y,x} : T^*_x M \to T^*_y M$ be the parallel displacement along the geodesic $\gamma_{y,x}$ and $\Upsilon_{y,x} = |\det \Phi_{y,x}|$. Obviously, $\Phi_{x,y} = \Phi_{y,x}^{-1}$, $\Upsilon_{x,y} = \Upsilon_{y,x}^{-1}$, and $\Upsilon_{y,x}$ is a density in y and a (-1)-density in x.

Let $y = \{y^k\}$ be the n.c.s. with origin x associated with coordinates $\{x^k\}$. We denote by $\Phi_x(y)$ and $\Upsilon_x(y) = |\det \Phi_x(y)|$ the $n \times n$-matrix-function and the function which represent in the chosen coordinates $\Phi_{y,x}$ and $\Upsilon_{y,x}$ respectively. Since the curve $(\gamma_{y,x}, \dot\gamma_{y,x})$ is horizontal, the explicit formulae (1.4) imply

$$\sum_j (y^j - x^j)\,(\Phi_x)^i_j(y) \;=\; (y^i - x^i)\,. \tag{2.4}$$

By the definition of parallel displacement

$$r\,\partial_r(\Phi_x)^i_j(y) \;=\; \sum_{k,l} (y^k - x^k)\,\Gamma^l_{kj}(y)\,(\Phi_x)^i_l(y)\,, \qquad \Phi_x(y)\big|_{y=x} \;=\; I\,, \tag{2.5}$$

and then by the Liouville formula

$$r\,\partial_r\Upsilon_x(y) \;=\; \sum_{k,l} (y^k - x^k)\,\Gamma^j_{kj}(y)\,\Upsilon_x(y)\,, \qquad \Upsilon_x(y)\big|_{y=x} \;=\; 1\,, \tag{2.6}$$

where

$$r \;=\; |y - x|\,, \qquad\qquad \partial_r \;=\; r^{-1}\sum_k (y^k - x^k)\,\partial_{y^k}\,.$$

From (2.5) and (1.3) we obtain by straightforward calculation

$$\begin{aligned} r\,\partial_r^2\Big(r\,(\Phi_x)^i_j(y)\Big) \;&=\; r\,\partial_r\Big(\sum_k (y^k - x^k)\,T^l_{kj}(y)\,(\Phi_x)^i_l(y)\Big) \\ &\quad - \sum_{k,m} (y^k - x^k)\,(y^m - x^m)\,R^l{}_{kjm}(y)\,(\Phi_x)^i_l(y)\,. \end{aligned} \tag{2.7}$$

The sets of values of the derivatives $\{\partial_y^\alpha \Phi_x(y)\}_{|\alpha|=q}$ and $\{\partial_y^\alpha \Upsilon_x(y)\}_{|\alpha|=q}$ at the point $y = x$ are $(1, q+1)$-tensors and $(0, q)$-tensors respectively. By Lemma 1.1 these tensors are polynomials in the curvature and torsion tensors and their symmetric covariant differentials. From (2.5) and (2.6) it follows that

$$\partial_{y^k}(\Phi_x)^i_j(y)\big|_{y=x} \;=\; \Gamma^i_{kj}(y)\big|_{y=x} \;=\; \frac{1}{2}\,T^i_{kj}(x)\,, \tag{2.8}$$

$$\partial_{y^k}\Upsilon_x(y)\big|_{y=x} \;=\; \sum_j \Gamma^j_{kj}(y)\Big|_{y=x} \;=\; \frac{1}{2}\sum_j T^j_{kj}(x)\,. \tag{2.9}$$

The values of the higher order derivatives of $\Phi_x(y)$ and $\Upsilon_x(y)$ at $y = x$ can be easily derived from (2.7). In particular, if Γ is symmetric then we obtain

$$\partial_{y^k}\partial_{y^l}(\Phi_x)^i_j(y)\big|_{y=x} \;=\; -\frac{1}{6}\left(R^i{}_{kjl}(x) \;+\; R^i{}_{ljk}(x)\right), \tag{2.10}$$

$$\partial_{y^k}\partial_{y^l}\Upsilon_x(y)\big|_{y=x} \;=\; -\frac{1}{6}\left(R_{kl}(x) \;+\; R_{lk}(x)\right). \tag{2.11}$$

Remark 2.3. Let M be a pseudo-Riemannian manifold and Γ be the Levi–Civita connection. Then in an arbitrary coordinate system $\Phi^T_{y,x}\cdot\{g^{ij}(y)\}\cdot\Phi_{y,x}=\{g^{ij}(x)\}$ and, consequently, $\Upsilon_{y,x}=g^{-1}(x)\,g(y)$. Differentiating these equalities with respect to y and taking into account (2.8)–(2.11), we obtain

$$\partial_{y^k}g^{ij}(y)\big|_{y=x} \;=\; \partial_{y^k}g_{ij}(y)\big|_{y=x} \;=\; 0\,,\qquad \partial_{y^k}g(y)\big|_{y=x} \;=\; 0\,,$$

$$\partial_{y^k}\partial_{y^l}g_{ij}(y)\big|_{y=x} \;=\; -\,\frac{1}{3}\left(R_{ikjl}(x)+R_{iljk}(x)\right),$$

$$\partial_{y^k}\partial_{y^l}g(y)\big|_{y=x} \;=\; -\,\frac{1}{3}\,g(x)\,R_{kl}(x)$$

in the n.c.s. $y=\{y^k\}$ with origin x associated with coordinates $x=\{x^k\}$.

3. Let $a\in C^\infty(T^*M)$ and

$$\nabla_x^\alpha a(x,\xi) \;=\; \frac{d^\alpha}{dy^\alpha}a(y,\Phi_x(y)\xi)\bigg|_{y=x}\,. \tag{2.12}$$

Then $\{\nabla_x^\alpha a\}_{|\alpha|=q}$ is a symmetric $(0,q)$-tensor, which is called the q-th *symmetric horizontal differential* of a. The symmetric $(p,0)$-tensor $\{\partial_\xi^\alpha a(x,\xi)\}_{|\alpha|=p}$ is said to be the p-th *vertical differential* of the function a.

Lemma 2.4. *In an arbitrary coordinate system $\{x^k\}$ and the associated n.c.s. $y=\{y^k\}$ with origin x we have*

$$\begin{aligned} q!\sum_{|\alpha|=q}\frac{1}{\alpha!}\,(y-x)^\alpha\,\frac{d^\alpha}{dy^\alpha}a\big(y,\Phi_x(y)\xi\big)& \\ =\sum_{1\leqslant i_1,\dots,i_q\leqslant n}(y^{i_1}-x^{i_1})\dots(y^{i_q}-x^{i_q})&\left(\nabla^{(x)}_{i_1}\dots\nabla^{(x)}_{i_q}a\right)\big(y,\Phi_x(y)\xi\big)\,, \end{aligned}\tag{2.13}$$

where $\nabla^{(x)}_k=\nabla^{(x)}_k(y,\eta,\partial_y,\partial_\eta)$ are the horizontal lifts of the vector fields ∂_{y^k}.

Proof. Taking into account (2.5) and the identity

$$\frac{d}{dy^k}\,a\big(y,\Phi_x(y)\xi\big) \;=\; \big(\partial_{y^k}a\big)\big(y,\Phi_x(y)\xi\big) \;+\; \sum_{i,j}\partial_{y^k}(\Phi_x)^i_j(y)\,\xi_i\,\big(\partial_{\eta_j}a\big)\big(y,\Phi_x(y)\xi\big)\,,$$

we see that

$$\sum_k(y^k-x^k)\,\frac{d}{dy^k}\,a\big(y,\Phi_x(y)\xi\big) \;=\; \sum_k(y^k-x^k)\,\big(\nabla^{(x)}_k a\big)\big(y,\Phi_x(y)\xi\big)\,.$$

From here by induction with respect to q we obtain

$$
\begin{aligned}
&q!\sum_{|\alpha|=q}\frac{1}{\alpha!}\,(y-x)^\alpha\,\frac{d^\alpha}{dy^\alpha}a\big(y,\Phi_x(y)\xi\big)\\
&\qquad=\sum_{1\leqslant i_1,\dots,i_q\leqslant n}(y^{i_1}-x^{i_1})\dots(y^{i_q}-x^{i_q})\,\frac{d^q}{dy^{i_1}\dots dy^{i_q}}a\big(y,\Phi_x(y)\xi\big)\\
&\qquad\qquad=\sum_{1\leqslant i_1,\dots,i_q\leqslant n}(y^{i_1}-x^{i_1})\dots(y^{i_q}-x^{i_q})\,\big(\nabla^{(x)}_{i_1}\dots\nabla^{(x)}_{i_q}a\big)\big(y,\Phi_x(y)\xi\big)
\end{aligned}
$$

for all $q\in\mathbf{N}$. $\square$

Corollary 2.5. *The q-th symmetric horizontal differential of a coincides with the symmetrization of the tensor $\{\big(\nabla^{(x)}_{i_1}\dots\nabla^{(x)}_{i_q}a\big)(x,\xi)\}$.*

Proof. In view of (2.13)

$$
\begin{aligned}
&q!\sum_{|\alpha|=q}\frac{1}{\alpha!}\,(y-x)^\alpha\,\nabla_x^\alpha a(x,\xi)\\
&\qquad\qquad=\sum_{1\leqslant i_1,\dots,i_q\leqslant n}(y^{i_1}-x^{i_1})\dots(y^{i_q}-x^{i_q})\,\big(\nabla^{(x)}_{i_1}\dots\nabla^{(x)}_{i_q}a\big)(x,\xi)
\end{aligned}
$$

modulo $O(|y-x|^{q+1})$. Multiplying both sides of this equality by $|y-x|^{-q}$ we obtain that

$$
q!\sum_{|\alpha|=q}\frac{1}{\alpha!}\,c_\alpha\,\nabla_x^\alpha a(x,\xi)=\sum_{1\leqslant i_1,\dots,i_q\leqslant n}c^{i_1,\dots,i_q}\,\big(\nabla^{(x)}_{i_1}\dots\nabla^{(x)}_{i_q}a\big)(x,\xi)
$$

for an arbitrary symmetric tensor $\{c^{i_1,\dots,i_q}\}_{1\leqslant i_1,\dots,i_q\leqslant n}=\{c_\alpha\}_{|\alpha|=q}$. This immediately implies the corollary. $\square$

From Corollary 2.5 it follows that a function $a\in C^\infty(T^*M)$ is constant along any horizontal curve if and only if all its symmetric horizontal differentials are equal to zero.

Remark 2.6. By (2.3) $\nabla_x^\alpha\,\mathcal{F}(x,\xi)=\sum_{i_1,\dots,i_p}\mathbf{D}^\alpha F^{i_1,\dots,i_p}(x)\,\xi_{i_1}\dots\xi_{i_p}$ for any $(p,0)$-tensor F and the corresponding polynomial $\mathcal{F}(x,\xi)$.

Remark 2.7. If the connection Γ is flat then $\nabla_x^\alpha\nabla_x^\beta=\nabla_x^{\alpha+\beta}$ and $\partial_\xi^\alpha\nabla_x^\beta=\nabla_x^\beta\partial_\xi^\alpha$. In the general case this is not true. For example,

$$
\begin{aligned}
(\nabla_{x^k}\nabla_{x^l}-\nabla_{x^l}\nabla_{x^k})\,a(x,\xi)&=\sum_{i,j}R^i{}_{jkl}(x)\,\xi_i\,a_{\xi_j}(x,\xi)\,,\\
(\partial_{\xi^k}\nabla_{x^l}-\nabla_{x^l}\partial_{\xi^k})\,a(x,\xi)&=\frac{1}{2}\sum_j T^k_{lj}(x)\,a_{\xi_j}(x,\xi)\,.
\end{aligned}
$$

3. Classes of symbols

1. Let us fix a positive function $w(y,\eta) \in C^\infty(T^*M \setminus 0)$ homogeneous in η of degree one and define $\langle\eta\rangle_y := (1+w^2(y,\eta))^{1/2}$.

Remark 3.1. All the further definitions and results are independent of the choice of w. If M is a Riemannian manifold then we can take, for example, $w(y,\eta) = |\eta|_y$.

For an arbitrary coordinate system $y = \{y^k\}$ let ∇_k be the horizontal lifts of the coordinate vector fields ∂_{y^k}. We denote by $\mathrm{S}^m_{\rho,\delta}(\Gamma)$ the class of functions $a \in C^\infty(T^*M)$ such that in any coordinates y for all α and $i_1,\dots,i_q$

$$\left|\partial_\eta^\alpha \nabla_{i_1}\dots\nabla_{i_q} a(y,\eta)\right| \;\leqslant\; \mathrm{const}_{K,\alpha,i_1,\dots,i_q}\, \langle\eta\rangle_y^{m+\delta q-\rho|\alpha|} \tag{3.1}$$

when y runs over a compact set $K \subset M$. Such functions are said to be *symbols.* Analogously, we define the class $\mathrm{S}^m_{\rho,\delta}(\Gamma) \subset C^\infty(M\times T^*M)$ of functions $a(z;y,\eta)$ such that in any coordinates y and z for all α, β and $i_1,\dots,i_q$

$$\left|\partial_z^\beta\partial_\eta^\alpha \nabla_{i_1}\dots\nabla_{i_q} a(z;y,\eta)\right| \;\leqslant\; \mathrm{const}_{K,\alpha,\beta,i_1,\dots,i_q}\, \langle\eta\rangle_x^{m+\delta|\beta|+\delta q-\rho|\alpha|} \tag{3.2}$$

when (y,z) runs over a compact set $K \subset M\times M$. We call these functions *amplitudes.* For the sake of simplicity by $\mathrm{S}^m_{\rho,\delta}(\Gamma)$ we denote also the classes of tensors, and in this case we mean that the estimates (3.1) or (3.2) hold for all the components of the tensors in any local coordinate system.

If $a \in \mathrm{S}^{m_1}_{\rho,\delta}(\Gamma)$ and $b \in \mathrm{S}^{m_2}_{\rho,\delta}(\Gamma)$ then $ab \in \mathrm{S}^{m_1+m_2}_{\rho,\delta}(\Gamma)$ and $a+b \in \mathrm{S}^m_{\rho,\delta}(\Gamma)$, where $m = \max\{m_1,m_2\}$. Moreover,

$$\partial_\eta^\alpha a \;\in\; \mathrm{S}^{m-\rho|\alpha|}_{\rho,\delta}(\Gamma)\,, \quad \nabla_{\nu_1}\dots\nabla_{\nu_q} a \;\in\; \mathrm{S}^{m+\delta q}_{\rho,\delta}(\Gamma)\,, \qquad \forall\, a \in \mathrm{S}^m_{\rho,\delta}(\Gamma)\,, \tag{3.3}$$

for any smooth vector fields $\nu_1,\dots,\nu_q$ on M. In particular,

$$\nabla^\alpha a \;\in\; \mathrm{S}^{m+\delta|\alpha|}_{\rho,\delta}(\Gamma)\,, \qquad \forall a \in \mathrm{S}^m_{\rho,\delta}(\Gamma)\,. \tag{3.4}$$

We shall always assume that

$$0 \;\leqslant\; \delta \;<\; \rho \;\leqslant\; 1\,.$$

Obviously, if $a \in \mathrm{S}^m_{\rho,\delta}(\Gamma)$ for some Γ then $a \in \mathrm{S}^m_{\rho,\delta'}(\Gamma')$ with $\delta' = \max\{\delta, 1-\rho\}$ for any other linear connection Γ'. Therefore under the traditional condition $1-\rho \leqslant \delta$ (which implies $\rho > 1/2$) the definition of the classes $\mathrm{S}^m_{\rho,\delta}$ does not depend on the choice of Γ. In this case $\mathrm{S}^m_{\rho,\delta}$ coincides with the standard class defined in local coordinates.

By S^m we denote the class of symbols $a \in \mathrm{S}^m_{1,0}$ which admit an asymptotic expansion

$$a(y,\eta) \;\sim\; \sum_{i=0}^\infty a_{m-i}(y,\eta)\,, \qquad \langle\eta\rangle_y \to \infty\,, \tag{3.5}$$

with a_{m-i} positively homogeneous in η of degree $m-i$. By analogy, we define the class S^m of amplitudes (or tensor amplitudes) from $C^\infty(M\times T^*M)$. For all ρ,δ and Γ the intersection $S^{-\infty} = \cap_m \mathrm{S}^m_{\rho,\delta}(\Gamma) = \cap_m S^m$ consists of functions (or tensors) which vanish with all their derivatives faster than any power of $\langle\eta\rangle_y$ as $\langle\eta\rangle_y \to \infty$.

2. We shall make frequent use of the following two lemmas. The first of them is a simple modification of [Proposition 18.1.3, H] and is proved in the same way (analogous lemmas are proved in [Sh] and [T]).

Lemma 3.2. *Let $a_k \in \mathrm{S}^{m_k}_{\rho,\delta}(\Gamma)$ where $m_k \to -\infty$ as $k \to \infty$, and $m = \max\{m_k\}$. Then there exists a function $a \in \mathrm{S}^{m}_{\rho,\delta}(\Gamma)$ unique modulo $S^{-\infty}$, such that $a \sim \sum_k a_k$. The functions a_k may depend on some parameters λ; in this case we assume in addition that $\partial^\alpha_\lambda a_k \in \mathrm{S}^{m_k}_{\rho,\delta}(\Gamma)$, $\forall\alpha$.*

Lemma 3.3. *Let $a \in \mathrm{S}^{m}_{\rho,\delta}(\Gamma)$. Then for all non-negative integers q*

$$a(y,\Phi_{y,x}\xi) \;=\; \sum_{|\alpha|\leqslant q} \frac{1}{\alpha!}\,\dot\gamma^\alpha_{y,x}\,\nabla^\alpha_x a(x,\xi) \;+\; \sum_{|\alpha|=q+1} \dot\gamma^\alpha_{y,x}\,\tilde a_\alpha(y;x,\xi)\,, \tag{3.6}$$

where $\tilde a_\alpha \in \mathrm{S}^{m+\delta|\alpha|}_{\rho,\delta'}(\Gamma)$, $\delta' = \max\{\delta, 1-\rho\}$.

Proof. Let $\{x^k\}$ be some local coordinates and $y = \{y^k\}$ be the associated n.c.s. with origin x. By Taylor's formula

$$\begin{aligned} a(y,\Phi_x(y)\xi) \;=\; & \sum_{|\alpha|\leqslant q} \frac{1}{\alpha!}\,(y-x)^\alpha\,\nabla^\alpha_x a(x,\xi) \\ & +\;(q+1)\sum_{|\alpha|=q+1} \frac{1}{\alpha!}\,(y-x)^\alpha \int_0^1 (1-t)^q\,\frac{d^\alpha}{dy^\alpha} a(y,\Phi_x(y)\xi)\Big|_{y=z_t}\,dt\,, \end{aligned}$$

where $z_t = x + t(y-x)$. Since $y - x = t^{-1}(z_t - x)$ we can apply (2.13) with z_t instead of y. Then we obtain (3.6) with

$$\tilde a_\alpha(y;x,\xi) \;=\; (q!)^{-1} \sum_{i_1,\dots i_{q+1}} \int_0^1 (1-t)^q\,\big(\nabla^{(x)}_{i_1}\dots\nabla^{(x)}_{i_{q+1}} a\big)(z_t,\Phi_x(z_t)\xi)\,dt\,,$$

where the sum is taken over all the ordered sets of indices $i_1,\dots i_{q+1}$ corresponding to the multi-index α. In view of (3.3) these functions belong to $\mathrm{S}^{m+\delta|\alpha|}_{\rho,\delta'}(\Gamma)$. □

Example 3.4. Let M be a Riemannian manifold and Γ be a metric connection. By (2.3) the function $|\xi|^2_x$ is constant along the horizontal curves (the same is valid when M is a pseudo-Riemannian manifold). Therefore for an arbitrary $f \in C^\infty(\mathbf{R}^1)$ all the horizontal derivatives of the function $f(|\xi|_x)$ are equal to zero. Let $f(r) = 0$ in a neighbourhood of $r = 0$ and $|d^k f/dr^k| \leqslant \mathrm{const}_k$ for all $k = 0,1,\dots$ Then $f(|\xi|^{1-\rho}_x) \in \mathrm{S}^0_{\rho,0}(\Gamma)$. However, when the curvature tensor is not equal to zero, even the first differential $d_x(|\xi|_x)$ in local coordinates x can not vanish on an open set. Therefore in any local coordinates we have only $f(|\xi|^{1-\rho}_x) \in \mathrm{S}^0_{\rho,1-\rho}$.

Remark 3.5. In [H] L. Hörmander has introduced very general classes of ψDOs in $\mathbf{R}^n$. The corresponding classes of symbols are defined in terms of a slowly varying Riemannian metric on the Euclidean space $\mathbf{R}^n_x \times \mathbf{R}^n_\xi$. This Riemannian metric determines a weight function h (which plays the role of $\langle\xi\rangle_x^{-1}$), and the main results in [H] are obtained under the assumption $h \leqslant 1$. In particular, the standard classes $\mathrm{S}^m_{\rho,\delta}$ are generated by the metric

$$\langle\xi\rangle^{2\delta}\,|dx|^2 \;+\; \langle\xi\rangle^{-2\rho}\,|d\xi|^2\,, \tag{3.7}$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. In this case the weight function coincides with $\langle\xi\rangle^{-1}$.

When $M = \mathbf{R}^n$, the classes $\mathrm{S}^m_{\rho,\delta}(\Gamma)$ can be defined in the same manner. The corresponding metric has the form

$$\langle\xi\rangle^{2\delta}\,|dx|^2 \;+\; \langle\xi\rangle^{-2\rho} \sum_j |\, d\xi_j - \sum_{i,k} \Gamma^{\,i}_{kj}(x)\, \xi_i\, dx^k\,|^2\,, \tag{3.8}$$

where $\Gamma^{\,i}_{kj}$ are the Christoffel symbols. This means that the length of a vector $(y,\eta) \in T_{(x,\xi)}(\mathbf{R}^n_x \times \mathbf{R}^n_\xi)$ is defined by the equality

$$|(y,\eta)|^2_{(x,\xi)} \;=\; \langle\xi\rangle^{2\delta}\,|y|^2 \;+\; \langle\xi\rangle^{-2\rho} \sum_j |\,\eta_j - \sum_{i,k} \Gamma^{\,i}_{kj}(x)\, \xi_i\, y^k\,|^2\,. \tag{3.9}$$

Assume that $\Gamma^{\,i}_{kj}$ are uniformly bounded with all their derivatives. It can be easily seen from (3.9) that the metrics (3.7) and (3.8) are equivalent when $\rho \geqslant 1-\delta$. If $1-\rho \leqslant 2\delta$ and, consequently, $\rho \geqslant 1/3$ then the metric (3.8) is *slowly varying* (i.e., there exist positive constants c and C such that $|(y,\eta)|_{(x,\xi)} \leqslant c$ implies $|(z,\zeta)|_{(x+y,\xi+\eta)} \leqslant C\,|(z,\zeta)|_{(x,\xi)}$ for all $(z,\zeta) \in \mathbf{R}^{2n}$). However, when $\rho < 1/2$ the weight function h corresponding to the canonical Euclidean structure on $T^*\mathbf{R}^n = \mathbf{R}^n_x \times \mathbf{R}^n_\xi$, generally speaking, is not bounded. It is quite possible that the technique developed in [H] can be adapted to the classes $\mathrm{S}^m_{\rho,\delta}(\Gamma)$ (and other classes of symbols depending on a connection Γ) by introducing a "proper" structure on $T^*\mathbf{R}^n$.

4. Representation of ψDOs by invariant oscillatory integrals

1. Let $A : C_0^\infty(M;\Omega^\varkappa) \to C^\infty(M;\Omega^\varkappa)$ be a linear operator with the Schwartz kernel $\mathcal{A}(x,y)$. The operator A is said to be *pseudodifferential* if

(1) $\mathcal{A}(x,y)$ is smooth outside the diagonal in $M \times M$;
(2) in each coordinate patch $U \times U \subset M \times M$ the kernel $\mathcal{A}(x,y)$ is represented modulo a smooth function by an oscillatory integral of the form

$$\int_{\mathbf{R}^n} e^{(x-y)\cdot\theta}\, a(x,y,\theta)\, d\theta\,,$$

where $a(x,y,\theta)$ is an amplitude from some coordinate class $\mathrm{S}^m_{\rho,\delta'}$ with $\rho,\delta' \in [0,1]$ (we do not assume that $\rho \geqslant \delta'$).

A ψDO A is said to be *properly supported* if both projections $\operatorname{supp}\mathcal{A} \to M$ are proper maps (a continuous map is called proper if the inverse image of any compact set is compact). A properly supported ψDO acts from $C_0^\infty(M;\Omega^\varkappa)$ into $C_0^\infty(M;\Omega^\varkappa)$ and from $C^\infty(M;\Omega^\varkappa)$ into $C^\infty(M;\Omega^\varkappa)$.

We denote by $\Psi^{-\infty}(\Omega^\varkappa)$ the class of operators with smooth kernels acting in the space of $\varkappa$-densities. Clearly, an arbitrary ψDO can be represented as the sum of a properly supported ψDO and an operator from $\Psi^{-\infty}(\Omega^\varkappa)$ (whose kernel is equal to zero in a neighbourhood of the diagonal).

Let $\Psi^m(\Omega^\varkappa)$ be the class of ψDOs acting in the space of $\varkappa$-densities such that the amplitudes in the corresponding local oscillatory integrals belong to S^m. These ψDOs are said to be *classical.* In particular, differential operators are properly supported classical ψDO, and their local amplitudes are polynomials with respect to the variables θ.

2. Let V be some sufficiently small neighbourhood of the diagonal in $M\times M$, and $z_\tau = z_\tau(x,y) = \gamma_{y,x}(\tau)$ where $\tau\in[0,1]$ (here and further on τ is considered as a parameter). We introduce the *phase functions*

$$\varphi_\tau(x,\zeta,y) \;=\; -\langle\dot\gamma_{y,x}(\tau),\zeta\rangle\,, \qquad (x,y)\in V\,,\ \zeta\in T^*_{z_\tau}M\,.$$

Obviously, the phase functions φ_τ are linear in ζ. If $\{y^k\}$ is the same coordinate system as $\{x^k\}$ then by (1.5)

$$\begin{aligned}\varphi_0(x,\zeta,y) \;\sim\; (x-y)\cdot\zeta \;-\; \frac12\sum_{i,j,k}\Gamma^k_{ij}(x)\,(y^i-x^i)\,(y^j-x^j)\,\zeta_k\\ -\sum_k\sum_{|\alpha|\geqslant 3}\frac{1}{\alpha!}\,\tilde\Gamma^k_\alpha(x)\,(y-x)^\alpha\,\zeta_k\,,\qquad y\to x\,. \quad (4.1)\end{aligned}$$

If $\{y^k\}$ is the n.c.s. with origin x associated with coordinates $\{x^k\}$ then by (1.4)

$$\varphi_\tau(x,\zeta,y) \;=\; (x-y)\cdot\zeta\,, \qquad \forall\tau\in[0,1]\,, \tag{4.2}$$

where we take the components of ζ corresponding to the y-coordinates. Since the curve $(\gamma_{y,x},\dot\gamma_{y,x})$ is horizontal, we have for all $\tau,s\in[0,1]$

$$\varphi_\tau(x,\zeta,y) \;=\; -\,\varphi_{1-\tau}(y,\zeta,x,)\,,\quad \varphi_\tau(x,\zeta,y) \;=\; \varphi_s(x,\Phi_{z_s,z_\tau}\zeta,y)\,. \tag{4.3}$$

Remark 4.1. In the classical theory of ψDOs one deals with phase functions of the form (4.2) assuming, however, that the coordinates $\{y^k\}$ are the same as $\{x^k\}$.

We associate with a symbol $a\in \mathrm{S}^m_{\rho,\delta}(\Gamma)$ the oscillatory integral

$$\mathcal A(x,y) \;=\; p_{\varkappa,\tau}\int e^{i\varphi_\tau(x,\zeta,y)}a(z_\tau,\zeta)\,đ\zeta\,, \qquad (x,y)\in V\,, \tag{4.4}$$

where $đ\zeta=(2\pi)^{-n}d\zeta$ and

$$p_{\varkappa,\tau} \;=\; p_{\varkappa,\tau}(x,y) \;=\; \Upsilon^{1-\varkappa}_{y,z_\tau}\,\Upsilon^{-\varkappa}_{z_\tau,x}$$

is a weight factor. In any local coordinates x and y the oscillatory integral (4.4) admits the standard regularization (see, for example, [Sh]) which allows us to interpret it as a distribution defined on V. This distribution is also denoted by $\mathcal{A}(x,y)$. It is easy to see that (when we change coordinates) $\mathcal{A}(x,y)$ behaves as a $\varkappa$-density in x and a $(1-\varkappa)$-density in y.

Let $U\times U\subset V$ be a coordinate patch, and $\{y^k\}$ be the same coordinates as $\{x^k\}$. From (4.1) and (4.3) it follows that

$$\varphi_\tau(x,\zeta,y) \;=\; (x-y)\cdot\Psi_\tau\zeta\,, \qquad \forall(x,y)\in U\times U\,,$$

where $\Psi_\tau=\Psi_\tau(x,y)$ is a smooth non-degenerate $n\times n$-matrix-function. Changing variables $\tilde\zeta=\Psi_\tau\zeta$ in (4.4), we obtain

$$\mathcal{A}(x,y) \;=\; p_{\varkappa,\tau}\,|\det\Psi_\tau|^{-1}\int e^{i(x-y)\cdot\zeta}\,a(z_\tau,\Psi_\tau^{-1}\tilde\zeta)\,d\!\!{}^-\tilde\zeta\,, \qquad (x,y)\in U\times U\,.$$

The amplitude $p_{\varkappa,\tau}\,|\det\Psi_\tau|^{-1}\,a(z_\tau,\Psi_\tau^{-1}\tilde\zeta)$ belongs to the coordinate class $\mathrm{S}^m_{\rho,\delta'}$ with $\delta'=\max\{\delta,1-\rho\}$. Therefore $\mathcal{A}(x,y)$ coincides in V with the Schwartz kernel of a ψDO A. This ψDO acts in the space of $\varkappa$-densities on M, and it is determined uniquely modulo an operator with smooth kernel.

Definition 4.2. We denote by $\Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma,\tau)$ the class of ψDOs A acting in the space of $\varkappa$-densities whose Schwartz kernels are represented in a neighbourhood of the diagonal by oscillatory integrals (4.4) with $a\in\mathrm{S}^m_{\rho,\delta}(\Gamma)$. The function a from (4.4) is called the τ-*symbol* of the ψDO A and it is denoted by $\sigma_{A,\tau}$. The functions $\sigma_A=\sigma_{A,0}$ and $\sigma_A^W=\sigma_{A,1/2}$ are said to be the *symbol* and the *Weyl symbol* of the ψDO A respectively.

We can replace the symbol in (4.4) by an amplitude $a(z_s;z_\tau,\zeta)\in\mathrm{S}^m_{\rho,\delta}(\Gamma)$ where $s\in[0,1]$. In this case $\mathcal{A}$ coincides in V with the Schwartz kernel of a ψDO as well. If the amplitude (or symbol) a is from $S^{-\infty}$ then the oscillatory integral (4.4) determines a smooth density and the corresponding ψDO belongs to $\Psi^{-\infty}(\Omega^\varkappa)$.

Remark 4.3. Since the definition of geodesics depends only on the symmetric part of the Christoffel symbols, the phase functions φ_τ are independent of the torsion tensor T, and the τ-symbols $\sigma_{A,\tau}$ of a ψDO A are independent of T modulo some inessential factor coming from $p_{\varkappa,\tau}$. However, when $\rho<1/2$ the classes $\mathrm{S}^m_{\rho,\delta}(\Gamma)$ and, consequently, $\Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma,\tau)$ depend on T.

Proposition 4.4. *For all $\tau,s\in[0,1]$ the classes $\Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma,\tau)$ and $\Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma,s)$ coincide, and*

$$\sigma_{A,s}(x,\xi)\sim\sum\frac{1}{\alpha!}\,(\tau-s)^{|\alpha|}\,D_\xi^\alpha\nabla_x^\alpha\sigma_{A,\tau}(x,\xi)\,, \qquad \langle\xi\rangle_x\to\infty\,.$$

Proof. Let us change variables $\zeta=\Phi_{z_\tau,z_s}\zeta'$ in (4.4). Then, in view of (4.3), we obtain the oscillatory integral

$$\mathcal{A}(x,y) \;=\; p_{\varkappa,s}\int e^{i\varphi_s(x,\zeta',y)}\,a(z_\tau,\Phi_{z_\tau,z_s}\zeta')\,d\!\!{}^-\zeta'\,. \tag{4.5}$$

We substitute in (4.5) the expansion (3.6) with $x = z_s$ and $y = z_\tau$. Then using the equality

$$\dot\gamma_{z_\tau,z_s}\, e^{i\varphi_s} \;=\; (\tau-s)\,\dot\gamma_{y,x}(s)\, e^{i\varphi_s} \;=\; (s-\tau)\, D_{\zeta'} e^{i\varphi_s}\,,$$

we replace $\dot\gamma^\alpha_{z_\tau,z_s}$ by $(s-\tau)^{|\alpha|} D^\alpha_{\zeta'} e^{i\varphi_s}$ and integrate by parts with respect to ζ'. This procedure transforms the oscillatory integral (4.5) into an oscillatory integral with the same phase function and with an amplitude of the form

$$\sum_{|\alpha|\leqslant q} \frac{1}{\alpha!}\,(\tau-s)^{|\alpha|}\, D^\alpha_{\zeta'} \nabla^\alpha_z a(z_s,\zeta') \;+\; r_{q+1}(z_s,z_\tau,\zeta')\,,$$

where $r_{q+1} \in \mathrm{S}^{m-(\rho-\delta)(q+1)}_{\rho,\delta'}$. Since $\rho > \delta$ the oscillatory integral with the amplitude r_{q+1} defines a smoother and smoother density as $q \to \infty$. $\square$

In the same way one can prove

Proposition 4.5. *Let $\tau, s \in [0,1]$ and a be an amplitude from $\mathrm{S}^m_{\rho,\delta}(\Gamma)$. Then the oscillatory integral*

$$\mathcal{A}(x,y) \;=\; p_{\varkappa,\tau} \int e^{i\varphi_\tau(x,\zeta,y)} a(z_s; z_\tau, \zeta)\, đ\zeta$$

coincides with the Schwartz kernel of a ψDO $A \in \Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma,\tau)$ such that

$$\sigma_{A,\tau}(x,\xi) \sim \sum \frac{1}{\alpha!}\,(s-\tau)^{|\alpha|}\;\; D^\alpha_\xi \nabla^\alpha_y a(y;x,\xi)\big|_{y=x}\,, \qquad \langle\xi\rangle_x \to \infty\,, \tag{4.7}$$

$$\sigma_{A,s}(x,\xi) \sim \sum \frac{1}{\alpha!}\;(\tau-s)^{|\alpha|}\, D^\alpha_\eta \nabla^\alpha_y a(x;y,\eta)\Big|_{(y,\eta)=(x,\xi)}\,, \qquad \langle\xi\rangle_x \to \infty\,.$$

By Proposition 4.4 the classes $\Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma,\tau)$ do not depend on the parameter τ, and further on we denote them by $\Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma)$. If $1-\rho \leqslant \delta$ then, in addition, the classes $\Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma)$ are independent of Γ and coincide with the standard classes of ψDOs $\Psi^m_{\rho,\delta}(\Omega^\varkappa)$ defined in local coordinates. In this case the choice of the connection Γ determines only the definition of the full symbols. For $\delta < 1-\rho$, generally speaking, the classes of ψDOs corresponding to different connections Γ are different.

The symbol σ_A of a ψDO $A \in \Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma)$ is determined uniquely modulo $S^{-\infty}$. Indeed, we can easily find σ_A calculating the asymptotic behaviour of the Fourier transform $\mathcal{F}_{y\to\eta}\big(p^{-1}_{\varkappa,0}\,\mathcal{A}(x,y)\big)$ in the n.c.s. y with origin x associated with some coordinates $\{x^k\}$. Therefore Proposition 4.4 implies the following

Corollary 4.6. *For any $\varkappa \in \mathbf{R}^1$, $\tau \in [0,1]$ the map $A \to \sigma_{A,\tau}$ is an isomorphism of the factor-classes $\Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma)\,/\,\Psi^{-\infty}(\Omega^\varkappa)$ and $\mathrm{S}^m_{\rho,\delta}(\Gamma)\,/\,S^{-\infty}$.*

5. Symbols of differential operators

If A is a differential operator in the space of $\varkappa$-densities then its τ-symbols are polynomials in ζ. In an arbitrary coordinate system $\{x^k\}$ and the associated n.c.s. $y=\{y^k\}$ with origin x we have

$$(Au)(x) \;=\; \sigma_A(y,D_y)\left(\Upsilon_x^{1-\varkappa}(y)\,u(y)\right)\big|_{y=x}\,, \qquad \forall u\in C_0^\infty(M;\Omega^\varkappa)\,, \tag{5.1}$$

$$\sigma_A(x,\xi) \;=\; A(y,D_y)\left(e^{i(y-x)\cdot\xi}\,\Upsilon_{x,\varkappa}^{\varkappa-1}(y)\right)\Big|_{y=x}\,, \tag{5.2}$$

where $\Upsilon_{x,\varkappa}^{\varkappa-1}$ is the $\varkappa$-density which coincides in the coordinates y with $\Upsilon_x^{\varkappa-1}$. This observation enables us to calculate the symbols of differential operators.

Example 5.1. If A is the operator of multiplication by a smooth function q then all its τ-symbols are equal to $q(x)$.

Example 5.2. Let M be a pseudo-Riemannian manifold, Γ be the Levi–Civita connection, and Δ be the Laplace operator in the space of $\varkappa$-densities. Taking into account Remark 2.3, we obtain from (5.2) that for all $\varkappa\in\mathbf{R}^1$

$$\sigma_\Delta(x,\xi) \;=\; -\,|\xi|_x^2 \;+\; \frac{1}{3}\,S(x)$$

where $S(x)$ is the scalar curvature. Since the function $|\xi|_x^2$ is constant along the horizontal curves (Example 3.3), by Proposition 4.4 $\sigma_{\Delta,\tau}=\sigma_\Delta$, $\forall\tau$.

Example 5.3. Let $A^{(\varkappa)}=A^{(\varkappa)}(y,D_y)$ be the Lie differentiation along a vector field $\nu(y)=\{\nu^j(y)\}$ in the space of $\varkappa$-densities. In an arbitrary local coordinate system $y=\{y^k\}$

$$\begin{aligned} A^{(\varkappa)}u(y) \;:=\; &\sum_j \nu^j(y)\,\partial_{y^j}u(y) \;+\; \varkappa\Big(\sum_j \partial_{y^j}\nu^j(y)\Big)\,u(y)\\ =\; &\sum_j \nu^j(y)\,\partial_{y^j}u(y) \;+\; \varkappa\Big(\sum_j \mathbf{D}_j\nu^j(y) \;-\; \sum_{j,k}\Gamma^j_{jk}(y)\,\nu^k(y)\Big)\,u(y)\,, \end{aligned}$$

where $\mathbf{D}_j\nu^j(y) \;=\; \partial_{y_j}\nu^j(y)+\sum_k\Gamma^j_{jk}(y)\,\nu^k(y)$ is the j-th component of the covariant derivative of ν with respect to the coordinate vector field ∂_{y_j}. From (5.2) and (2.9) it follows that

$$\sigma_{A^{(\varkappa)}}(x,\xi) \;=\; i\sum_j \nu^j(x)\,\xi_j \;+\; \varkappa\sum_j \mathbf{D}_j\nu^j(x) \;+\; (\varkappa-1/2)\sum_{j,k}T^j_{kj}(x)\,\nu^k(x)\,,$$

and then by Proposition 4.4

$$\sigma_{\tau,A^{(\varkappa)}}(x,\xi) \;=\; i\sum_j \nu^j(x)\,\xi_j + \;(\varkappa-\tau)\sum_j \mathbf{D}_j\nu^j(x) \;+\; (\varkappa-1/2)\sum_{j,k}T^j_{kj}(x)\,\nu^k(x)\,.$$

Example 5.4. Let us assume that on M there exist n smooth linearly independent vector fields $\nu_1,\dots,\nu_n$. We denote by $A_l^{(\varkappa)} = A_l^{(\varkappa)}(y,D_y)$ the Lie differentiations along ν_l in the space of $\varkappa$-densities (see Example 5.3) and define

$$A^\alpha_{(\varkappa)}(y,D_y) \;=\; \sum_{i_1,\dots,i_q} A^{(\varkappa)}_{i_1}\dots A^{(\varkappa)}_{i_q}\,, \qquad q=|\alpha|\,,$$

where the sum is taken over all the ordered sets of indices $i_1,\dots,i_q$ corresponding to the multi-index $\alpha=\{\alpha_1,\dots\alpha_n\}$. In other words, $A^\alpha_{(\varkappa)}$ is the symmetrized composition of the operators $A^{(\varkappa)}_{i_k}$.

For $s\in\mathbf{R}^1$ let ${}_s\Gamma$ be the linear connection of M such that ${}_s\mathbf{D}_{\nu_k}\nu_j = s\,[\nu_k,\nu_j]$. Here and later on we mark the objects corresponding to ${}_s\Gamma$ with the left lower index s which is omitted when $s=0$.

In an arbitrary coordinate system the Christoffel symbols of ${}_s\Gamma$ are equal to

$${}_s\Gamma^i_{kj}(y) \;=\; (s-1)\sum_l \tilde\nu^l_j(y)\,\partial_{y^k}\nu^i_l(y) \;-\; s\sum_l \tilde\nu^l_k(y)\,\partial_{y^j}\nu^i_l(y)\,.$$

Here ν^i_l are the components of vector fields ν_l and $\tilde\nu^l_k$ are the elements of inverse matrix, i.e., $\sum_l \tilde\nu^l_k(y)\,\nu^i_l(y)\equiv\delta^i_k$ where δ^i_k are the Kronecker symbols. By (1.6) and (1.7) we have

$${}_sT^i_{kj}(y) \;=\; (2s-1)\sum_{i',j',k'} C^{i'}_{k'j'}(y)\,\nu^i_{i'}(y)\,\tilde\nu^{j'}_j(y)\,\tilde\nu^{k'}_k(y)\,, \tag{5.3}$$

$${}_sR^{\,i}{}_{kjl}(y) \;=\; s\,(1-s)\sum_{i',j',k',l'} S^{\,i'}{}_{k'j'l'}(y)\,\nu^i_{i'}(y)\,\tilde\nu^{j'}_j(y)\,\tilde\nu^{k'}_k(y)\,\tilde\nu^{l'}_l(y)\,, \tag{5.4}$$

where C^i_{jk} and $S^{\,i}{}_{jkl}$ are the functions on M defined by the equalities

$$[\nu_j,\nu_k] \;=\; \sum_i C^i_{jk}\,\nu_i\,, \qquad [\,\nu_j,[\nu_k,\nu_l]\,] \;=\; \sum_i S^{\,i}{}_{jkl}\,\nu_i\,.$$

For all s the connections ${}_s\Gamma$ generate the same geodesics and n.c.s. The geodesics coincide with the integral curves of the vector fields $\sum_l c^l\nu_l$, where c_l are arbitrary constants.

If $s=0$ then all the covariant differentials $\mathbf{D}^\alpha\nu_l$ of the vector fields ν_l are equal to zero. In this case the parallel displacement along a curve connecting points x and y is independent of the curve and is determined only by these points. In arbitrary coordinates $\{x^k\}$ and $\{y^j\}$ we have $(\Phi_{y,x})^k_j = \sum_l \tilde\nu^l_j(y)\,\nu^k_l(x)$.

Let $\{x^k\}$ be a coordinate system such that $\nu^k_l(x)=\delta^k_l$, and $y=\{y^k\}$ be the associated n.c.s. with origin x. Then by (2.4)

$$\sum_l (y^l-x^l)\,\nu^k_l(y) \;=\; (y^k-x^k)\,. \tag{5.5}$$

The equalities (5.2) and (5.5) imply that the ${}_s\Gamma$-symbol of $A^\alpha_{(\varkappa)}$ is equal to

$$ {}_s\sigma_{A^\alpha_{(\varkappa)}}(x,\xi) \;=\; \sum_{i_1,\dots,i_q} \tilde\partial_{i_1}\dots\tilde\partial_{i_q}\left(e^{i(y-x)\cdot\xi}\big({}_\varkappa\Upsilon_x(y)\big)^{-1}\right)\Big|_{y=x}\,,\qquad q=|\alpha|\,, \tag{5.6}$$

where $\tilde\partial_k=\partial_{y^k}+s\,(\varkappa-1)\sum_j C^j_{kj}(y)$ and the sum is taken over all the ordered sets of indices $i_1,\dots,i_q$ corresponding to α. Indeed, from the definition of Lie differentiation and (5.5) it follows that

$$\sum_l (y^l-x^l)\,A_l(y,D_y)\Big(f(y)\,{}_s\Upsilon^{\varkappa-1}_{x,\varkappa}(y)\Big)$$
$$=\;{}_s\Upsilon^{\varkappa-1}_{x,\varkappa}(y)\,{}_\varkappa\Upsilon_x(y)\,\sum_l (y^l-x^l)\,\tilde\partial_l\left(f(y)\big({}_\varkappa\Upsilon_x(y)\big)^{-1}\right)$$

for all $f\in C^\infty(M)$. By induction with respect to q we derive

$$\sum_{|\alpha|=q}\frac{1}{\alpha!}\,(y-x)^\alpha\,A^\alpha_{(\varkappa)}(y,D_y)\Big(\,e^{i(y-x)\cdot\xi}\,{}_s\Upsilon^{\varkappa-1}_{x,\varkappa}(y)\,\Big)\;=\;{}_s\Upsilon^{\varkappa-1}_{x,\varkappa}(y)\,{}_\varkappa\Upsilon_x(y)$$
$$\times\sum_{i_1,\dots,i_q}(y^{i_1}-x^{i_1})\dots(y^{i_q}-x^{i_q})\,\tilde\partial_{i_1}\dots\tilde\partial_{i_q}\left(e^{i(y-x)\cdot\xi}\big({}_\varkappa\Upsilon_x(y)\big)^{-1}\right)\,,\qquad\forall q\in\mathbf{N}\,.$$

Now (5.6) is proved by the same procedure as Corollary 2.5.

From (2.7) and (5.5) it follows that

$$r\,\partial_r^2\Big(r\,\tilde\nu^l_j(y)\Big)\;=\;-\,r\,\partial_r\Big(\sum_{i,k}(y^k-x^k)\,C^l_{ki}(y)\,\tilde\nu^i_j(y)\Big) \tag{5.7}$$

(here we take $s=0$). Let us assume that

$$C^i_{jk}\;\equiv\;0\,,\qquad\forall i\geqslant k\,. \tag{5.8}$$

Then the solution $\{\tilde\nu^i_j(y)\}$ of the equation (5.7) with the initial condition $\{\tilde\nu^i_j(x)\}=I$ is a triangular matrix whose diagonal elements are equal to 1. Therefore in our coordinates

$$T^i_{jk}\;\equiv\;0\,,\quad\forall i\geqslant j\,;\qquad R^{\;i}_{\;jkl}\;\equiv\;0\,,\quad\forall i\geqslant l\,.$$

Now (2.7) implies that for all s the matrices ${}_s\Phi_x(y)$ are of the same form as $\{\tilde\nu^i_j(y)\}$. Thus, ${}_s\Upsilon_x\equiv 1$, $\forall s\in\mathbf{R}^1$, and by (5.6)

$$ {}_s\sigma_{A^\alpha_{(\varkappa)}}(x,\xi)\;=\;i^{|\alpha|}\sigma^\alpha(x,\xi)\,,\qquad\forall s,\varkappa\in\mathbf{R}^1\,, \tag{5.9}$$

where

$$\sigma^\alpha\;=\;(\sigma_1)^{\alpha_1}\dots(\sigma_n)^{\alpha_n}\,,\qquad\sigma_l\;=\;\sigma_l(x,\xi)\;=\;\langle\nu_l(x),\xi\rangle\,. \tag{5.10}$$

Note that the functions σ^α are constant along the horizontal curves generated by the connection Γ. Therefore under the condition (5.8) all the τ-symbols of $A^\alpha_{(\varkappa)}$ corresponding to Γ coincide with $i^{|\alpha|}\sigma^\alpha$.

Remark 5.5. When the functions C^i_{jk} are constant, the vector fields ν_l form the basis of a Lie algebra, and then C^i_{jk} are said to be the structure constants. In this case the condition (5.8) implies that the corresponding Lie algebra is nilpotent. Conversely, any nilpotent n-dimensional Lie algebra admits a basis $\{\nu_l\}$ satisfying (5.8) (see, for example, [V]).

6. Transformation formulae for symbols

1. All the τ-symbols of a classical ψDO $A \in \Psi^m(\Omega^\varkappa)$ admit asymptotic expansions of the form (3.5). In the classical theory of ψDOs it is proved that the leading homogeneous term in these expansions does not depend on τ, and that it is a correctly defined function on the cotangent bundle. This function is said to be the *principal symbol* of A. The second term in the expansion of the Weyl symbol σ_A^W is called the *subprincipal symbol*. The subprincipal symbol is a correctly defined function on T^*M if $\varkappa = 1/2$, i.e., if A is a ψDO in the space of half-densities.

Analogous results are also obtained under our approach. We have defined all the symbols as functions on the cotangent bundle. In the next subsection we prove a transformation formula which implies that the principal symbol does not depend on Γ, and the subprincipal symbol is independent of Γ when $\varkappa = 1/2$.

2. Let $\tilde\Gamma$ be another linear connection of M and $\mathcal{D}^{\,l}_{kj} = \Gamma^{\,l}_{kj} - \tilde\Gamma^{\,l}_{kj}$ be the deformation tensor. Further on all the objects corresponding to $\tilde\Gamma$ are marked by $\tilde{}$.

Let $y = \{y^k\}$ and $\tilde y = \{\tilde y^k\}$ be the n.c.s. with origin x associated with some coordinates $\{x^k\}$, and $J(\tilde y) = \{\,\partial y^j/\partial \tilde y^k\,\}$ be the Jacobi matrix. We introduce the functions

$$
\begin{aligned}
\Theta(x,\tilde y,\xi) &= e^{i(y(\tilde y)-\tilde y)\cdot\xi}\ \Upsilon_x^{\varkappa-1}(y(\tilde y))\ \tilde\Upsilon_x^{1-\varkappa}(\tilde y)\ |\det J(\tilde y)|^{\varkappa}\,,\\
P_\alpha^{(\varkappa)}(x,\xi) &= \partial_{\tilde y}^\alpha \Theta(x,\tilde y,\xi)\big|_{\tilde y = x}\,.
\end{aligned}
$$

Obviously, $P_\alpha^{(\varkappa)}(x,\xi)$ are polynomials in ξ defined on T^*M. Since $y(\tilde y)-\tilde y$ has a the second order zero at the point $\tilde y = x$, the degrees $d_\alpha^{(\varkappa)}$ of the polynomials $P_\alpha^{(\varkappa)}$ are estimated from above by $|\alpha|/2$.

Proposition 6.1. *Let $A \in \Psi^m_{\rho,\delta}(\Omega^\varkappa,\tilde\Gamma)$ with $\rho > 1/2$. Then $A \in \Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma)$ and*

$$
\sigma_A(x,\xi) \ \sim\ \sum_\alpha \frac{1}{\alpha!}\, P_\alpha^{(\varkappa)}(x,\xi)\, D_\xi^\alpha \tilde\sigma_A(x,\xi)\,. \tag{6.1}
$$

In particular,

$$
\begin{aligned}
\sigma_A(x,\xi) \ =\ \tilde\sigma_A(x,\xi) \ &+\ (2i)^{-1}\sum_{j,k,l}\partial_{\xi_k}\Big(\mathcal{D}^{\,j}_{kl}(x)\xi_j\,\partial_{\xi_l}\tilde\sigma_A(x,\xi)\Big)\\
&+\ (2i)^{-1}\,(2\varkappa-1)\sum_{j,l}\mathcal{D}^{\,l}_{jl}(x)\,\partial_{\xi_j}\tilde\sigma_A(x,\xi) \qquad (\mathrm{mod}\ \mathrm{S}^{m+2-4\rho}_{\rho,\delta})\,,
\end{aligned} \tag{6.2}
$$

$$
\begin{aligned}
\sigma_A^W(x,\xi) \ =\ \tilde\sigma_A^W(x,\xi)&\\
&+\ (2i)^{-1}\,(2\varkappa-1)\sum_{j,l}\mathcal{D}^{\,l}_{jl}(x)\,\partial_{\xi_j}\tilde\sigma_A^W(x,\xi) \qquad (\mathrm{mod}\ \mathrm{S}^{m-2r}_{\rho,\delta})\,,
\end{aligned} \tag{6.3}
$$

where $r = \min\{\rho-\delta, 2\rho-1\}$.

Proof. First of all, let us note that (6.1) is an asymptotic series because $\rho > 1/2$ and $d_\alpha^{(\varkappa)} \leqslant |\alpha|/2$. The first two terms of this series are written in (6.2); they are easily obtained from (2.9) and the equality

$$\left.\frac{\partial^2 y^j}{\partial \tilde{y}^k\,\partial \tilde{y}^l}\right|_{\tilde{y}=x} \;=\; \mathcal{D}^j_{kl}(x) \;+\; \frac{1}{2}\,\tilde{T}^j_{kl}(x) \;-\; \frac{1}{2}\,T^j_{kl}(x)$$

which follows from (1.1). Proposition 4.4 and (6.2) immediately imply (6.3). Thus, it remains to prove (6.1).

Let $\mathcal{A}$ be the Schwartz kernel of the operator A, and for sufficiently close x and y let $\tilde{\Psi}(x,y)$ be the non-degenerate matrix defined by the formula

$$x \;-\; \tilde{y}(y) \;=\; \tilde{\Psi}^T(x,y)\,(x-y)\,.$$

For close x and y we have in the coordinates $\tilde{y}$

$$\mathcal{A}(x,\tilde{y}) \;=\; \tilde{\Upsilon}_x^{1-\varkappa}(\tilde{y}) \int e^{i(x-\tilde{y})\cdot\xi}\,\tilde{\sigma}_A(x,\xi)\,đ\xi\,.$$

Therefore $\mathcal{A}(x,y)$ coincides in the coordinates y with

$$\tilde{\Upsilon}_x^{1-\varkappa}(\tilde{y}(y))\;|\det J^{-1}(\tilde{y}(y))|^{1-\varkappa} \int e^{i(x-y)\cdot\tilde{\Psi}(x,y)\xi}\,\tilde{\sigma}_A(x,\xi)\,đ\xi \;=\; \tilde{\Upsilon}_x^{1-\varkappa}(\tilde{y}(y))$$
$$\times\,|\det J(\tilde{y}(y))|^{\varkappa-1}\;|\det\tilde{\Psi}(x,y)|^{-1}\int e^{i(x-y)\cdot\xi}\,\tilde{\sigma}_A\big(x,\tilde{\Psi}^{-1}(x,y)\xi\big)\,đ\xi\,, \qquad (6.4)$$

i.e., with an oscillatory integral of the form (4.4) with the phase function corresponding to the connection Γ and the amplitude

$$\tilde{\Upsilon}_x^{1-\varkappa}(\tilde{y}(y))\;|\det J(\tilde{y}(y))|^{\varkappa-1}\;\Upsilon_x^{\varkappa-1}(y)\;|\det\tilde{\Psi}(x,y)|^{-1}\;\tilde{\sigma}_A\big(x,\tilde{\Psi}^{-1}(x,y)\,\xi\big)$$

This amplitude belongs to $\mathrm{S}^m_{\rho,\delta'}(\Gamma)$ where $\delta' = \max\{\delta, 1-\rho\}$. Since $\delta' < \rho$, from Proposition 4.5 it follows that $A \in \Psi^m_{\rho,\delta'}(\Omega^\varkappa,\Gamma)$, and by (4.7)

$$\sigma_A(x,\xi) \;\sim\; \sum_\alpha \tilde{P}_\alpha^{(\varkappa)}(x,\xi)\,D_\xi^\alpha\tilde{\sigma}_A(x,\xi)\,, \qquad (6.5)$$

where $\tilde{P}_\alpha^{(\varkappa)}$ are some polynomials in ξ independent of the symbol $\tilde{\sigma}_A$. Clearly, (6.5) implies $\sigma_A \in \mathrm{S}^m_{\rho,\delta}(\Gamma)$, and so $A \in \Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma)$.

In order to prove the explicit formula (6.1) let us assume that A is a differential operator. By (5.2) we have

$$\sigma_A(x,\xi) \;=\; A(y,D_y)\left(e^{i(y-x)\cdot\xi}\,\Upsilon_{x,\varkappa}^{\varkappa-1}(y)\right),$$

and then by (5.1)

$$\sigma_A(x,\xi) \;=\; \tilde{\sigma}_A(\tilde{y},D_{\tilde{y}})\left(e^{i(\tilde{y}-x)\cdot\xi}\,\Theta(x,\tilde{y},\xi)\right)\Big|_{\tilde{y}=x} \;=\; \frac{1}{\alpha!}\sum_\alpha D_\xi^\alpha\tilde{\sigma}_A(x,\xi)\;\partial_{\tilde{y}}^\alpha\,\Theta\big|_{\tilde{y}=x}\,.$$

Since $\tilde{P}_\alpha$ are independent of A, from here and (6.5) it follows that $\tilde{P}_\alpha^{(\varkappa)} = P_\alpha^{(\varkappa)}$. $\square$

3. The diffeomorphism $G : M \to M$ is said to be an affine transformation if the induced transformation of the cotangent bundle

$$T^*M \ni (x,\xi) \;\to\; \big(G(x),(dG^T)^{-1}(x)\,\xi\big) \in T^*M \tag{6.6}$$

transfers any horizontal curve into a horizontal curve. In this case G completely preserves all the objects generated by the linear connection such as the geodesics, the horizontal differentials, etc. (see [KN]). In particular,

$$\begin{aligned} z_\tau\big(G(x),G(y)\big) &= G\big(z_\tau(x,y)\big)\,, && (6.7)\\ \varphi_\tau\big(G(x),(dG^T)^{-1}\zeta,G(y)\big) &= \varphi_\tau(x,\zeta,y)\,, && (6.8)\\ \Upsilon_{G(y),G(x)} &= |\det dG(y)|^{-1}\;\Upsilon_{y,x}\;|\det dG(x)|\,. && (6.9)\end{aligned}$$

Proposition 6.2. *Let $G : M \to M$ be an affine transformation and*

$$\mathcal{G}_k \;:\; u(x) \;\to\; |\det dG(x)|^{\varkappa}\,u(G(x))$$

be the corresponding operator acting in the space of $\varkappa$-densities on M. Then for any $A \in \Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma)$ the operator $\mathcal{G}_\varkappa\,A\,\mathcal{G}_\varkappa^{-1}$ also belongs to $\Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma)$, and its τ-symbol is equal to $\sigma_{A,\tau}\big(G(x),(dG^T)^{-1}(x)\,\xi\big)$.

Proof. The Schwartz kernel of the operator $\mathcal{G}_\varkappa\,A\,\mathcal{G}_\varkappa^{-1}$ is smooth outside the diagonal and coincides in some neighbourhood of the diagonal with

$$\begin{aligned}|\det dG(x)|^{\varkappa}\;|\det dG(y)|^{1-\varkappa}\;\mathcal{A}\big(G(x),G(y)\big) &= |\det dG(x)|^{\varkappa}\;|\det dG(y)|^{1-\varkappa}\\ &\times p_{\varkappa,\tau}\big(G(x),G(y)\big)\int e^{i\varphi_\tau\left(G(x),\zeta',G(y)\right)}\,\sigma_{A,\tau}(z'_\tau,\zeta')\,đ\zeta'\,,\end{aligned}$$

where $z'_\tau(x,y) = z_\tau\big(G(x),G(y)\big)$ and $\zeta' \in T^*_{z'_\tau}M$.

Now changing variables $\zeta' = (dG^T)^{-1}(z'_\tau)\,\zeta$ and taking into account (6.7)–(6.9), we obtain (4.4) with $a(z_\tau,\zeta) = \sigma_{A,\tau}\big(G(z_\tau),(dG^T)^{-1}(z_\tau)\,\zeta\big)$. $\square$

Corollary 6.3. *Let $G : M \to M$ be an affine transformation of M and A be a differential operator on M acting in the space of $\varkappa$-densities. Then $A = \mathcal{G}_\varkappa\,A\,\mathcal{G}_\varkappa^{-1}$ if and only if the τ-symbol $\sigma_{A,\tau}$ is invariant with respect to the induced transformation (6.6) for some (and then for any) $\tau \in [0,1]$.*

7. Adjoint operators

1. Let A be a ψDO in the space of $\varkappa$-densities. We denote by A^* the formally adjoint operator (with respect to the form $(u,v) = \int_M u\,\bar v\,dx$) acting in the space of $(1-\varkappa)$-densities on M. When $\varkappa = 1/2$ and $A = A^*$, we say that the operator A is formally self-adjoint.

Theorem 7.1. *If* $A \in \Psi^m_{\rho,\delta}(\Omega^\varkappa, \Gamma)$ *then* $A^* \in \Psi^m_{\rho,\delta}(\Omega^{1-\varkappa}, \Gamma)$ *and*

$$\sigma_{A^*,\tau}(x,\xi) \ \sim \ \sum_\alpha \frac{1}{\alpha!}\,(1-2\tau)^{|\alpha|}\, D_\xi^\alpha \nabla_x^\alpha\, \overline{\sigma_{A,\tau}(x,\xi)}\,, \qquad \langle\xi\rangle_x \to \infty\,.$$

In particular, $\sigma^W_{A^*}(x,\xi) = \overline{\sigma^W_A(x,\xi)}$.

Proof. Let $\mathcal{A}(x,y)$ and $\mathcal{A}^*(x,y)$ be the Schwartz kernels of the operators A and A^*. Then $\mathcal{A}^*(x,y) = \overline{\mathcal{A}(y,x)}$. Since $z_\tau(x,y) = z_{1-\tau}(y,x)$ and $\Upsilon_{y,x} = \Upsilon^{-1}_{x,y}$, we obtain from (4.4) that for close x and y

$$\mathcal{A}^*(x,y) \ = \ p_{1-\varkappa,1-\tau}(x,y) \int e^{i\varphi_{1-\tau}(x,\zeta,y)}\, \overline{\sigma_{A,\tau}(z_{1-\tau},\zeta)}\, đ\zeta\,,$$

where $\zeta \in T^*_{z_{1-\tau}}M$. Therefore $A^* \in \Psi^m_{\rho,\delta}(\Omega^{1-\varkappa},\Gamma)$ and $\sigma_{A^*,1-\tau} = \overline{\sigma_{A,\tau}}$. Now the theorem follows from Proposition 4.4. $\square$

Theorem 7.1 immediately implies

Corollary 7.2. *A differential operator is formally self-adjoint if and only if its Weyl symbol is real.*

8. Composition of ψDOs

1. Let

$$\Upsilon_\varkappa(x,y,z) \ = \ \Upsilon^{1-\varkappa}_{y,z}\,\Upsilon^{2-\varkappa}_{z,x}\,\Upsilon^{1-\varkappa}_{x,y}\,,$$

$$\psi(x,\xi;y,z) \ = \ \langle\dot\gamma_{y,x},\xi\rangle \ - \ \langle\dot\gamma_{z,x},\xi\rangle \ - \ \langle\dot\gamma_{y,z},\Phi_{z,x}\xi\rangle\,.$$

It is easy to see that

$$\Upsilon_\varkappa(x,y,x) \ \equiv \ 1\,, \qquad \Upsilon_\varkappa(x,x,z) \ = \ \Upsilon_\varkappa(x,z,z) \ = \ \Upsilon_{z,x}\,, \tag{8.1}$$

$$\psi(x,\xi;y,x) \ = \ \psi(x,\xi;x,z) \ = \ \psi(x,\xi;z,z) \ = \ 0\,. \tag{8.2}$$

Let $y = \{y^k\}$ and $z = \{z^k\}$ be the normal coordinate systems with origin x associated with some coordinates $\{x^k\}$. We define

$$P^{(\varkappa)}_{\beta,\gamma}(x,\xi) \ = \ \left((\partial_y+\partial_z)^\beta \partial_y^\gamma \sum_{|\beta'|\leqslant|\beta|} \frac{1}{\beta'!} D_\xi^{\beta'} \partial_y^{\beta'} \left(e^{i\psi}\Upsilon_\varkappa\right)\right)\Big|_{y=z=x}\,. \tag{8.3}$$

The functions $P^{(\varkappa)}_{\beta,\gamma} \in C^\infty(T^*M)$ are polynomials in ξ; we denote their degrees by $d^{(\varkappa)}_{\beta,\gamma}$. Obviously, $P^{(\varkappa)}_{0,0} \equiv 1$. From Lemma 1.1, (1.5) and (2.7) it follows that for $|\beta|+|\gamma| \geqslant 1$ the coefficients of a polynomial $P^{(\varkappa)}_{\beta,\gamma}$ are components of some tensors, which are polynomials in the curvature and torsion tensors and their symmetric covariant differentials.

Lemma 8.1. *For an arbitrary connection* Γ

$$d^{(\varkappa)}_{\beta,\gamma} \;\leqslant\; \min\{\,|\beta|,|\gamma|\,\}\,. \tag{8.4}$$

If Γ *is symmetric then*

$$d^{(\varkappa)}_{\beta,\gamma} \;\leqslant\; \min\{\,|\beta|,|\gamma|,(|\beta|+|\gamma|)/3\,\}\,. \tag{8.5}$$

If Γ *is flat then* $P^{(\varkappa)}_{\beta,\gamma}\equiv 0$ *as* $|\alpha|+|\beta|\geqslant 1$ *.*

Proof. By (8.2) $\partial_z^\beta\partial_y^\gamma\psi\big|_{y=z=x}=0$ if $|\beta|=0$ or $|\gamma|=0$. If $|\beta|\geqslant 1$ and $|\gamma|\geqslant 1$ then

$$\partial_z^\beta\partial_y^\gamma\psi(x,\xi;y,z)\big|_{z=x} \;=\; -\,\partial_y^\gamma\nabla_x^\beta\,\langle\dot\gamma_{y,x}\,,\,\xi\rangle \;=\; -\,\partial_y^\gamma\,\left\langle\mathbf{D}_x^\beta\dot\gamma_{y,x}\,,\,\xi\right\rangle\,, \tag{8.6}$$

where ∇_x^β is the symmetric horizontal differential and $\mathbf{D}^\beta$ is the symmetric covariant differential. Taking into account (1.5), we obtain from (8.6) $\partial_{z^j}\partial_{y^k}\psi\big|_{y=z=x}=\sum_p T^p_{kj}(x)\,\xi_p\,/2$. Thus,

$$\begin{aligned}\psi(x,\xi;y,z) \;=\; &\frac12\sum_{j,k,p}T^p_{kj}(x)\,(y^k-x^k)\,(z^j-x^j)\,\xi_p\\ &+\;\frac12\sum_{j,k,p}\psi^p_{kj}(x,y,z)\,(y^k-x^k)\,(z^j-x^j)\,\xi_p\,,\end{aligned} \tag{8.7}$$

where $\psi^p_{kj}=0$ at the point $y=z=x$.

From (8.7) it is clear that in the right hand side of (8.3) ξ^α can appear (with a factor which does not vanish at $y=z=x$) only when we take at least $|\alpha|$ derivatives ∂_y and at least $|\alpha|$ derivatives ∂_z for each fixed β'. This implies (8.4). If $T^p_{kj}\equiv 0$ then, in addition, the total number of these derivatives must be greater than or equal to $3|\alpha|$, and therefore (8.5) holds. The third statement of the lemma follows from the fact that for a flat connection $\psi\equiv 0$ and $\Upsilon_\varkappa\equiv 1$ in any n.c.s. $\square$

Proposition 8.2. *For an arbitrary connection* Γ

$$P^{(\varkappa)}_{0,\gamma}\equiv 0\,,\quad \forall\gamma\neq 0\,,\qquad\qquad P^{(\varkappa)}_{\beta,0}\equiv 0\,,\quad\forall\beta\neq 0\,, \tag{8.8}$$

and

$$P^{(\varkappa)}_{j,k}(x,\xi) \;=\; (2i)^{-1}\sum_p T^p_{jk}(x)\,\xi_p \tag{8.9}$$

modulo a function of x *. If* Γ *is symmetric then*

$$P^{(\varkappa)}_{j,k} \;\equiv\; \frac16\,(R_{jk}+R_{kj}) \;-\; \frac{\varkappa}{2}\,(R_{jk}-R_{kj})\,, \tag{8.10}$$

and

$$P^{(\varkappa)}_{j,kl}(x,\xi) \;=\; -\,\frac{i}{3}\,\sum_p \bigl(R^p{}_{kjl}(x)+R^p{}_{ljk}(x)\bigr)\,\xi_p\,, \tag{8.11}$$

$$P^{(\varkappa)}_{jk,l}(x,\xi) \;=\; \frac{i}{6}\,\sum_p \bigl(R^p{}_{jlk}(x)+R^p{}_{klj}(x)\bigr)\,\xi_p \tag{8.12}$$

modulo some functions of x .

Proof. The first identity (8.8) immediately follows from (8.1) and (8.2).

In order to prove the second identity (8.8) let us note that

$$P^{(\varkappa)}_{\beta,0}(x,\xi) \;=\; \partial_z^\beta \Bigl(\sum_{|\beta'|\leqslant|\beta|} \frac{1}{\beta'!}\, D_\xi^{\beta'}\partial_y^{\beta'}\,(e^{i\psi}\Upsilon_\varkappa)\Big|_{y=z}\,\Bigr)\Big|_{z=x}$$

and, by (4.1) and (4.2), $\nabla_y\psi(x,\xi;z,z) = (I-\Phi_x(z))\,\xi$. Taking into account (8.1) and (8.2) we obtain

$$\sum_{|\beta'|\leqslant|\beta|} \frac{1}{\beta'!}\, D_\xi^{\beta'}\partial_y^{\beta'}\,(e^{i\psi}\Upsilon_\varkappa)\Big|_{y=z} \;=\; \Upsilon_x(z) \sum_{|\beta'|\leqslant|\beta|} \frac{1}{\beta'!}\, D_\xi^{\beta'}\partial_y^{\beta'}\, e^{i\tilde\psi}\Big|_{y=z}\,,$$

where $\tilde\psi = \tilde\psi(x,\xi;y,z) = (y-z)\cdot\nabla_y\psi(x,\xi;z,z) = (y-z)\cdot(I-\Phi_x(z))\xi$. It is clear that

$$\begin{aligned}&\sum_{|\beta'|\leqslant|\beta|} \frac{1}{\beta'!}\, D_\xi^{\beta'}\partial_y^{\beta'}\, e^{i\tilde\psi}\Big|_{y=z}\\ &= \sum_{0\leqslant j_1+j_2+\dots+j_k\leqslant|\beta|} c_{j_1,j_2,\dots,j_k}\,\mathrm{Tr}\,(I-\Phi_x(z))^{j_1}\,\mathrm{Tr}\,(I-\Phi_x(z))^{j_2}\dots\mathrm{Tr}\,(I-\Phi_x(z))^{j_k}\,,\end{aligned}$$

where $c_{j_1,j_2,\dots,j_k}$ are some constants depending only on the dimension n and $|\beta|$. Therefore

$$\sum_{|\beta'|\leqslant|\beta|} \frac{1}{\beta'!}\, D_\xi^{\beta'}\partial_y^{\beta'}\,(e^{i\psi}\Upsilon_\varkappa)\Big|_{y=z} \;=\; \Upsilon_x(z) \sum_{|\beta'|\leqslant|\beta|} \frac{1}{\beta'!}\, D_\xi^{\beta'}\partial_y^{\beta'}\, e^{i\tilde\psi}\Big|_{y=z}\,,$$

is a polynomial in the eigenvalues $\lambda_1,\dots,\lambda_n$ of the matrix $I-\Phi_x(z)$ whose coefficients depend only on n and $|\beta|$. In order to find these coefficients we may assume that $\Phi_x(z)$ is a diagonal matrix, and then we obtain

$$\Upsilon_x(z) \sum_{|\beta'|\leqslant|\beta|} \frac{1}{\beta'!}\, D_\xi^{\beta'}\partial_y^{\beta'}\, e^{i\tilde\psi}\Big|_{y=z} \;=\; (1-\lambda_1)\dots(1-\lambda_n) \sum_{0\leqslant k_1+\dots+k_n\leqslant|\beta|} \lambda_1^{k_1}\dots\lambda_n^{k_n}$$

(of course, the same equality holds for an arbitrary matrix $\Phi_x(z)$). By induction in n one can easily prove that the right hand side coincides with 1 modulo a polynomial in

λ_j which contains only the terms of degree higher than $|\beta|$. Since $\Phi_x(x) = I$, we have $\lambda_j = O(|x-z|)$ for all j and, consequently,

$$\sum_{|\beta'|\leqslant|\beta|} \frac{1}{\beta'!}\, D_\xi^{\beta'} \partial_y^{\beta'} (e^{i\psi}\Upsilon_\varkappa)\Big|_{y=z} \;=\; 1 + O(|x-z|^{|\beta|+1})\,.$$

This implies the second equality (8.8).

Since ψ has a second order zero at $y = z = x$, the degrees of the polynomials (8.3) corresponding to $|\beta'| = 1$ are equal to zero. For $|\beta'| = 0$ we derive from (8.7)

$$(\partial_{y^j} + \partial_{z^j})\partial_{y^k}\,(e^{i\psi}\,\Upsilon_\varkappa)\big|_{y=z=x} \;=\; (2i)^{-1} \sum_p T^p_{jk}(x)\,\xi_p$$

modulo a function of x. This proves (8.9).

Let Γ be symmetric. Then ψ has a third order zero at $y = z = x$ and the first derivatives of Υ and Φ are equal to zero at the origin of a n.c.s. Therefore

$$\begin{aligned} P^{(\varkappa)}_{j,k} \;&\equiv\; (\partial_{y^j} + \partial_{z^j})\partial_{y^k}\Upsilon_\varkappa\big|_{y=z=x} \;+\; \sum_p (\partial_{y^j} + \partial_{z^j})\partial_{y^k}\; D_{\xi_p}\partial_{y^p}(e^{i\psi})\big|_{y=z=x} \\ &\equiv \partial_{y^j}\partial_{y^k}\big(\Upsilon^{\varkappa-1}_{y,x}\big)\big|_{y=x} + \partial_{z^j}\Big(\partial_{y^k}\Upsilon^{1-\varkappa}_{y,z}\big|_{y=z}\Big)\Big|_{z=x} + \sum_p \partial_{z^j}\Big(\partial_{y^k}\partial_{y^p}\psi_{\xi_p}\big|_{y=z}\Big)\Big|_{z=x}\,, \end{aligned}$$

and

$$P^{(\varkappa)}_{j,kl}(x,\xi) \;=\; (\partial_{y^j} + \partial_{z^j})\partial_{y^k}\partial_{y^l}(e^{i\psi})\big|_{y=z=x} = \; i\,\partial_{z^j}\Big(\partial_{y^k}\partial_{y^l}\psi\big|_{y=z}\Big)\Big|_{z=x}\,,$$

$$P^{(\varkappa)}_{jk,l}(x,\xi) \;=\; (\partial_{y^j} + \partial_{z^j})(\partial_{y^k} + \partial_{z^k})\partial_{y^l}(e^{i\psi})\big|_{y=z=x} = i\,\partial_{z^j}\partial_{z^k}\Big(\partial_{y^l}\psi\big|_{y=z}\Big)\Big|_{z=x}$$

modulo a a function of x. Now (4.1), (4.2) and (1.8) imply

$$\partial_{z^j}\Big(\partial_{y^k}\partial_{y^l}\psi\big|_{y=z}\Big)\Big|_{z=x} \;=\; -\frac{1}{3}\sum_p \big(R^p{}_{kjl}(x) + R^p{}_{ljk}(x)\big)\,\xi_p\,,$$

and (4.1), (4.2) and (2.10) imply

$$\partial_{z^j}\partial_{z^k}\Big(\partial_{y^l}\psi\big|_{y=z}\Big)\Big|_{z=x} \;=\; \frac{1}{6}\sum_p \big(R^p{}_{jlk}(x) + R^p{}_{klj}(x)\big)\,\xi_p\,.$$

Thus, we obtain (8.11) and (8.12).

We have

$$\begin{aligned} &\sum_p \partial_{z^j}\Big(\partial_{y^k}\partial_{y^p}\psi_{\xi_p}\big|_{y=z}\Big)\Big|_{z=x} \\ &\qquad\qquad = \;-\frac{1}{3}\sum_p \big(R^p{}_{kjp}(x) + R^p{}_{pjk}(x)\big) \;=\; \frac{1}{3}\big(2\,R_{kj}(x) - R_{jk}(x)\big)\,, \end{aligned}$$

and, by (2.11),

$$\partial_{y^j}\partial_{y^k}\big(\Upsilon_{y,x}^{\varkappa-1}\big)\big|_{y=x} \;=\; (\varkappa-1)\;\partial_{y^j}\partial_{y^k}\Upsilon_x(y)\big|_{y=x} \;=\; \frac{1-\varkappa}{6}\,\big(R_{kj}(x)+R_{jk}(x)\big)\,.$$

Therefore in order to obtain (8.10) it remains to prove that

$$\partial_{z^j}\Big(\partial_{y^k}\Upsilon_{y,z}^{1-\varkappa}\big|_{y=z}\Big)\Big|_{z=x} \;=\; \frac{\varkappa-1}{3}\,\big(2\,R_{kj}-R_{jk}(x)\big) \tag{8.13}$$

Let $\{\tilde y^k\}$ be the n.c.s. with origin z associated with coordinates $\{y^k\}$. Then

$$\begin{aligned}\partial_{y^k}\Upsilon_{y,z}^{1-\varkappa}\big|_{y=z} \;&=\; \partial_{\tilde y^k}\Big(|\det\{\partial y/\partial\tilde y\}|^{\varkappa-1}\Upsilon_z^{1-\varkappa}(\tilde y)\Big)\Big|_{\tilde y=z}\\ &=\; \partial_{\tilde y^k}\Big(|\det\{\partial y/\partial\tilde y\}|^{\varkappa-1}\Big)\Big|_{\tilde y=z} \;=\; (\varkappa-1)\sum_p \frac{\partial^2 y^p}{\partial\tilde y^k\,\partial\tilde y^p}\bigg|_{\tilde y=z}\,.\end{aligned}$$

Now from (1.1) it follows that $\partial_{y^k}\Upsilon_{y,z}^{1-\varkappa}\big|_{y=z} = (1-\varkappa)\sum_p\Gamma_{kp}^p(z)$. This equality and (1.8) yield (8.13). $\square$

2. In this subsection we prove the following two theorems.

Theorem 8.3. *Let* $A\in\Psi_{\rho,\delta}^{m_1}(\Omega^\varkappa,\Gamma)$, $B\in\Psi_{\rho,\delta}^{m_2}(\Omega^\varkappa,\Gamma)$, *and let at least one of these* ψ*DOs be properly supported. Assume that at least one of the following conditions is fulfilled:*

(1) $\rho>1/2$;
(2) *the connection* Γ *is symmetric and* $\rho>1/3$;
(3) *the connection* Γ *is flat.*

Then $AB\in\Psi_{\rho,\delta}^{m_1+m_2}(\Omega^\varkappa,\Gamma)$ *and*

$$\sigma_{AB}(x,\xi)\;\sim\;\sum_{\alpha,\beta,\gamma}\frac{1}{\alpha!}\,\frac{1}{\beta!}\,\frac{1}{\gamma!}\;P_{\beta,\gamma}^{(\varkappa)}(x,\xi)\;D_\xi^{\alpha+\beta}\sigma_A(x,\xi)\;D_\xi^\gamma\nabla_x^\alpha\sigma_B(x,\xi) \tag{8.14}$$

as $\langle\xi\rangle_x\to\infty$.

Theorem 8.4. *Let* $A\in\Psi_{\rho,\delta}^{m_1}(\Omega^\varkappa,\Gamma)$, $B\in\Psi_{\rho,\delta}^{m_2}(\Omega^\varkappa,\Gamma)$, *and let at least one of these* ψ*DOs be properly supported. Assume that* $A\in\Psi_{1,0}^{m_1}(\Omega^\varkappa)$ *or* $B\in\Psi_{1,0}^{m_2}(\Omega^\varkappa)$. *Then* $AB\in\Psi_{\rho,\delta}^{m_1+m_2}(\Omega^\varkappa,\Gamma)$ *and* σ_{AB} *admits the asymptotic expansion (8.14).*

Remark 8.5. Lemma 8.1 implies that under the conditions of Theorem 8.3 or Theorem 8.4 the terms in the right hand side of (8.14) form an asymptotic series.

Remark 8.6. Substituting in (8.14) $\sigma_A\equiv 1$ or $\sigma_B\equiv 1$ we obtain (8.8).

Proof of Theorem 8.3. Since at least one of the operators A and B is properly supported, the composition AB is a well-defined operator, acting from $C_0^\infty(M;\Omega^\varkappa)$ into $C^\infty(M;\Omega^\varkappa)$, whose Schwartz kernel is smooth outside the diagonal. Let $\chi\in C_0^\infty(M\times M)$ be a cut-off function which is equal to 0 outside some small neighbourhood of the diagonal and to 1 in a smaller neighbourhood. Then the kernel of the operator AB is represented modulo C^∞ by the oscillatory integral

$$\int e^{i\varphi_0(x,\xi;z)+i\varphi_0(z,\zeta;y)}\,\chi(x,z)\,\chi(z,y)\,\sigma_A(x,\xi)\,\sigma_B(z,\zeta)\,\Upsilon_{y,z}^{1-\varkappa}\,\Upsilon_{z,x}^{1-\varkappa}\,đ\zeta\,dz\,đ\xi\,. \tag{8.15}$$

Changing variables $\zeta=\Phi_{z,x}\tilde\xi$, $\xi=(\tilde\xi+\xi')$, we obtain from (8.15)

$$\begin{aligned}\Upsilon_{y,x}^{1-\varkappa}\int e^{i\varphi_0(x,\xi';z)+i\varphi_0(x,\tilde\xi;y)}\,\sigma_A(x,\xi'+\tilde\xi)\;\sigma_B(z,\Phi_{z,x}\tilde\xi)&\\ \times\;\chi(x,z)\;\chi(z,y)\;e^{i\psi(x,\tilde\xi;y,z)}\;\Upsilon_\varkappa(x,y,z)\;đ\tilde\xi\,dz\,đ\xi'\,.&\end{aligned} \tag{8.16}$$

Let us fix a coordinate system $\{x^k\}$, and let $y=\{y^k\}$ and $z=\{z^k\}$ be the associated n.c.s. with origin x. We denote by $\hat\Psi$ the matrix-function with elements $\hat\Psi_k^p=\sum_k(T_{kj}^p+\psi_{kj}^p)(z^j-x^j)/2$, where ψ_{kj}^p are defined by (8.7), and set

$$\Psi(x,y,z)\;=\;\big(I-\hat\Psi(x,y,z)\big)^{-1}.$$

Then

$$\varphi_0(x,\Psi\xi,y)\;+\;\psi(x,\Psi\xi;y,z)\;=\;\varphi_0(x,\xi,y)\,.$$

If the connection Γ is flat then $\hat\Psi\equiv 0$ and $\Psi\equiv I$. In the general case (8.7) implies

$$\Psi|_{z=x}=\;I\,,\qquad \partial_{z^j}\Psi_k^p|_{z=y=x}=\;\frac12\,T_{kj}^p(x)\,. \tag{8.17}$$

Now we change variables $\tilde\xi=\Psi\xi''$ in (8.16). Then in our coordinates this integral takes the form

$$\begin{aligned}\Upsilon_x^{1-\varkappa}(y)\int e^{i(x-z)\cdot\xi'}\;e^{i(x-y)\cdot\xi''}\;\sigma_A(x,\xi'+\Psi\xi'')\;\sigma_B(z,\Phi_x(z)\Psi\xi'')&\\ \times\;\chi(x,z)\;\chi(z,y)\;\Upsilon_\varkappa\;|\det\Psi|\;đ\xi'\,dz\,đ\xi''\,.&\end{aligned} \tag{8.18}$$

Substituting in (8.18) the Taylor expansion of the function $\sigma_A(x,\xi'+\Psi\xi'')$ at the point $\xi'=0$ we obtain

$$\begin{aligned}&\Upsilon_x^{1-\varkappa}(y)\int e^{i(x-z)\cdot\xi'}\;e^{i(x-y)\cdot\xi''}\sum_{|\alpha|\leqslant p}\frac{1}{\alpha!}\;(\xi')^\alpha\;\partial_\xi^\alpha\sigma_A(x,\Psi\xi'')\;\sigma_B(z,\Phi_x(z)\Psi\xi'')\\ &\qquad\times\;\chi(x,z)\;\chi(z,y)\;\Upsilon_\varkappa\;|\det\Psi|\;đ\xi'\,dz\,đ\xi''\\ &+\;(p+1)\;\Upsilon_x^{1-\varkappa}(y)\int e^{i(x-z)\cdot\xi'}\;e^{i(x-y)\cdot\xi''}\sum_{|\alpha|=p+1}\frac{1}{\alpha!}\;(\xi')^\alpha\int_0^1(1-t)^p\\ &\times\partial_\xi^\alpha\sigma_A(x,\Psi\xi''+t\xi')\;\sigma_B(z,\Phi_x(z)\Psi\xi'')\;\chi(x,z)\;\chi(z,y)\;\Upsilon_\varkappa\;|\det\Psi|\;dt\,đ\xi'\,dz\,đ\xi''\,.\end{aligned}$$

In the first term we integrate with respect to ξ' and z, and in the second term we replace $(\xi')^\alpha e^{i(x-z)\cdot\xi'}$ by $(-D_z)^\alpha e^{i(x-z)\cdot\xi'}$ and then integrate by parts in z. Then we obtain

$$\begin{aligned} &\Upsilon_x^{1-\varkappa}(y) \int e^{i(x-y)\cdot\xi''} \sum_{|\alpha|\leqslant p} \frac{1}{\alpha!}\, Q_\alpha^{A,B}(x,y,x,0,\xi'')\, đ\xi'' \\ &\qquad + (p+1)\ \Upsilon_x^{1-\varkappa}(y) \int e^{i(x-y)\cdot\xi''}\, e^{i(x-z)\cdot\xi'} \int_0^1 (1-t)^p \\ &\qquad\qquad \times \sum_{|\alpha|=p+1} \frac{1}{\alpha!}\, Q_{\alpha,\Psi}^{A,B}(x,y,z,t\xi',\xi'')\, dt\, đ\xi'\, dz\, đ\xi'' , \end{aligned} \tag{8.19}$$

where

$$\begin{aligned} &Q_{\alpha,\Psi}^{A,B}(x,y,z,t\xi',\xi'') \\ &\quad = D_z^\alpha\Big(\partial_\xi^\alpha \sigma_A(x,\Psi\xi''+t\xi')\ \sigma_B(z,\Phi_x(z)\Psi\xi'')\ \chi(x,z)\ \chi(z,y)\ \Upsilon_\varkappa\ |\det\Psi|\Big). \end{aligned} \tag{8.20}$$

Obviously, the function $\sum_{\alpha\leqslant p} Q_{\alpha,\Psi}^{A,B}(x,y,x,0,\xi'')/\alpha!$ can be written as a finite sum of the form

$$\sum_{\alpha,\beta,\gamma} Q_{\beta,\gamma}^\alpha(y,x,\xi'')\ D_{\xi''}^\beta \sigma_A(x,\xi'')\ D_{\xi''}^\gamma \nabla_x^\alpha \sigma_B(x,\xi''),$$

where $Q_{\beta,\gamma}^\alpha(y,x,\xi'')$ are some polynomials in ξ'' independent of the symbols σ_A and σ_B. Therefore this function is an amplitude from $\mathrm{S}_{\rho,\delta}^{m_1+m_2}(\Gamma)$, and the first integral in (8.19) defines a ψDO from $\Psi_{\rho,\delta}^{m_1+m_2}(\Omega^\varkappa,\Gamma)$. By (4.7) its symbol has the form

$$\sum_{\alpha,\beta,\gamma} \tilde P_{\beta,\gamma}^\alpha(x,\xi)\ D_\xi^\beta \sigma_A(x,\xi)\ D_\xi^\gamma \nabla_x^\alpha \sigma_B(x,\xi),$$

where $\tilde P_{\beta,\gamma}^\alpha$ are polynomials in ξ which are also independent of σ_A and σ_B.

In order to find these polynomials we assume A and B to be differential operators. In this case σ_A and σ_B are polynomials in ξ. We substitute in (8.16) instead of $\sigma_A(x,\tilde\xi+\xi')$ its Taylor expansion at the point $\xi'=0$ and integrate with respect to ξ' and z. Then (8.16) takes the form

$$\Upsilon_x^{1-\varkappa}(y) \int e^{i(x-y)\cdot\xi} \sum_\alpha \frac{1}{\alpha!}\ D_\xi^\alpha \sigma_A(x,\xi)\ \partial_z^\alpha\Big(\sigma_B(z,\Phi_x(z)\xi)\, e^{i\psi}\, \Upsilon_\varkappa\Big)\Big|_{z=x} đ\xi \tag{8.21}$$

(we have replaced $\tilde\xi$ by ξ and omitted the inessential factor $\chi(x,z)\,\chi(z,y)$). The integrand in (8.21) is also a polynomial in ξ. We expand its coefficients by Taylor's formula at the point $y=x$, then replace $(y-x)^\alpha e^{i(x-y)\cdot\xi}$ by $(-D_\xi)^\alpha e^{i(x-y)\cdot\xi}$ and integrate by parts in ξ. This transforms (8.21) into the integral

$$\begin{aligned} &\Upsilon_x^{1-\varkappa}(y) \int e^{i(x-y)\cdot\xi} \sum_{\alpha,\beta} \frac{1}{\alpha!}\,\frac{1}{\beta!} \\ &\qquad \times\ D_\xi^\beta\Big(D_\xi^\alpha \sigma_A(x,\xi)\ \partial_z^\alpha\Big(\sigma_B(z,\Phi_x(z)\xi)\ \partial_y^\beta\big(e^{i\psi}\Upsilon_\varkappa\big)\Big)\Big|_{y=z=x}\Big)\, đ\xi\,. \end{aligned}$$

We have

$$\sum_{\alpha,\beta} \frac{1}{\alpha!}\,\frac{1}{\beta!}\, D_\xi^\beta \Big(D_\xi^\alpha \sigma_A(x,\xi)\ \partial_z^\alpha \Big(\sigma_B(z,\Phi_x(z)\xi)\ \partial_y^\beta \big(e^{i\psi}\Upsilon_\varkappa\big)\Big)\Big|_{y=z=x}\Big)$$
$$= \sum_{\alpha,\alpha_1,\beta} \frac{1}{\alpha!}\,\frac{1}{\alpha_1!}\,\frac{1}{\beta!}\ D_\xi^\beta \Big(D_\xi^{\alpha+\alpha_1}\sigma_A(x,\xi)\ \nabla_x^\alpha \sigma_B(x,\xi)\ \partial_z^{\alpha_1}\partial_y^\beta \big(e^{i\psi}\Upsilon_\varkappa\big)\big|_{y=z=x}\Big)\,,$$

$$\sum_{\alpha,\alpha_1,\beta} \frac{1}{\alpha!}\,\frac{1}{\alpha_1!}\,\frac{1}{\beta!}\ D_\xi^\beta \Big(D_\xi^{\alpha+\alpha_1}\sigma_A\ \nabla_x^\alpha \sigma_B\ \partial_z^{\alpha_1}\partial_y^\beta \big(e^{i\psi}\Upsilon_\varkappa\big)\Big)$$
$$= \sum_{\alpha,\alpha_1,\beta_1,\beta',\gamma} \frac{1}{\alpha!}\,\frac{1}{\alpha_1!}\,\frac{1}{\beta'!}\,\frac{1}{\beta_1!}\,\frac{1}{\gamma!}\ D_\xi^{\alpha+\alpha_1+\beta_1}\sigma_A\ D_\xi^\gamma \nabla_x^\alpha \sigma_B\ D_\xi^{\beta'}\partial_z^{\alpha_1}\partial_y^{\beta_1+\beta'+\gamma}\big(e^{i\psi}\Upsilon_\varkappa\big)$$
$$= \sum_{\alpha,\beta,\gamma} \frac{1}{\alpha!}\,\frac{1}{\beta!}\,\frac{1}{\gamma!}\ D_\xi^{\alpha+\beta}\sigma_A\ D_\xi^\gamma \nabla_x^\alpha \sigma_B\ \partial_y^\gamma (\partial_z+\partial_y)^\beta \sum_{\beta'} \frac{1}{\beta'!}\, D_\xi^{\beta'}\partial_y^{\beta'}\big(e^{i\psi}\Upsilon_\varkappa\big)\,.$$

Since ψ has a second order zero at the point $z=y=x$, in the last sum all the terms with $|\beta'|>|\beta|$ are equal to zero at this point. Thus, we obtain (8.3).

Now it remains to prove that the remainder term in (8.19) gets smoother and smoother as $p\to\infty$. Changing variables $\xi''=\Psi^{-1}\tilde\xi$ we can write it in the form

$$(p+1)\ \Upsilon_x^{1-\varkappa}(y) \int e^{i(x-y)\cdot\Psi^{-1}\tilde\xi}\, e^{i(x-z)\cdot\xi'} \int_0^1 (1-t)^p$$
$$\times \sum_{|\alpha|=p+1} \frac{1}{\alpha!}\ Q_\alpha^{A,B}(x,y,z,t\xi',\tilde\xi)\ dt\, đ\xi'\, dz\, đ\tilde\xi\,,$$

where $Q_\alpha^{A,B}(x,y,z,t\xi',\tilde\xi) = Q_{\alpha,\Psi}^{A,B}(x,y,z,t\xi',\xi'')\Big|_{\xi''=\Psi^{-1}\tilde\xi}$. We split this integral into the sum

$$(p+1)\ \Upsilon_x^{1-\varkappa}(y) \int e^{i(x-y)\cdot\Psi^{-1}\tilde\xi}\, e^{i(x-z)\cdot\xi'} \int_0^1 (1-t)^p$$
$$\times \sum_{|\alpha|=p+1} \frac{1}{\alpha!}\ \big(1-\varsigma(\xi',\tilde\xi)\big)\ Q_\alpha^{A,B}(x,y,z,t\xi',\tilde\xi)\ dt\, đ\xi'\, dz\, đ\tilde\xi$$
$$+\ (p+1)\ \Upsilon_x^{1-\varkappa}(y) \int e^{i(x-y)\cdot\Psi^{-1}\tilde\xi}\, e^{i(x-z)\cdot\xi'} \int_0^1 (1-t)^p$$
$$\times \sum_{|\alpha|=p+1} \frac{1}{\alpha!}\ \varsigma(\xi',\tilde\xi)\ Q_\alpha^{A,B}(x,y,z,t\xi',\tilde\xi)\ dt\, đ\xi'\, dz\, đ\tilde\xi\,, \tag{8.22}$$

where ς is a smooth function bounded with all its derivatives and such that

$$\operatorname{supp}(1-\varsigma)\ \subset\ \{2C\langle\xi'\rangle_x \geqslant \langle\tilde\xi\rangle_x\} \cup \{\langle\xi'\rangle_x + \langle\tilde\xi\rangle_x \leqslant C\}\,,$$
$$\operatorname{supp}\varsigma\ \subset\ \{C\langle\xi'\rangle_x \leqslant \langle\tilde\xi\rangle_x\} \cup \{\langle\xi'\rangle_x + \langle\tilde\xi\rangle_x \leqslant C\}$$

for a sufficiently large constant C.

Assuming that $|x-y|$ is sufficiently small on $\operatorname{supp} Q_\alpha^{A,B}$ (one can always achieve this by shrinking $\operatorname{supp}\chi$), in the first integral we replace $e^{i(x-y)\cdot\Psi^{-1}\tilde\xi}e^{i(x-z)\cdot\xi'}$ by

$$\frac{\left((x-y)\cdot\Psi^{-1}\tilde\xi+(x-z)\cdot\xi'\right)_z\cdot D_z\left(e^{i(x-y)\cdot\Psi^{-1}\tilde\xi}e^{i(x-z)\cdot\xi'}\right)}{\left|\left((x-y)\cdot\Psi^{-1}\tilde\xi+(x-z)\cdot\xi'\right)_z\right|^2}$$

and then integrate by parts in z. This reduces the order of the amplitude by one. Repeating this procedure N times we obtain an integral with the same phase function and an amplitude which is estimated by const $\langle\xi'\rangle^{m_{p,N}}$ with

$$m_{\alpha,N} \;=\; m_1 \;+\; m_2 \;+\; (p+1) \;-\; N\max\{1-\rho,\delta\}\,.$$

Since $m_{p,N}\to-\infty$ as $N\to\infty$, the first integral in (8.22) defines an infinitely smooth function for each fixed p.

Let us consider the second integral in (8.22). Since y and z are close to x, on $\operatorname{supp}(\varsigma Q_\alpha^{A,B})$ we have the uniform estimates

$$\left\langle\Phi_x(z)\tilde\xi\right\rangle_x\asymp\langle\tilde\xi\rangle_x\,,\qquad\left\langle\tilde\xi+t\xi'\right\rangle\asymp\tilde\xi\,.$$

This implies that the function $\varsigma(\xi',\tilde\xi)\,Q_\alpha^{A,B}(x,y,z,t\xi',\tilde\xi)$ is estimated by $\mathrm{const}\langle\tilde\xi\rangle_x^{m_\alpha}$ where

$$m_\alpha \;=\; m_1 \;+\; m_2 \;-\; |\alpha|\,\rho \;+\; |\alpha|\max\{1-\rho,\delta\}\,, \tag{8.23}$$

and analogous estimates hold for its derivatives with respect to x and y. If $\rho>1/2$ then $m_\alpha\to-\infty$ as $|\alpha|\to\infty$, so for sufficiently large p the second integral in (8.22) defines a C^{n_p}-function where $n_p\to\infty$ as $p\to\infty$.

If $\rho\leqslant 1/2$ then m_α do not tend to $-\infty$ as $|\alpha|\to\infty$. For any symbols a and b we have

$$\frac{d}{dz^k}\,a(x,\Psi\xi+t\xi') \;=\; (\Psi)_{z^k}\xi\cdot a_\xi(x,\Psi\xi+t\xi')\,, \tag{8.24}$$

$$\frac{d}{dz^k}\,b(z,\Phi_x(z)\Psi\xi) \;=\; \nabla_{z^k}b(z,\Phi_x(z)\Psi\xi) \;+\; F_{(k)}\xi\cdot b_\zeta(z,\Phi_x(z)\Psi\xi)\,, \tag{8.25}$$

where $F_{(k)}=F_{(k)}(x,y,z)$ is some smooth matrix-function. Therefore, generally speaking, each differentiation ∂_{z^k} in (8.20) increases the order of the amplitude by $\max\{1-\rho,\delta\}$. However, if $T\equiv0$ then, in view of (2.8) and (8.17), the matrix-functions $(\Psi)_{z^k}$ and $F_{(k)}$ in (8.24) and (8.25) are equal to zero at the point $y=z=x$. We can single out the factor $(x-z)$ or $(x-y)$, replace $(x-z)e^{i(x-z)\cdot\xi'}$ by $D_{\xi'}e^{i(x-z)\cdot\xi'}$ or $(x-y)e^{i(x-y)\cdot\Psi^{-1}\tilde\xi}$ by $\Psi^TD_{\tilde\xi}e^{i(x-y)\cdot\Psi^{-1}\tilde\xi}$, and then integrate by parts. Clearly, each of these factors allows us to decrease the order of the amplitude by ρ, and then the differentiation with respect to z leads to an increase of the order only by $1-2\rho$. Certainly, the factor $(x-z)$ or $(x-y)$ can disappear when we differentiate $(\Psi)_{z^k}$ or $F_{(k)}$ once more, but then the further differentiation does not increase the order of the amplitude. In this case only two (or more) differentiations in z increase the order by $1-\rho$. As a result, after all the possible integrations by parts we obtain in the second integral in (8.22) a new amplitude of the order

$$m_{(p+1)} \;=\; m_1 \;+\; m_2 \;-\; (p+1)\,\rho \;+\; (p+1)\max\{\delta, 1-2\rho, (1-\rho)/2\}\,.$$

If $\rho > 1/3$ then $m_{(p+1)} \to -\infty$ as $p\to\infty$. Thus, if $T\equiv 0$ and $\rho>1/3$ then the second integral in (8.22) also defines a C^{n_p}-function with $n_p\to\infty$ as $p\to\infty$.

Finally, if the connection Γ is flat then in the flat coordinates $\Phi_x(z)\equiv\Psi\equiv I$. In this case the function defined by the second integral gets smoother and smoother as $p\to\infty$ without any additional restriction on ρ and δ. $\square$

Proof of Theorem 8.4. We apply the same procedure as in the proof of Theorem 8.3. Then we obtain instead of (8.23)

$$m_\alpha \;=\; m_1 \;+\; m_2 \;-\; |\alpha| \;+\; |\alpha|\max\{1-\rho,\delta\} \qquad \text{if } \ A\in\Psi^{m_1}_{1,0}\,,$$

$$m_\alpha \;=\; m_1 \;+\; m_2 \;-\; |\alpha|\,\rho \qquad \text{if } \ B\in\Psi^{m_2}_{1,0}\,.$$

Obviously, in both these cases $m_\alpha\to-\infty$ as $|\alpha|\to\infty$. Therefore AB is a ψDO whose symbol is defined modulo $S^{-\infty}$ by the asymptotic expansion (8.14). $\square$

3. Propositions 8.2 and 4.4 immediately imply

Corollary 8.7. *Let the conditions of Theorem 8.3 or Theorem 8.4 be fulfilled, and*

$r=\min\{\rho-\delta\,,\,2\rho-1\}$ *under condition (1) of Theorem 8.3,*
$r=\min\{\rho-\delta\,,\,(3\rho-1)/2\}$ *under condition (2) of Theorem 8.3,*
$r=\rho-\delta$ *in the other cases.*

Then

$$\sigma_{AB} \;=\; \sigma_A\,\sigma_B \;+\; \sum_k D_{\xi_k}\sigma_A\;\nabla_{x^k}\sigma_B \;-\; (2i)^{-1}\sum_{j,k,p} T^p_{jk}\,\xi_p\;\partial_{\xi_j}\sigma_A\;\partial_{\xi_k}\sigma_B\,, \tag{8.26}$$

$$\sigma_{[A,B]} \;=\; -i\,\{\sigma_A,\sigma_B\}\,, \qquad \sigma^W_{AB} \;=\; \sigma^W_A\,\sigma^W_B \;-\; (2i)^{-1}\,\{\sigma^W_A,\sigma^W_B\} \tag{8.27}$$

modulo $\mathrm{S}^{m_1+m_2-2r}_{\rho,\delta}(\Gamma)$. *Moreover, if* Γ *is symmetric then*

$$\begin{aligned}\sigma_{AB} \;=\; \sum_{|\alpha|\leqslant 2}\frac{1}{\alpha!}\;D^\alpha_\xi\sigma_A\;\nabla^\alpha_x\sigma_B \qquad\qquad&\\ +\;\frac16\sum_{j,k,l,p}\left(R^p{}_{kjl}+R^p{}_{ljk}\right)\xi_p\;\partial_{\xi_j}\sigma_A\;\partial_{\xi_k}\partial_{\xi_l}\sigma_B&\\ -\;\frac1{12}\sum_{j,k,l,p}\left(R^p{}_{jlk}+R^p{}_{klj}\right)\xi_p\;\partial_{\xi_j}\partial_{\xi_k}\sigma_A\;\partial_{\xi_l}\sigma_B&\\ -\;\frac16\sum_{j,k}\left(R_{jk}+R_{kj}\right)\,\partial_{\xi_j}\sigma_A\;\partial_{\xi_k}\sigma_B\;+\;\frac{\varkappa}{2}\sum_{j,k}\left(R_{jk}-R_{kj}\right)\,\partial_{\xi_j}\sigma_A\;\partial_{\xi_k}\sigma_B&\end{aligned} \tag{8.28}$$

modulo $\mathrm{S}^{m_1+m_2-3r}_{\rho,\delta}(\Gamma)$.

9. L_2-ESTIMATES

Let H^s be the Sobolev space W_2^s (defined in local coordinates), $H^s_{\mathrm{comp}}(M;\Omega^\varkappa)$ be the space of H^s-densities on M with compact supports, and $H^s_{\mathrm{loc}}(M;\Omega^\varkappa)$ be the space of densities whose restrictions to compact subsets of M belong to H^s. In this section we prove

Theorem 9.1. *Let at least one of the conditions (1)–(3) of Theorem 8.3 be fulfilled. Then a ψDO $A\in\Psi^m_{\rho,\delta}(\Omega^\varkappa,\Gamma)$ is bounded from $H^s_{\mathrm{comp}}(M;\Omega^\varkappa)$ to $H^{s-m}_{\mathrm{loc}}(M;\Omega^\varkappa)$ for all $s\in\mathbf{R}^1$.*

Remark 9.2. It is known that $A: H^s_{\mathrm{comp}}(M;\Omega^\varkappa)\to H^{s-m}_{\mathrm{loc}}(M;\Omega^\varkappa)$ if in local coordinates the Schwartz kernel of A can be represented by an oscillatory integral with the standard phase function $(x-y)\cdot\theta$ and an amplitude from $\mathrm{S}^m_{\rho,\delta}$ with $\delta\leqslant\rho$ [T], [Tr]. Therefore Theorem 9.1 gives a new result only when $1/3<\rho<1/2$ and the connection Γ is symmetric.

Proof of Theorem 9.1. In view of Theorem 8.4 it is sufficient to prove that

$$\rho_1\,A\,\rho_2\;:\;L_2(M;\Omega^\varkappa)\to L_2(M;\Omega^\varkappa) \tag{9.1}$$

for all $\rho_1,\rho_2\in C_0^\infty(M)$ and $A\in\Psi^0_{\rho,\delta}(\Omega^\varkappa,\Gamma)$. Fixing a smooth density, we can always identify $\varkappa$-densities with half-densities. Therefore it is sufficient to consider only $\varkappa=1/2$.

If $A\in\Psi^0_{\rho,\delta}(\Omega^{1/2},\Gamma)$ then $\sup|\rho_1(x)\,\rho_2(x)\,\sigma^W_A(x,\xi)|<2C$ for some positive constant C. We are going to prove that

$$C^2\;-\;(\rho_1\,A\,\rho_2)^*(\rho_1\,A\,\rho_2)\;=\;B^*B\;+\;R\,, \tag{9.2}$$

where $B\in\Psi^0_{\rho,\delta}(\Omega^{1/2},\Gamma)$ and R is an operator with smooth Schwartz kernel. Obviously, (9.2) implies (9.1).

From Propositions 4.4 and 4.5 it follows that the Weyl symbol of the ψDO $\rho_1\,A\,\rho_2$ is equal to $\rho_1(x)\,\rho_2(x)\,\sigma^W_A(x,\xi)$ modulo $\mathrm{S}^{\delta-\rho}_{\rho,\delta}(\Gamma)$. Let $B_0\in\Psi^0_{\rho,\delta}(\Omega^{1/2},\Gamma)$ be a properly supported ψDO with the Weyl symbol

$$\sigma^W_{B_0}\;=\;\left(C^2-|\rho_1(x)\,\rho_2(x)\,\sigma^W_A(x,\xi)|^2\right)^{1/2}\,.$$

Then Theorems 7.1 and 8.3 imply

$$C^2\;-\;(\rho_1\,A\,\rho_2)^*(\rho_1\,A\,\rho_2)\;-\;B_0^*B_0\;\in\;\Psi^{-r}_{\rho,\delta}(\Omega^{1/2},\Gamma)$$

with some positive r.

Assume that we have constructed ψDOs $B_j\in\Psi^{-rj}_{\rho,\delta}(\Omega^{1/2},\Gamma)$ for $j=0,1,\dots,k-1$ such that the operator

$$R_k\;=\;C^2\;-\;(\rho_1\,A\,\rho_2)^*(\rho_1\,A\,\rho_2)\;-\;(B_0+\cdots+B_{k-1})^*(B_0+...+B_{k-1})$$

belongs to $\Psi^{-kr}_{\rho,\delta}(\Omega^{1/2},\Gamma)$. Since R_k is formally self-adjoint, its Weyl symbol is real. Let $B_k \in \Psi^{-kr}_{\rho,\delta}(\Omega^{1/2},\Gamma)$ be a properly supported ψDO whose symbol coincides with $\sigma^W_{R_k}\left(2\,\overline{\sigma^W_{B_0}}\right)^{-1}$ for sufficiently large $\langle\xi\rangle_x$. Then

$$C^2 \;-\; (\rho_1\,A\,\rho_2)^*(\rho_1\,A\,\rho_2) \;-\; (B_0+\cdots+B_k)^*(B_0+\ldots+B_k) \;\in\; \Psi^{-(k+1)r}_{\rho,\delta}(\Omega^{1/2},\Gamma)\,. \tag{9.3}$$

Thus, step by step we can find ψDOs $B_j \in \Psi^{-jr}_{\rho,\delta}(\Omega^{1/2},\Gamma)$ satisfying (9.3) for all k. Then (9.2) holds for any properly supported ψDO B whose Weyl symbol admits the asymptotic expansion $\sigma^W_B \sim \sum_{j=0}^\infty \sigma^W_{B_j}$, $\langle\xi\rangle_x \to \infty$. $\square$

10. Operators with pseudodifferential parametrices

1. Let $\mathrm{HS}^{m,m_0}_{\rho,\delta}(\Gamma)$ be the subclass of symbols $a \in \mathrm{S}^m_{\rho,\delta}(\Gamma)$ satisfying the following condition: for any compact subset $K\subset M$ there exists a positive constant c_K such that

$$\langle\eta\rangle_y^{m_0} \;\leqslant\; \mathrm{const}_K\,|a(y,\eta)|\,,\qquad \forall (y,\eta)\in T^*M\;:\;y\in K\,,\;\langle\eta\rangle_y \geqslant c_K\,, \tag{10.1}$$

and

$$|\partial^\alpha_\eta \nabla_{i_1}\ldots\nabla_{i_q} a(x,\xi)| \;\leqslant\; \mathrm{const}_{K,\alpha,i_1,\ldots i_q}\,\langle\eta\rangle_y^{\delta q-\rho|\alpha|}\;|a(y,\eta)|\,,$$
$$\forall\,\alpha,\;\;\forall\, i_1,\ldots,i_q\,,\qquad \forall (y,\eta)\in T^*M\;:\;y\in K\,,\;\langle\eta\rangle_y \geqslant c_K\,. \tag{10.2}$$

We denote by $\mathrm{H}\Psi^{m,m_0}_{\rho,\delta}(\Omega^\varkappa,\Gamma)$ the class of ψDOs with symbols from $\mathrm{HS}^{m,m_0}_{\rho,\delta}(\Gamma)$.

In the same way as in the classical theory of ψDOs (see, for example, [Sh]) one can prove the following lemmas.

Lemma 10.1. *Let $a\in \mathrm{HS}^{m,m_0}_{\rho,\delta}(\Gamma)$, $b \in \mathrm{HS}^{m',m_0'}_{\rho,\delta}(\Gamma)$, and $\tilde a \in \mathrm{S}^{\tilde m}_{\rho,\delta}(\Gamma)$ with $\tilde m < m_0$. Then $ab \in \mathrm{HS}^{m+m',m_0+m_0'}_{\rho,\delta}(\Gamma)$ and $a+\tilde a \in \mathrm{HS}^{m,m_0}_{\rho,\delta}(\Gamma)$.*

Lemma 10.2. *Let $\sigma_A \in \mathrm{HS}^{m,m_0}_{\rho,\delta}(\Gamma)$, $\sigma_A^{(-1)} \in C^\infty(T^*M)$ and*

$$\sigma_A^{(-1)}(y,\eta) \;=\; \sigma_A^{-1}(y,\eta)\,,\qquad \forall\forall (y,\eta)\in T^*M\;:\;y\in K\,,\;\langle\eta\rangle_y \geqslant c_K\,,$$

for all compact subsets $K\subset M$. Then $\sigma_A^{(-1)} \in \mathrm{HS}^{-m_0,-m}_{\rho,\delta}(\Gamma)$.

Lemma 10.1, Proposition 4.4, 6.1 and Theorems 7.1, 8.3, 8.4 immediately imply the following simple results.

(1) If $A\in \mathrm{H}\Psi^{m,m_0}_{\rho,\delta}(\Omega^\varkappa,\Gamma)$, $\tilde A \in \Psi^{\tilde m}_{\rho,\delta}(\Omega^\varkappa,\Gamma)$, $\tilde m < m_0$, then $A+\tilde A \in \mathrm{H}\Psi^{m,m_0}_{\rho,\delta}(\Omega^\varkappa,\Gamma)$.
(2) $A\in \mathrm{H}\Psi^{m,m_0}_{\rho,\delta}(\Omega^\varkappa,\Gamma)$ if and only if $\sigma_{A,\tau} \in \mathrm{HS}^{m,m_0}_{\rho,\delta}(\Gamma)$ for some (and then for all) $\tau\in[0,1]$.
(3) If $\rho > 1/2$ then the classes $\mathrm{H}\Psi^{m,m_0}_{\rho,\delta}(\Omega^\varkappa,\Gamma)$ do not depend on Γ.
(4) If $A\in \mathrm{H}\Psi^{m,m_0}_{\rho,\delta}(\Omega^\varkappa,\Gamma)$ then $A^*\in \mathrm{H}\Psi^{m,m_0}_{\rho,\delta}(\Omega^{1-\varkappa},\Gamma)$.
(5) If $A\in \mathrm{H}\Psi^{m,m_0}_{\rho,\delta}(\Omega^\varkappa,\Gamma)$, $B\in \mathrm{H}\Psi^{m',m_0'}_{\rho,\delta}(\Omega^\varkappa,\Gamma)$ then under the conditions of Theorem 8.3 or Theorem 8.4 $AB\in \mathrm{H}\Psi^{m+m',\,m_0+m_0'}_{\rho,\delta}(\Omega^\varkappa,\Gamma)$.

Remark 10.3. If $\rho \leqslant 1/2$ then, generally speaking, (3) is not true. For example, the heat operator $\partial_{x^{n+1}} - \partial^2_{x^1} - \ldots - \partial^2_{x^n}$ in $\mathbf{R}^{n+1}$ belongs to $\mathrm{H}\Psi^{2,1}_{1/2,0}$ in the standard coordinates (i.e., for the corresponding flat connection Γ). However, in other coordinates its full symbol can have a big zero set, and then the estimates (10.1), (10.2) do not hold (see [Tr, Ch.4.2, Example 2.1]).

An operator B such that $AB = BA = I \pmod{\Psi^{-\infty}}$ is said to be the *parametrix* of A. Obviously, if A has a pseudodifferential parametrix then A is *hypoelliptic*, i.e., $Au \in C^\infty(M)$ implies $u \in C^\infty(M)$.

Theorem 10.4. *Assume that at least one of the conditions (1)–(3) of Theorem 8.3 is fulfilled. Then any ψDO $A \in \mathrm{H}\Psi^{m,m_0}_{\rho,\delta}(\Omega^\varkappa, \Gamma)$ has a pseudodifferential parametrix $B \in \mathrm{H}\Psi^{-m_0,-m}_{\rho,\delta}(\Omega^\varkappa, \Gamma)$.*

Proof. Let $\sigma_A^{(-1)}$ be the symbol from Lemma 10.2, $b \in \mathrm{S}^{\tilde m}_{\rho,\delta}(\Gamma)$, and $\tilde B$ be a properly supported ψDO with the symbol $\sigma_{\tilde B}(y,\eta) = \sigma_A^{(-1)}(y,\eta)\, b(y,\eta)$. Then by (8.14)

$$\sigma_{A\tilde B} \;=\; b \,-\, b' , \qquad \sigma_{\tilde B A} \;=\; b \,-\, b'' , \tag{10.3}$$

where $b', b'' \in \mathrm{S}^{\tilde m - r}_{\rho,\delta}(\Gamma)$ with some positive r.

Let $b_1 \equiv 1$ and $b_{k+1} = b'_k$, $k = 0, 1, 2, \ldots$, where b'_k is the symbol which appears in the first equality (10.3) when we replace b by b_k. Then $b_k \in \mathrm{S}^{-kr}_{\rho,\delta}(\Gamma)$, and for any properly supported ψDO B whose symbol admits the asymptotic expansion

$$\sigma_B(y,\eta) \;\sim\; \sigma_A^{(-1)}(y,\eta) \sum_{k=1}^\infty b_k(y,\eta) , \qquad \langle\eta\rangle_y \to \infty ,$$

we obtain $AB = I \pmod{\Psi^{-\infty}}$.

By analogy (using the second equality (10.3)) we can construct a properly supported ψDO B' such that $B'A = I \pmod{\Psi^{-\infty}}$. Since $B' = B'AB = B$ modulo $\Psi^{-\infty}$, we also have $BA = I \pmod{\Psi^{-\infty}}$. $\square$

2. In this subsection we assume that there exist n smooth linearly independent vector fields $\nu_1, \ldots, \nu_n$ on M. Further on we use the notation introduced in Example 5.4; in particular, $\Gamma = {}_0\Gamma$ is the linear connection for which all the covariant differentials of ν_l are equal to zero.

Let $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$ be some positive numbers and $\mathbf{d} = (d_1, \ldots, d_n)$ be the corresponding n-vector. We say that a function $\tilde a(y,\theta) \in C^\infty(M \times \mathbf{R}^n \setminus \{0\})$ is $\mathbf{d}$-homogeneous of degree m if

$$\tilde a(y, \lambda^{1/d_1}\theta_1, \ldots, \lambda^{1/d_n}\theta_n) \;=\; \lambda^m\, \tilde a(y,\theta) , \qquad \forall \lambda > 0 .$$

Then the derivatives $\partial_\theta^\alpha \partial_y^\beta \tilde a$ are $\mathbf{d}$-homogeneous of degrees $m - |\alpha : \mathbf{d}|$ where

$$|\alpha : \mathbf{d}| \;:=\; \sum_k \alpha_k / d_k .$$

A typical example of a $\mathbf{d}$-homogeneous function of degree one is

$$|\theta|_{\mathbf{d}} \;:=\; \Big(\sum_k |\theta_k|^{2d_k}\Big)^{1/2}.$$

Obviously, if $\tilde a$ is $\mathbf{d}$-homogeneous of degree m and K is a compact subset of M then $|\tilde a(y,\theta)| \leqslant \mathrm{const}_K\,|\theta|_{\mathbf{d}}^m$ for all $(y,\theta)\in K\times\mathbf{R}^n$.

Let us introduce the vector-function $\vec\sigma := (\sigma_1,\dots,\sigma_n)$, where $\sigma_l\in C^\infty(T^*M)$ are defined in (5.10). We denote by $\mathrm{S}^m_{\mathbf{d}}(\Gamma)$ the class of functions $a\in C^\infty(T^*M)$ which admit the asymptotic expansion

$$a(y,\eta) \;\sim\; \sum_{k=0}^\infty \tilde a_k\big(y,\vec\sigma(y,\eta)\big)\,,\qquad |\vec\sigma(y,\eta)|_{\mathbf{d}}\to\infty\,, \tag{10.4}$$

with $\tilde a_k$ $\mathbf{d}$-homogeneous of degree m_k, where $m_0=m$ and $m_k\to-\infty$ as $k\to\infty$. Since the functions σ_l are constant along the horizontal curves generated by Γ, (10.4) implies that

$$|\partial_\eta^\alpha\nabla_y^\beta a(y,\eta)| \;\leqslant\; \mathrm{const}_K\,|\vec\sigma(y,\eta)|_{\mathbf{d}}^{m-|\alpha|/d_n}\,,\qquad \forall(y,\eta)\;:\; y\in K\,,\ |\vec\sigma(y,\eta)|_{\mathbf{d}}\geqslant 1\,, \tag{10.5}$$

for all compact subsets $K\subset M$.

Let $\Psi^m_{\mathbf{d}}(\Omega^\varkappa,\Gamma)$ be the class of ψDO acting in the space of $\varkappa$-densities whose Γ-symbols belong to $\mathrm{S}^m_{\mathbf{d}}(\Gamma)$. In view of (10.5) $\Psi^m_{\mathbf{d}}(\Omega^\varkappa,\Gamma)\subset\Psi^{md_n}_{d_1/d_n,\,0}(\Omega^\varkappa,\Gamma)$.

For a ψDO $A\in\Psi^m_{\mathbf{d}}(\Omega^\varkappa,\Gamma)$ we set $\sigma^0_A(y,\eta)=\tilde a_0(y,\vec\sigma)$, where a_0 is the leading term in the expansion (10.4) with $a=\sigma_A$. The function σ^0_A is said to be the *principal symbol* of the ψDO $A\in\Psi^m_{\mathbf{d}}(\Omega^\varkappa,\Gamma)$.

Example 10.5. Let

$$A(y,D_y) \;=\; \sum_{|\alpha:\mathbf{d}|\leqslant m} c_\alpha(y)\,A^\alpha_{(\varkappa)}(y,D_y)\,,$$

where $c_\alpha\in C^\infty(M)$ and $A^\alpha_{(\varkappa)}$ are the symmetrized compositions of the Lie differentiations $A^{(\varkappa)}_l$. Then $A\in\Psi^m_{\mathbf{d}}(\Omega^\varkappa,\Gamma)$ and (5.6) (with $s=0$) implies

$$\sigma^0_A(y,\eta) \;=\; \sum_{|\alpha:\mathbf{d}|=m} c_\alpha(y)\,i^{|\alpha|}\,\sigma^\alpha(y,\eta)\,.$$

Definition 10.6. A ψDO $A\in\Psi^m_{\mathbf{d}}(\Omega^\varkappa,\Gamma)$ is said to be *semi-elliptic* if there exists a positive constant c such that $|\sigma^0_A(y,\eta)|\geqslant c\,|\theta|^m_{\mathbf{d}}$.

Obviously, if $A\in\Psi^m_{\mathbf{d}}(\Omega^\varkappa,\Gamma)$ is semi-elliptic then $A\in\mathrm{H}\Psi^{md_n,md_1}_{d_1/d_n,\,0}(\Omega^\varkappa,\Gamma)$.

Theorem 10.7. *Assume that*

$$d_j^{-1} + d_k^{-1} > d_i^{-1}, \qquad \forall\, i,j,k \;:\; C_{jk}^{i} \not\equiv 0\,. \tag{10.6}$$

Then any semi-elliptic differential operator $A \in \Psi^m_{\mathbf{d}}(\Omega^\varkappa,\Gamma)$ has a pseudodifferential parametrix $B \in \Psi^{-m}_{\mathbf{d}}(\Omega^\varkappa,\Gamma)$.

Proof. The connection Γ is curvature free. Therefore the coefficients of the polynomials (8.3) are components of some polynomials in the torsion tensor and its symmetric covariant differentials. In other words, $P_{\beta,\gamma}(y,\eta)$ is a linear combination of terms of the form

$$\sum \mathbf{D}_y^{\alpha^{(1)}} T_{j_1k_1}^{i_1}(y)\ldots \mathbf{D}_y^{\alpha^{(q)}} T_{j_qk_q}^{i_q}(y)\,\eta_{i_1}\ldots\eta_{i_p}\,, \qquad p \leqslant q\,,$$

where the sum is taken over $i_1,\ldots,i_p$, and over the remaining upper indices $i_{p+1},\ldots i_q$ and some $(q-p)$ lower indices (which are j_l, k_l and those corresponding to $\alpha^{(l)}$). In view of (5.3) we have

$$\mathbf{D}_y^\alpha T_{jk}^i(y) \;=\; -\sum_{i',j',k'} \nu_{i'}^{i}(y)\,\tilde\nu_j^{j'}(y)\,\tilde\nu_k^{k'}(y)\,\mathbf{D}_y^\alpha C_{j'k'}^{i'}(y)\,, \tag{10.7}$$

where $\mathbf{D}_y^\alpha C_{j'k'}^{i'}$ are the symmetric covariant differentials of functions $C_{j'k'}^{i'}$.

Let $\varepsilon = \min\{d_j^{-1}+d_k^{-1}-d_i^{-1}\}$ where the minimum is taken over all i,j,k such that $C_{jk}^i \not\equiv 0$. The equality (10.7) implies that

$$\sum_{i,j,k} \mathbf{D}_y^\alpha T_{jk}^i(y)\,\eta_i\;\partial_{\eta_j}a_1(y,\eta)\;\partial_{\eta_k}a_2(y,\eta)$$

$$= \sum_{i,j,k} \mathbf{D}_y^\alpha C_{jk}^i(y)\,\sigma_i\;(\tilde a_1)_{\theta_j}(y,\vec\sigma)\;(\tilde a_2)_{\theta_k}(y,\vec\sigma)$$

for all functions $a_1(y,\eta) = \tilde a_1(y,\vec\sigma)$ and $a_2(y,\eta) = \tilde a_2(y,\vec\sigma)$. From here it follows that under condition (10.6)

$$P_{\beta,\gamma}(y,\eta)\,\partial_\eta^\beta a_1(y,\eta)\,\partial_\eta^\gamma a_2(y,\eta) \;\in\; \mathrm{S}_{\mathbf{d}}^{m_1+m_2-\varepsilon}(\Gamma) \tag{10.8}$$

for all $a_1 \in \mathrm{S}_{\mathbf{d}}^{m_1}(\Gamma)$, $a_2 \in \mathrm{S}_{\mathbf{d}}^{m_2}(\Gamma)$.

Let us fix a symbol $a \in \mathrm{S}_{\mathbf{d}}^{-m}(\Gamma)$ such that $a = \big(\sigma_A^0(y,\eta)\big)^{-1}$ as $|\vec\sigma|_{\mathbf{d}} \geqslant 1$. Let $\tilde B$ be a properly supported ψDO with the symbol $\sigma_{\tilde B}(y,\eta) = a(y,\eta)\,b(y,\eta)$ where $b \in \mathrm{S}_{\mathbf{d}}^{\tilde m}(\Gamma)$. Then by Theorem 8.4 the compositions $A\tilde B$ and $\tilde B A$ are pseudodifferential operators. In view of (10.8) their symbols satisfy (10.3) with $b', b'' \in \mathrm{S}_{\mathbf{d}}^{\tilde m-\varepsilon}(\Gamma)$. Therefore we can construct the parametrix $B \in \Psi_{\mathbf{d}}^{-m}(\Omega^\varkappa,\Gamma)$ in the same way as in the proof of Theorem 10.4. $\square$

Remark 10.8. If $a \in \mathrm{S}_{\mathbf{d}}^{m}(\Gamma)$ then in an arbitrary coordinate system

$$|\partial_\eta^\alpha \partial_y^\beta a(y,\eta)| \;\leqslant\; \mathrm{const}\,|\vec\sigma(y,\eta)|_{\mathbf{d}}^{m-\rho|\alpha|+\delta|\beta|}$$

with $\rho = d_n^{-1}$ and $\delta = \max_{j,k}|d_j^{-1}-d_k^{-1}|$. When $C_{kj}^i \not\equiv 0$ for all i,j,k, the condition (10.6) implies $\rho > \delta$. In this case the local parametrices of semi-elliptic differential operators belong to the coordinate classes of ψDO associated with the weight function $|\vec\sigma|$ (instead of $\langle\xi\rangle_x$).

11. Functions of second order differential operators

1. Let M be a closed Riemannian manifold, Δ be the Laplace operator, and ν be a first order differential operator (or a classical ψDO) on M. We assume that both operators Δ and ν act in the space of half-densities and that the operator ν is self-adjoint. We also assume that the operator $-\Delta+\nu$ is strictly positive (when M is compact, this can be always achieved by adding a large positive constant).

Let

$$A \;=\; (-\Delta+\nu)^{1/2}\,. \tag{11.1}$$

It is well known that A is a classical ψDO from $\Psi^1(\Omega^{1/2})$ whose principal symbol is equal to $|\xi|_x$. Moreover, $A^\lambda\in\Psi^\lambda(\Omega^{1/2})$ for all $\lambda\in\mathbf{R}^1$, all the homogeneous terms of the symbol σ_{A^λ} are analytic functions of λ, and $\sigma_{A^\lambda}(x,\xi)=|\xi|_x^\lambda$ modulo $S^{\lambda-1}$. Certainly, these facts are independent of the choice of the linear connection.

If Γ is the Levi–Civita connection, then the function $|\xi|_x$ is constant along the horizontal curves (see Example 3.4) and

$$\sigma_{A^2}(x,\xi) \;=\; |\xi|_x^2 \;+\; \sigma_\nu(x,\xi) \;-\; \frac13\, S(x)\,, \tag{11.2}$$

where σ_ν is the symbol of ν and S is the scalar curvature (Example 5.2). Let

$$\hat\xi^i \;:=\; \partial_{\xi_i}(|\xi|_x) \;=\; \sum_j g^{ij}(x)\,\xi_j/|\xi|_x\,,\qquad \hat\xi_\gamma \;:=\; (\hat\xi^1)^{\gamma_1}\dots(\hat\xi^n)^{\gamma_n}\,. \tag{11.3}$$

Lemma 11.1. *Let Γ be the Levi–Civita connection. Then*

$$\sigma_A(x,\xi) \;=\; |\xi|_x \;+\; \frac1{12}\,|\xi|_x^{-1}\Big(\sum_{j,k} R_{jk}(x)\,\hat\xi^j\,\hat\xi^k - 2\,S(x)\Big) \;+\; \sigma'_\nu(x,\xi) \qquad (\mathrm{mod}\ S^{-2})$$

where

$$\sigma'_\nu(x,\xi) \;=\; \frac12\,|\xi|_x^{-1}\,\sigma_\nu(x,\xi) \;+\; \frac i4\,|\xi|_x^{-2}\sum_j \hat\xi^j\,\nabla_{x^j}\sigma_\nu(x,\xi) \;-\; \frac18\,|\xi|_x^{-3}\,\sigma_\nu^2(x,\xi)\,.$$

Proof. Let B be a properly supported ψDO such that $\sigma_B(x,\xi)=|\xi|_x+\sigma(x,\xi)$ as $|\xi|_x>1$, where $\sigma\in S^0$. Since $\sum_{j,k,l} R^{\;p}_{\;jkl}\,\hat\xi^j\,\hat\xi^k\,\hat\xi^l\equiv 0$, we obtain from (8.28) that

$$\begin{aligned}\sigma_{B^2} \;=\; |\xi|_x^2 \;+\; 2\,|\xi|_x\,\sigma \;+\; \sigma^2 \;-\; i\sum_j \hat\xi^j\,\nabla_{x^j}\sigma\\ -\;\frac13\sum_{j,k} R_{jk}\,\hat\xi^j\,\hat\xi^k \;+\; \frac1{12}\,|\xi|_x^{-1}\sum_{j,k,l,p}(R^{\;p}_{\;kjl}+R^{\;p}_{\;ljk})\,g^{kl}\,\xi_p\,\hat\xi^j\end{aligned}$$

modulo S^{-1}. We have

$$\begin{aligned}|\xi|_x^{-1}\sum_{j,k,l,p}(R^{\;p}_{\;klj}+R^{\;p}_{\;lkj})\,g^{kl}\,\xi_p\,\hat\xi^j \;&=\; \sum_{j,k,l,p}(R_{pkjl}+R_{pljk})\,g^{kl}\,\hat\xi^p\,\hat\xi^j\\ &=\; \sum_{j,k,l,p}(R_{kplj}+R_{lpkj})\,g^{kl}\,\hat\xi^p\,\hat\xi^j \;=\; 2\sum_{j,p} R_{pj}\,\hat\xi^p\,\hat\xi^j\,,\end{aligned}$$

and therefore

$$\sigma_{B^2} \;=\; |\xi|_x^2 \;+\; 2\,|\xi|_x\,\sigma \;+\; \sigma^2 \;-\; i\sum_j \hat\xi^j\,\nabla_{x^j}\sigma \;-\; \frac{1}{6}\sum_{j,k} R_{jk}\,\hat\xi^j\,\hat\xi^k$$

modulo S^{-1}. Taking $\sigma = \sigma'_\nu + |\xi|_x^{-1}\big(\sum_{j,k} R_{jk}\,\hat\xi^j\,\hat\xi^k - 2\,S\big)/12$ we obtain $\sigma_{B^2} = \sigma_{A^2}$ (mod S^{-1}). This implies the lemma. $\square$

2. Let $\mathrm{S}_\rho^m(\mathbf{R}^1)$ be the class of functions $\omega \in C^\infty(\mathbf{R}^1)$ such that

$$|\,\partial_s^k\omega(s)\,| \;\leqslant\; \mathrm{const}_k\,(1+|s|)^{m-k\rho}\,, \qquad \forall k = 0,1,\dots$$

If $\omega \in \mathrm{S}_\rho^m(\mathbf{R}^1)$ then $t^k\hat\omega(t) \in C^{N_k}(\mathbf{R}^1)$, where $N_k \to +\infty$ as $k \to +\infty$. Therefore the Fourier transform $\hat\omega(t)$ is a distribution which coincides with a smooth rapidly decreasing function outside any neighbourhood of $t = 0$.

The main result of this section is the following

Theorem 11.2. *Let $\omega \in \mathrm{S}_\rho^m(\mathbf{R}^1)$, $\rho \in (0,1]$. Then $\omega(A) \in \Psi_{\rho,0}^m(\Omega^{1/2},\Gamma)$, where Γ is the Levi–Civita connection. The symbol of the ψDO $\omega(A)$ admits the asymptotic expansion*

$$\sigma_{\omega(A)}(x,\xi) \;\sim\; \omega(|\xi|_x) \;+\; \sum_{j=1}^\infty c_j(x,\xi)\,\omega^{(j)}(|\xi|_x)\,, \qquad |\xi|_x \to \infty\,, \tag{11.4}$$

where $c_j \in S^0$ and $\omega^{(j)}(s) := \partial_s^j\omega(s)$. The functions c_j are determined recursively from the system of equations

$$\sigma_{A^k}(x,\xi) \;=\; |\xi|_x^k \;+\; \sum_{j=1}^k \frac{k!}{(k-j)!}\,|\xi|_x^{k-j}\,c_j(x,\xi)\,. \tag{11.5}$$

In particular,

$$c_1(x,\xi) \;=\; \frac{1}{12}\,|\xi|_x^{-1}\Big(\sum_{j,k} R_{jk}(x)\,\hat\xi^j\,\hat\xi^k - 2\,S(x)\Big) \;+\; \sigma'_\nu(x,\xi) \quad (\mathrm{mod}\ S^{-2}) \tag{11.6}$$

where σ'_ν is defined in Lemma 11.1, and

$$c_2(x,\xi) \;=\; -\,\frac{1}{12}\sum_{j,k} R_{jk}(x)\,\hat\xi^j\,\hat\xi^k \\ \qquad -\;\frac{i}{4}\,|\xi|_x^{-1}\sum_j \hat\xi^j\,\nabla_{x^j}\sigma_\nu(x,\xi) \;+\; \frac{1}{8}\,|\xi|_x^{-2}\,\sigma_\nu^2(x,\xi) \quad (\mathrm{mod}\ S^{-1})\,. \tag{11.7}$$

Remark 11.3. Since $|\xi|_x$ is constant along the horizontal curves generated by the connection Γ, we have $\omega^{(j)}(|\xi|_x) \in \mathrm{S}_{\rho,0}^{m-j\rho}$.

Remark 11.4. In [T, Ch. XII.3] it has been proved that $\omega(A)$ belongs to the coordinate class $\Psi_{\rho,1-\rho}^m(\Omega^{1/2})$ when $\rho > 1/2$. The author has also conjectured that $\omega(A)$ can be included in some appropriate classes of ψDOs when $0 < \rho \leqslant 1/2$. By Theorem 11.2 the appropriate classes corresponding to operators of the form (11.1) are those generated by the Levi–Civita connection.

Theorem 11.2 immediately implies

Corollary 11.5. *Let A_1 and A_2 be positive differential operators of the form (11.1) with $\nu = \nu_1$ and $\nu = \nu_2$ respectively. Let $\omega \in \mathrm{S}^m_\rho(\mathbf{R}^1)$ and $B = \omega(A_1) - \omega(A_2)$. Then $B \in \Psi^{m-\rho}_{\rho,0}(\Omega^{1/2},\Gamma)$ where Γ is the Levi–Civita connection, and*

$$\begin{aligned}\sigma_B \;=\; & \frac{1}{2}\,|\xi|_x^{-1}\,\omega'(|\xi|_x)\left(\sigma_{\nu_1} - \sigma_{\nu_2}\right) \\ & + \frac{i}{4}\,|\xi|_x^{-1}\left(|\xi|_x^{-1}\,\omega'(|\xi|_x) - \omega''(|\xi|_x)\right)\sum_j \hat{\xi}^j\,\nabla_{x^j}\left(\sigma_{\nu_1} - \sigma_{\nu_2}\right) \\ & - \frac{1}{8}\,|\xi|_x^{-2}\left(|\xi|_x^{-1}\,\omega'(|\xi|_x) - \omega''(|\xi|_x)\right)\left(\sigma^2_{\nu_1} - \sigma^2_{\nu_2}\right)\end{aligned}$$

modulo $\mathrm{S}^{m-3\rho}_{\rho,0}(\Gamma)$.

3. From now on we assume that Γ is the Levi–Civita connection. We will deduce Theorem 11.2 from

Proposition 11.6. *Let $\lambda \in [0,1)$ and $\mathbf{U}_\lambda(t) = \exp(itA^\lambda)$. Then*

$$\mathbf{U}_\lambda(t) \in \Psi^0_{1-\lambda,0}(\Omega^{1/2},\Gamma)\,, \qquad \sigma_{\mathbf{U}_\lambda(t)} \;=\; e^{it|\xi|_x^\lambda}\,b^{(\lambda)}(t;x,\xi)\,, \qquad \forall t \in \mathbf{R}^1\,,$$

where $b^{(\lambda)} \in C^\infty(\mathbf{R}^1 \times T^*M)$ and $D_t^k b^{(\lambda)} \in S^0$, $\forall t \in \mathbf{R}^1$, $\forall k = 0,1,\dots$ Moreover,

$$b^{(\lambda)}(t;x,\xi) \;\sim\; 1 \;+\; \sum_{j=1}^{\infty}(it)^j\,b_j^{(\lambda)}(x,\xi)\,, \qquad |\xi|_x \to \infty\,, \tag{11.8}$$

where $b_j^{(\lambda)} \in S^{-j(1-\lambda)}$, $\forall k = 0,1,\dots$

The proof of Proposition 11.6 is based on the following technical lemmas.

Lemma 11.7. *Let $P^{(1/2)}_{\beta,\gamma}$ be the polynomials of degree $d^{(1/2)}_{\beta,\gamma}$ defined by (8.3). Then*

$$\sum_{|\gamma|=d}\frac{\hat{\xi}_\gamma}{\gamma!}\,P^{(1/2)}_{\beta,\gamma}(x,\xi) \;\equiv\; 0$$

for all d and β such that $|\beta| < 2d^{(1/2)}_{\beta,\gamma}$.

Proof. Since the coefficients of polynomials $P^{(1/2)}_{\beta,\gamma}$ are components of some polynomials in the curvature tensor and its symmetric covariant differentials, the sum

$$\sum_{|\gamma|=d} P^{(1/2)}_{\beta,\gamma}(x,\xi)\,\hat{\xi}_\gamma/\gamma!$$

coincides with a linear combination of sums of the form

$$\sum \mathbf{D}^{\alpha^{(1)}}R^{\,i_1}_{\;j_1k_1l_1}(x)\dots\mathbf{D}^{\alpha^{(q)}}R^{\,i_q}_{\;j_qk_ql_q}(x)\,\xi_{i_1}\dots\xi_{i_p}\,\hat{\xi}^{n_1}\dots\hat{\xi}^{n_d}\,, \tag{11.9}$$

where $p \leqslant d^{(1/2)}_{\beta,\gamma}$, $q \geqslant p$ and

$$d + q + |\beta| = 3q + p + |\alpha^{(1)}| + \dots + |\alpha^{(q)}|. \tag{11.10}$$

In (11.9) the sum is taken over the indices $i_1, \dots, i_p$, and over the remaining upper indices $i_p, \dots, i_q$, $n_1, \dots, n_d$ and some $(q+d-p)$ lower indices (which are j_s , k_s , l_s and those corresponding to $\alpha^{(s)}$).

When $|\beta| < 2p$, (11.10) implies that

$$d > 2q - p + |\alpha^{(1)}| + \dots + |\alpha^{(q)}|.$$

Then (11.9) contains at least one partial sum of the form $\sum\limits_{j,k,l} \mathbf{D}^{\alpha^{(s)}} R^{\,i}_{\;jkl}\, \hat\xi^j\, \hat\xi^k\, \hat\xi^l$ or $\sum\limits_{i,k,l} \mathbf{D}^{\alpha^{(s)}} R^{\,i}_{\;jkl}\, \xi_i\, \hat\xi^k\, \hat\xi^l$ or $\sum\limits_{i,j,k} \mathbf{D}^{\alpha^{(s)}} R^{\,i}_{\;jkl}\, \xi_i\, \hat\xi^j\, \hat\xi^k$, which vanishes due to the symmetries of the curvature tensor (see Section 1). This proves the lemma. $\square$

Lemma 11.8. *Let $A_1 \in \Psi^{m_1}(\Omega^{1/2})$, $\omega \in \mathrm{S}^0_\rho(\mathbf{R}^1)$, and B be a properly supported ψDO with a symbol of the form $\sigma_B(x,\xi) = \omega(|\xi|_x)\, b(x,\xi)$ where $b \in S^m$. Then*

$$\sigma_{A_1 B}(x,\xi) \sim \sum_{k=0}^{\infty} \omega^{(k)}(|\xi|_x)\, b_k(x,\xi)\,, \tag{11.11}$$

where $b_0 - \sigma_{A_1} b \in S^{m+m_1-1}$ and $b_j \in S^{m+m_1-1}$ for $j \geqslant 1$. The symbols b_k admit the asymptotic expansions

$$b_k(x,\xi) \sim \sum_{\alpha,\beta,\gamma} \sum_{\substack{\gamma' \leqslant \gamma, \\ |\gamma'| \geqslant k}} \frac{1}{\alpha!}\,\frac{1}{\beta!}\,\frac{1}{\gamma'!}\,\frac{(-i)^{|\gamma'|}}{(\gamma-\gamma')!}\; P^{(1/2)}_{\beta,\gamma}(x,\xi)$$
$$\times\, a_{k,\gamma'}(x,\xi)\; D_\xi^{\alpha+\beta}\sigma_{A_1}(x,\xi)\; D_\xi^{\gamma-\gamma'}\nabla_x^\alpha b(x,\xi)\,, \tag{11.12}$$

where $a_{k,\gamma'}$ are functions homogeneous in ξ of degree $k - |\gamma'|$ depending only on the Riemannian metric $\{g^{ij}\}$. In particular, $a_{|\gamma'|,\gamma'} = \hat\xi_{\gamma'}$.

Proof. Obviously, $B \in \Psi^m_{\rho,0}(\Omega^{1/2},\Gamma)$. By Theorem 8.4 $A_1 B \in \Psi^{m+m_1}_{\rho,0}(\Omega^{1/2},\Gamma)$ and

$$\sigma_{A_1 B}(x,\xi) \sim \sum_{\alpha,\beta,\gamma} \frac{1}{\alpha!}\,\frac{1}{\beta!}\,\frac{1}{\gamma!}\; P^{(1/2)}_{\beta,\gamma}(x,\xi)\; D_\xi^{\alpha+\beta}\sigma_{A_1}(x,\xi)\; D_\xi^{\gamma}\nabla_x^\alpha \sigma_B(x,\xi)$$
$$= \sum_{\alpha,\beta,\gamma} \sum_{\gamma' \leqslant \gamma} \frac{1}{\alpha!}\,\frac{1}{\beta!}\,\frac{1}{\gamma'!}\,\frac{(-i)^{|\gamma'|}}{(\gamma-\gamma')!}\; P^{(1/2)}_{\beta,\gamma}(x,\xi)$$
$$\times\, D_\xi^{\alpha+\beta}\sigma_{A_1}(x,\xi)\; \partial_\xi^{\gamma'}\omega(|\xi|_x)\; D_\xi^{\gamma-\gamma'}\nabla_x^\alpha b(x,\xi)\,.$$

Differentiating $\omega(|\xi|_x)$ we obtain

$$\partial_\xi^{\gamma'}\omega(|\xi|_x) \;=\; \sum_{k\leqslant|\gamma'|} a_{k,\gamma'}(x,\xi)\,\omega^{(k)}(|\xi|_x)\,,$$

where $a_{k,\gamma'}$ are functions homogeneous in ξ of degree $|\gamma'|-k$ depending only on the Riemannian metric, and $a_{k,\gamma'}=\hat\xi_{\gamma'}$ as $k=|\gamma'|$. Therefore the asymptotic expansion for σ_{A_1B} can be rewritten in the form (11.11) with b_k satisfying (11.12). From (11.12) it follows that $b_0-\sigma_{A_1}b\in S^{m+m_1-1}$. It remains to prove that $b_k\in S^{m+m_1-1}$ for $k\geqslant 1$.

In the asymptotic expansions (11.12) with $k>1$ the orders of the terms are

$$m\;+\;m_1\;+\;d^{(1/2)}_{\beta,\gamma}\;-\;|\alpha|\;-\;|\beta|\;-\;|\gamma|\;+\;k\,, \tag{11.13}$$

where $d^{(1/2)}_{\beta,\gamma}$ are the degrees of the polynomials $P^{(1/2)}_{\beta,\gamma}$. By (8.4) we have $d^{(1/2)}_{\beta,\gamma}\leqslant|\beta|$. Therefore (11.13) is estimated by $m+m_1-1$ if $k<|\gamma|$.

Let us consider the terms with $k=|\gamma|$. Then $\gamma'=\gamma$ and $a_{k,\gamma'}=\hat\xi_{\gamma'}$. By Lemma 11.7 the sum of the terms with $|\beta|<2d^{(1/2)}_{\beta,\gamma}$ is equal to zero. Since $P^{(1/2)}_{0,\gamma}\equiv 0$ for $\gamma\neq 0$ (Proposition 8.2), all the terms with $\beta=0$ are also equal to zero. Now the estimates $|\beta|\geqslant 1$ and $|\beta|\geqslant 2d^{(1/2)}_{\beta,\gamma}$ imply that (11.13) is not greater than $m+m_1-1$. $\square$

Proof of Proposition 11.6. $\mathbf{U}_\lambda(t)$ is a unique solution of the Cauchy problem

$$D_t\mathbf{U}_\lambda(t)\;-\;A^\lambda\,\mathbf{U}_\lambda(t)\;=\;0\,, \tag{11.14}$$

$$\mathbf{U}_\lambda(0)\;=\;I\,. \tag{11.15}$$

We will construct a properly supported ψDO $U_\lambda(t)\in\Psi^0_{1-\lambda,0}(\Omega^{1/2},\Gamma)$ smoothly dependent on t which satisfies (11.14) and (11.15) modulo $\Psi^{-\infty}$. Then from the well-known a priori estimates it follows that $D_t^k\big(\mathbf{U}_\lambda(t)-U_\lambda(t)\big)\in\Psi^{-\infty}$ for all k and, consequently, $\mathbf{U}_\lambda(t)\in\Psi^0_{1-\lambda,0}(\Omega^{1/2},\Gamma)$ and

$$D_t^k(\sigma_{\mathbf{U}_\lambda(t)}-\sigma_{U_\lambda(t)})\in S^{-\infty}\,,\qquad \forall t\in\mathbf{R}^1\,,\ \forall k=0,1\dots$$

Obviously, for all $t\in\mathbf{R}^1$ the functions $f_t(s)=e^{its^\lambda}$ belong to $\mathrm{S}^0_{1-\lambda}(\mathbf{R}^1)$ (outside a neighbourhood of $s=0$). Let B be a properly supported ψDO with a symbol of the form $e^{it|\xi|_x^\lambda}\,b(t;x,\xi)$, where $D_t^kb\in S^m$, $\forall t\in\mathbf{R}^1$, $\forall k=0,1\dots$ Then by Lemma 11.8

$$\sigma_{A^\lambda B}\;=\;e^{it|\xi|_x^\lambda}\,|\xi|_x^\lambda\,b(t;x,\xi)\;+\;e^{it|\xi|_x^\lambda}\,\mathcal{L}_\lambda b(t;x,\xi)\,, \tag{11.16}$$

where $D_t^k(\mathcal{L}_\lambda b)\in S^{m+\lambda-1}$, $\forall t\in\mathbf{R}^1$, $\forall k=0,1\dots$ Moreover, (11.11) and (11.12) imply that

$$\mathcal{L}_\lambda b(t;x,\xi)\;\sim\;\sum_{k=0}^{\infty}t^k\,\mathcal{L}^{(k)}_\lambda b(t;x,\xi)\,,\qquad |\xi|_x\to\infty\,, \tag{11.17}$$

where $\mathcal{L}_\lambda^{(k)} b \in S^{m-1-k(1-\lambda)}$ and

$$t^j D_t^l(\mathcal{L}_\lambda^{(k)} b) \;=\; \mathcal{L}_\lambda^{(k)}(t^j D_t^l b)\,, \qquad \forall t \in \mathbf{R}^1\,, \quad \forall j,k,l = 0,1\dots \tag{11.18}$$

Let $U_\lambda(t)$ be a properly supported ψDO such that $\sigma_{U_\lambda(t)} = e^{it|\xi|_x^\lambda}\, b^{(\lambda)}(t;x,\xi)$, where

$$b^{(\lambda)} \;\sim\; \tilde b_0^{(\lambda)} \;+\; \tilde b_1^{(\lambda)} \;+\; \tilde b_2^{(\lambda)} \;+\; \dots\,, \qquad |\xi|_x \to \infty\,,$$

$$\tilde b_0^{(\lambda)} \equiv 1\,, \quad \tilde b_k^{(\lambda)}(t;x,\xi) \;=\; i\int_0^t \mathcal{L}_\lambda \tilde b_{k-1}^{(\lambda)}(s;x,\xi)\,ds\,, \qquad |\xi|_x \geqslant 1\,, \tag{11.19}$$

and $\mathcal{L}_\lambda$ is the operator defined by (11.16). Then $b^{(\lambda)}(0;x,\xi) = 1$, and so $U_\lambda(t)$ satisfies (11.15). The operator $(D_t - A^\lambda)U_\lambda(t)$ is a ψDO with the symbol

$$\begin{aligned} &e^{it|\xi|_x^\lambda}\left(D_t b^{(\lambda)} - \mathcal{L}_\lambda b^{(\lambda)}\right) \\ &\quad\sim\; e^{it|\xi|_x^\lambda}\, D_t \tilde b_0^{(\lambda)} \;+\; e^{it|\xi|_x^\lambda}\left(D_t \tilde b_1^{(\lambda)} - \mathcal{L}_\lambda \tilde b_0^{(\lambda)}\right) \;+\; e^{it|\xi|_x^\lambda}\left(D_t \tilde b_2^{(\lambda)} - \mathcal{L}_\lambda \tilde b_1^{(\lambda)}\right) \;+\dots \end{aligned}$$

By (11.19) all the terms in this asymptotic series are equal to zero as $|\xi|_x \geqslant 1$. Therefore $(D_t - A^\lambda)U_\lambda(t) \in \Psi^{-\infty}$.

Finally, substituting (11.17) in (11.19) and taking into account (11.18), we obtain the asymptotic expansion (11.8). $\square$

Proof of Theorem 11.2. The operator A is positive, and therefore we can assume that $\operatorname{supp}\omega \in (0,+\infty)$. Let us fix $\lambda \in (1-\rho,1)$ and set $\omega_\lambda(s) = \omega(s^{1/\lambda})$. Then $\omega_\lambda \in \mathrm{S}^{m/\lambda}_{1-(1-\rho)/\lambda}(\mathbf{R}^1)$, where $1-(1-\rho)/\lambda > 0$.

Since the Fourier transform $\hat\omega_\lambda(t)$ coincides with a rapidly decreasing function for large t, we have

$$\omega(A) \;=\; \int \hat\omega_\lambda(t)\, e^{itA^\lambda}\, dt\,,$$

where the integral converges in the weak operator topology. Let $\varsigma \in C_0^\infty(\mathbf{R}^1)$, $\varsigma(t) = 1$ in a neighbourhood of $t = 0$ and $\varsigma(t) = 0$ for large t. Then

$$\int \big(1-\varsigma(t)\big)\hat\omega_\lambda(t)\, e^{itA^\lambda}\, dt \;=\; A^{-k\lambda}\int (-D_t)^k \Big(\big(1-\varsigma(t)\big)\,\hat\omega_\lambda(t)\Big)\, e^{itA^\lambda}\, dt$$

for all positive integer k, and therefore

$$\omega(A) \;=\; \int \varsigma(t)\,\hat\omega_\lambda(t)\, e^{itA^\lambda}\, dt \qquad (\mathrm{mod}\ \Psi^{-\infty})\,.$$

Now from Proposition 11.6 it follows that $\omega(A)$ is a ψDO with the symbol

$$\sigma_{\omega(A)} \;=\; \int e^{it|\xi|_x^\lambda}\,\hat\omega_\lambda(t)\,\varsigma(t)\, b^{(\lambda)}(t;x,\xi)\, dt\,. \tag{11.20}$$

Substituting in (11.20) the asymptotic expansion (11.8) we get

$$\sigma_{\omega(A)} \;\sim\; \omega(|\xi|_x) \;+\; \sum_{j=1}^\infty \omega_{\lambda,j}(|\xi|_x)\, b_j^{(\lambda)}(x,\xi)\,, \qquad |\xi|_x \to \infty\,,$$

where $\omega_{\lambda,j}(s) = \left. d^j\big(\omega(r^{1/\lambda})\big)\big/\, dr^j \right|_{r=s^\lambda}$. This implies (11.4) with some symbols $c_j \in S^0$. Taking $\omega(s) = s^k$ we obtain from (11.4) the equations (11.5). The equalities (11.6) and (11.7) immediately follow from (11.2), Lemma 11.1 and (11.5). $\square$

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Department of Mathematics, King's College,
Strand, London WC2R 2LS, England
E-mail address: ysafarov@mth.kcl.ac.uk