# Estimates for the counting function of the Laplace operator on domains with rough boundaries

Y. Netrusov and Y. Safarov

This is a survey of results obtained by the authors in the last few years. Most of them were proved or implicitly stated in our papers [Ne], [NS] and [Sa]; we give precise references or outline proofs wherever it is possible. The results announced in Subsection 5.2 are new.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain in $\mathbb{R}^n$, and let $-\Delta_{\mathrm{B}}$ be the Laplacian on $\Omega$ subject to Dirichlet ($\mathrm{B}=\mathrm{D}$) or Neumann ($\mathrm{B}=\mathrm{N}$) boundary condition. Further on we use the lower index B in the cases where the corresponding statement refers to (or result holds for) both the Dirichlet and Neumann Laplacian. Let $N_{\mathrm{B}}(\Omega,\lambda)$ be the number of eigenvalues of $\Delta_{\mathrm{B}}$ lying below $\lambda^2$; if the number of these eigenvalues is infinite or $-\Delta_{\mathrm{B}}$ has essential spectrum below $\lambda^2$ then we define $N_{\mathrm{N}}(\Omega,\lambda) := +\infty$. Let

$$R_{\mathrm{B}}(\Omega,\lambda) \;:=\; N_{\mathrm{B}}(\Omega,\lambda) - (2\pi)^{-n}\,\omega_n\,|\Omega|\,\lambda^n\,,$$

where $\omega_n$ is the volume of the $n$-dimensional unit ball and $|\Omega|$ denotes the volume of $\Omega$. According to the Weyl formula, $R_{\mathrm{B}}(\Omega,\lambda) = o(\lambda^n)$ as $\lambda \to +\infty$. If $\mathrm{B}=\mathrm{D}$ then this is true for every bounded domain [BS]. If $\mathrm{B}=\mathrm{N}$ then the Weyl formula holds only for domains with sufficiently regular boundaries. In the general case $R_{\mathrm{N}}$ may well grow faster than $\lambda^n$; moreover, the Neumann Laplacian on a bounded domain may have a nonempty essential spectrum (see, for instance, Remark 5 or [HSS]). The necessary and sufficient conditions for the absence of the essential spectrum in terms of capacities were obtained by V. Maz'ya [M1].

The aim of this note is to present estimates for $R_{\mathrm{B}}(\Omega,\lambda)$, which involve only the most basic characteristics of $\Omega$ and constants depending only on the dimension $n$. The estimate from below (2) for $R_{\mathrm{B}}(\Omega,\lambda)$ and the estimate from above (3)

---

Y. Netrusov
Department of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK.
e-mail: y.netrusov@bristol.ac.uk

Y. Safarov
Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK.
e-mail: yuri.safarov@kcl.ac.uk

for $R_{\mathrm{D}}(\Omega,\lambda)$ hold for all bounded domains. The upper bound (4) for $R_{\mathrm{N}}(\Omega,\lambda)$ is obtained for domains $\Omega$ of class $C$, that is, under the following assumption:

- every point $x \in \partial\Omega$ has a neighbourhood $U_x$ such that $\Omega \cap U_x$ coincides (in a suitable coordinate system) with the subgraph a continuous function $f_x$.

If all the functions $f_x$ satisfy the Hölder condition of order $\alpha$, one says that $\Omega$ belongs to the class $C^\alpha$. For domains $\Omega \in C^\alpha$ with $\alpha \in (0,1)$, our estimates $R_{\mathrm{D}}(\Omega,\lambda) = O(\lambda^{n-\alpha})$ and $R_{\mathrm{N}}(\Omega,\lambda) = O(\lambda^{(n-1)/\alpha})$ are order sharp in the scale $C^\alpha$ as $\lambda \to \infty$. The latter estimate implies that the Weyl formula holds for the Neumann Laplacian whenever $\alpha > 1 - \frac{1}{n}$. If $\alpha \le 1 - \frac{1}{n}$ then there exist domains in which the Weyl formula for $N_{\mathrm{N}}(\Omega,\lambda)$ fails (see Remark 4 for details or [NS] for more advanced results).

For domains of class $C^\alpha$ with $\alpha \ge 1$, our methods only give the known remainder estimate $R_{\mathrm{B}}(\Omega,\lambda) = O(\lambda^{n-1}\log\lambda)$. To obtain the order sharp estimate $O(\lambda^{n-1})$ one has to use more sophisticated techniques. The most advanced results in this direction were obtained in [Iv], where the estimate $R_{\mathrm{B}}(\Omega,\lambda) = O(\lambda^{n-1})$ was established for domains which belong to a slightly better class than $C^1$.

Throughout the paper we shall be using the following notation.

- $d(x)$ is the Euclidean distance from the point $x \in \Omega$ to the boundary $\partial\Omega$.
- $\Omega^{\mathrm{b}}_\delta := \{x \in \Omega \,|\, d(x) \le \delta\}$ is the internal closed $\delta$-neighbourhood of $\partial\Omega$.
- $\Omega^{\mathrm{i}}_\delta := \Omega \setminus \Omega^{\mathrm{b}}_\delta$ is the interior part of $\Omega$.

## 1 Lower bounds

Denote by $\Pi_{\mathrm{B}}(\lambda)$ the spectral projection of the operator $-\Delta_{\mathrm{B}}$ corresponding to the interval $[0,\lambda^2)$, and let $e_{\mathrm{B}}(x,y;\lambda)$ be its integral kernel (the so-called *spectral function*). It is well known that $e_{\mathrm{B}}(x,y;\lambda)$ is a infinitely differentiable function on $\Omega\times\Omega$ for each fixed $\lambda$ and that $e_{\mathrm{B}}(x,x;\lambda)$ is a nondecreasing polynomially bounded function of $\lambda$ for each fixed $x \in \Omega$.

By the spectral theorem, the cosine Fourier transform of $\frac{\mathrm{d}}{\mathrm{d}\lambda} e_{\mathrm{B}}(x,y;\lambda)$ coincides with the fundamental solution $u_{\mathrm{B}}(x,y;t)$ of the wave equation in $\Omega$. On the other hand, due to the finite speed of propagation, $u_{\mathrm{B}}(x,x;t)$ is equal to $u_0(x,x;t)$ whenever $t \in (-d(x),d(x))$, where $u_0(x,y;t)$ is the fundamental solution of the wave equation in $\mathbb{R}^n$. By direct calculation, $u_0(x,x;t)$ is independent of $x$ and coincides with the cosine Fourier transform of the function $n\,(2\pi)^{-n}\omega_n\,\lambda_+^{n-1}$. Applying the Fourier Tauberian theorem proved in [Sa], we obtain

$$|e_{\mathrm{B}}(x,x;\lambda) - (2\pi)^{-n}\omega_n\,\lambda^n| \;\le\; \frac{2n(n+2)^2\,(2\pi)^{-n}\omega_n}{d(x)} \left(\lambda + \frac{(n+2)\,\sqrt[n+2]{3}}{d(x)}\right)^{n-1} \tag{1}$$

for all $x \in \Omega$ and $\lambda > 0$ [Sa, Corollary 3.1]. Since

$$N_{\mathrm{B}}(\Omega,\lambda) \;=\; \int_{\Omega} e_{\mathrm{B}}(x,x;\lambda)\,\mathrm{d}x \;\geq\; \int_{\Omega^{\mathrm{i}}_{\delta}} e_{\mathrm{B}}(x,x;\lambda)\,\mathrm{d}x$$

for all $\delta > 0$, integrating (1) over $\Omega^{\mathrm{i}}_{\lambda^{-1}}$, we arrive at

$$R_{\mathrm{B}}(\lambda,\Omega) \;\geq\; -2n(n+2)^2\,(2\pi)^{-n}\omega_n\left(1+(n+2)\,\sqrt[n+2]{3}\right)^{n-1}\lambda^{n-1}\int_{\Omega^{\mathrm{i}}_{\lambda^{-1}}}\frac{\mathrm{d}x}{d(x)}\,.$$

Estimating constants and taking into account the obvious inequality

$$\int_{\Omega^{\mathrm{i}}_{\delta}}\frac{\mathrm{d}x}{d(x)} \;=\; \int_{\delta}^{\infty} s^{-1}\,\mathrm{d}(|\Omega^{\mathrm{b}}_{s}|) \;\leq\; \int_{0}^{\delta^{-1}} |\Omega^{\mathrm{b}}_{t^{-1}}|\,\mathrm{d}t\,,$$

we see that

$$R_{\mathrm{B}}(\lambda,\Omega) \;\geq\; -C_{n,1}\,\lambda^{n-1}\int_{\lambda^{-1}}^{\infty} s^{-1}\,\mathrm{d}(|\Omega^{\mathrm{b}}_{s}|) \;\geq\; -C_{n,1}\,\lambda^{n-1}\int_{0}^{\lambda}|\Omega^{\mathrm{b}}_{t^{-1}}|\,\mathrm{d}t \qquad (2)$$

for all $\lambda > 0$, where $C_{n,1} := \frac{2\,(n+2)^{n+1}}{\pi^{n/2}\,\Gamma(n/2)}$ and $\Gamma$ is the gamma-function.

## 2 Variational formulae

In order to obtain upper bounds for $R_{\mathrm{B}}(\lambda,\Omega)$ we need to estimate the contribution of $\Omega^{\mathrm{b}}_{\delta}$. For the Neumann Laplacian, $\int_{\Omega^{\mathrm{b}}_{\delta}} e_{\mathrm{N}}(x,x;\lambda)\,\mathrm{d}x$ may well not be polynomially bounded, even if $\Omega \in C$. In this case the Fourier Tauberian theorems are not applicable. Instead, we use the variational technique.

The idea is to represent $\Omega$ as the union of relatively simple domains and estimate the counting functions for each of these domains. Then upper bounds for $N_{\mathrm{B}}(\lambda,\Omega)$ are obtained with the use of the following two lemmas.

Let $N_{\mathrm{N,D}}(\tilde{\Omega},\Upsilon,\lambda)$ be the counting function of the Laplacian on $\tilde{\Omega}$ with Dirichlet boundary condition on $\Upsilon \subset \partial\tilde{\Omega}$ and Neumann boundary condition on $\partial\tilde{\Omega}\setminus\Upsilon$.

**Lemma 1.** *If $\{\Omega_i\}$ is a countable family of disjoint open sets $\Omega_i \subset \Omega$ such that $|\Omega| = |\cup_i \Omega_i|$ then*

$$\sum_i N_{\mathrm{D}}(\Omega_i,\lambda) \;\leq\; N_{\mathrm{D}}(\Omega,\lambda) \;\leq\; N_{\mathrm{N}}(\Omega,\lambda) \;\leq\; \sum_i N_{\mathrm{N}}(\Omega_i,\lambda)$$

*and*

$$N_{\mathrm{N}}(\Omega,\lambda) \;\geq\; \sum_i N_{\mathrm{N,D}}(\Omega_i,\partial\Omega_i\setminus\partial\Omega,\lambda)\,.$$

*Proof.* The lemma is an elementary corollary of the Rayleigh–Ritz formula.

Given a collection of sets $\{\Omega_j\}$, let us denote by $\aleph\{\Omega_j\}$ the multiplicity of the covering $\{\Omega_j\}$, that is, the maximal number of the sets $\Omega_j$ containing a common element.

**Lemma 2.** *Let $\{\Omega_j\}$ be a countable family of open sets $\Omega_j \subset \Omega$ such that $|\Omega| = |\cup_j \Omega_j|$ and $\aleph\{\Omega_j\} \le \varkappa < +\infty$. If $\Upsilon \subset \partial\Omega$ and $\Upsilon_j := \partial\Omega_j \bigcap \Upsilon$ then*

$$N_{\mathrm{N,D}}(\Omega, \Upsilon, \varkappa^{-1/2}\lambda) \;\le\; \sum_j N_{\mathrm{N,D}}(\Omega_j, \Upsilon_j, \lambda).$$

*Proof.* See Lemma 2.2 in [NS].

*Remark 1.* Lemmas 1 and 2 remain valid for more general differential operator. This allows one to extend our results to some classes of higher order operators (see [NS]).

## 3 Partitions of $\Omega$

The following theorem is due to H. Whitney.

**Theorem 1.** *There exists a countable family $\{Q_{i,m}\}_{m\in\mathcal{M}_i,\, i\in\mathcal{I}}$ of mutually disjoint open $n$-dimensional cubes $Q_{i,m}$ with edges of length $2^{-i}$ such that*

$$\overline{\Omega} = \bigcup_{i\in\mathcal{I}} \bigcup_{m\in\mathcal{M}_i} \overline{Q_{i,m}} \quad \text{and} \quad Q_{i,m} \subset \left(\Omega^{\mathrm{b}}_{4\delta_i} \setminus \Omega^{\mathrm{b}}_{\delta_i}\right)$$

*where $\delta_i := \sqrt{n}\,2^{-i}$, $\mathcal{I}$ is a subset of $\mathbb{Z}$, and $\mathcal{M}_i$ are some finite index sets.*

*Proof.* See, for example, Chapter VI in [St].

**Lemma 3.** *For every $\delta > 0$ there exists a finite family of disjoint open sets $\{M_k\}$ such that*

*(i) each set $M_k$ coincides with the intersection of $\Omega$ and an open $n$-dimensional cube with edges of length $\delta$;*
*(ii) $\Omega^{\mathrm{b}}_{\delta_0} \subset \bigcup_k \overline{M_k} \subset \Omega^{\mathrm{b}}_{\delta_1} \bigcup \partial\Omega$, where $\delta_0 := \delta/\sqrt{n}$ and $\delta_1 := \sqrt{n}\,\delta + \delta/\sqrt{n}$.*

*Proof.* Consider an arbitrary covering of $\mathbb{R}^n$ by disjoint cubes of size $\delta$ and select the cubes which have nonempty intersections with $\Omega$.

Theorem 1 and Lemma 3 imply that $\Omega$ can be represented (modulo a set of measure zero) as the union of Whitney cubes and the subsets $M_k$ lying in cubes of size $\delta$. This is sufficient to estimate $R_{\mathrm{D}}(\lambda, \Omega)$. However, the condition (1) of Lemma 3 does not imply any estimates for $N_{\mathrm{N}}(\lambda, M_k)$. In order to obtain an upper bound for $R_{\mathrm{N}}(\lambda, \Omega)$, one has to consider a more sophisticated partition of $\Omega$.

If $\Omega'$ is an open $(d-1)$-dimensional set and $f$ is a continuous real-valued function on the closure $\overline{\Omega'}$, let

- $G_{f,b}(\Omega') := \{x \in \mathbb{R}^n \,|\, b < x_d < f(x'),\ x' \in \Omega'\}$, where $b$ is a constant such that $\inf f > b$;
- $\operatorname{Osc}(f, \Omega') := \sup_{x' \in \Omega'} f(x') - \inf_{x' \in \Omega'} f(x')$;
- $\mathscr{V}_\delta(f, \Omega')$ be the maximal number of disjoint $(n-1)$-dimensional cubes $Q'_i \subset \Omega'$ such that $\operatorname{Osc}(f, Q'_i) \geq \delta$ for each $i$.

If $n = 2$ then, roughly speaking, $\mathscr{V}_\delta(f, \Omega')$ coincides with the maximal number of oscillations of $f$ which are not smaller than $\delta$. Let

- $\mathbf{V}(\delta)$ be the class of domains $V$ which are represented in a suitable coordinate system in the form $V = G_{f,b}(Q')$, where $Q'$ is an $(n-1)$-dimensional cube with edges of length not greater than $\delta$, $f : \overline{Q'} \mapsto \mathbb{R}$ is a continuous function, $b = \inf f - \delta$ and $\operatorname{Osc}(f, Q') \leq \delta/2$;
- $\mathbf{P}(\delta)$ the set of $n$-dimensional rectangles such that the length of the maximal edge does not exceed $\delta$.

Assume that $\Omega \in C$. Then there is a finite collection of domains $\Omega_l \subset \Omega$ such that $\Omega_l = G_{f_l,b_l}(Q'_l) \in \mathbf{V}(\delta_l)$ with some $\delta_l > 0$ and $\partial\Omega \subset \bigcup_{l \in \mathscr{L}} \overline{\Omega_l}$. Let us fix such a collection, and let

- $n_\Omega$ be the number of the sets $\Omega_l$;
- $\mathscr{V}_\delta(\Omega) := \max\{1, \mathscr{V}_\delta(f_1, Q'_1), \mathscr{V}_\delta(f_2, Q'_2), \ldots\}$;
- $\delta_\Omega$ be the largest positive number such that $\Omega^{\mathrm{b}}_{\delta_\Omega} \subset \bigcup_{l \in \mathscr{L}} \Omega_l$, $\delta_\Omega \leq \operatorname{diam} Q'_l$ and $2\delta_\Omega \leq \inf f_l - b_l$ for all $l$.

**Theorem 2.** *Let $\Omega \in C$. Then for each $\delta \in (0, \delta_\Omega]$ there exist finite families of sets $\{P_j\}$ and $\{V_k\}$ satisfying the following conditions:*

*(i)* $P_j \in \mathbf{P}(\delta)$ *and* $V_k \in \mathbf{V}(\delta)$;
*(ii)* $\aleph\{P_j\} \leq 4^n n_\Omega$ *and* $\aleph\{V_k\} \leq 4^{n-1} n_\Omega$;
*(iii)* $\Omega^{\mathrm{b}}_{\delta_0} \subset \cup_{j,k} \left(\overline{P_j} \bigcup \overline{V_k}\right) \subset \Omega^{\mathrm{b}}_{\delta_1}$, *where* $\delta_0 := \delta/\sqrt{n}$ *and* $\delta_1 := \sqrt{n}\,\delta + \delta/\sqrt{n}$;
*(iv)* $\#\{V_k\} \leq 2^{3(n-1)} \left(3^{n-1} \mathscr{V}_{\delta/2}(\Omega) + n_\Omega\, \delta^{-n} |\Omega^{\mathrm{b}}_{\delta_1}|\right)$ *and*

$$\#\{P_j\} \leq 2^{3n-1} 3^{n-1} \delta^{-1} \int_{(2\operatorname{diam}\Omega)^{-1}}^{4/\delta} t^{-2} \mathscr{V}_{t^{-1}}(\Omega)\,\mathrm{d}t + 2^{3n} n^{n/2} n_\Omega\, \delta^{-n} |\Omega^{\mathrm{b}}_{\delta_1}|.$$

*Proof.* The theorem follows from Corollary 3.8 in [NS].

## 4 Upper bounds

The counting functions of the Laplacian on Whitney cubes can be evaluated explicitly. For other domains introduced in the previous section the counting functions are estimated as follows.

**Lemma 4.**

*(i) If* $P \in \mathbf{P}(\delta)$ *then* $N_{\mathrm{N}}(P, \lambda) = 1$ *for all* $\lambda \leq \pi\delta^{-1}$.

*(ii) If* $V \in \mathbf{V}(\delta)$ *then* $N_{\mathrm{N}}(V,\lambda)=1$ *for all* $\lambda \le (1+2\pi^{-2})^{-1/2}\delta^{-1}$.

*(iii) If* $M$ *is a subset of an* $n$*-dimensional cube* $Q$ *with edges of length* $\delta$ *and* $\Upsilon := \partial M \bigcap Q$ *then*
$N_{\mathrm{N,D}}(M,\Upsilon,\lambda)=0$ *for all* $\lambda \le (2^{-1}-2^{-1}\delta^{-n}|M|)^{1/2}\,\pi\delta^{-1}$ *and*
$N_{\mathrm{N,D}}(M,\Upsilon,\lambda)\le 1$ *for all* $\lambda \le \pi\delta^{-1}$.

*Proof.* See Lemma 2.6 in [NS].

*Remark 2.* The first estimate in Lemma 4(iii) is very rough. Much more precise results in terms of capacities were obtained in [M2, Chapter 10, Section 1].

Applying Theorem 1 and Lemmas 1, 2, 3, 4 and putting $\delta = C\lambda^{-1}$ with an appropriate constant $C$, we obtain

$$R_{\mathrm{D}}(\Omega,\lambda) \;\le\; 2^{7n}\,n^{2n}\,\lambda^{n-1}\int_0^{\lambda}|\Omega^{\mathrm{b}}_{t^{-1}}|\,\mathrm{d}t\,,\qquad \forall\lambda>0\,. \tag{3}$$

Similarly, if $\Omega \in C$ then Theorems 1, 2 and Lemmas 1, 2, 4 imply that

$$R_{\mathrm{N}}(\Omega,\lambda) \;\le\; 2^{7n}\,n_\Omega^{1/2}\lambda\int_{(2\,\mathrm{diam}\,\Omega)^{-1}}^{C_\Omega\lambda} t^{-2}\,\mathscr{V}_{t^{-1}}(\Omega)\,\mathrm{d}t \\ +\;2^{8n}\,n^{2n}\,n_\Omega\,\lambda^{n-1}\int_0^{C_\Omega\lambda}|\Omega^{\mathrm{b}}_{t^{-1}}|\,\mathrm{d}t \tag{4}$$

for all $\lambda \ge \delta_\Omega^{-1}$, where $C_\Omega := 2^{n+3}\,n_\Omega^{1/2}$ (see [NS] for details). Note that

$$|\Omega^{\mathrm{b}}_{t^{-1}}| \;\le\; 2^{2n-2}\,3^n\,n_\Omega\,(\mathrm{diam}\,\Omega)^{d-1}t^{-1}+2^{3n-3}\,3^{2n}\,t^{-n}\,\mathscr{V}_{t^{-1}}(\Omega)$$

for all $t>0$ [NS, Lemma 4.3]. Therefore (4) implies the estimate

$$R_{\mathrm{N}}(\Omega,\lambda) \;\le\; C'_\Omega\,\lambda^{n-1}\left(\log\lambda+\int_{(2\,\mathrm{diam}\,\Omega)^{-1}}^{C'_\Omega\lambda} t^{-n}\,\mathscr{V}_{t^{-1}}(\Omega)\,\mathrm{d}t\right) \tag{5}$$

with a constant $C'_\Omega$ depending on $\Omega$.

*Remark 3.* Assume that $\Omega$ belongs to the Hölder class $C^\alpha$ for some $\alpha \in (0,1)$. Then, by [NS, Lemma 4.5], there are constants $C'_1$ and $C'_2$ such that

$$\mathscr{V}_{t^{-1}}(\Omega) \;\le\; C'_1\,t^{(n-1)/\alpha}+C'_2\,.$$

Now (2) and (4) imply that

$$R_{\mathrm{N}}(\Omega,\lambda)=O\left(\lambda^{(n-1)/\alpha}\right),\qquad \lambda\to\infty\,.$$

This estimate is order sharp. More precisely, for each $\alpha \in (0,1)$ there exists a domain $\Omega$ with $C^\alpha$-boundary such that $R_{\mathrm{N}}(\Omega,\lambda) \ge c\,\lambda^{(n-1)/\alpha}$ for all sufficiently large $\lambda$, where $c$ is a positive constant [NS, Theorem 1.10 ]. The inequalities (2) and (3) imply the well known estimate

$$R_{\mathrm{D}}(\Omega,\lambda) \;=\; O\big(\lambda^{n-\alpha}\big)\,, \qquad \lambda\to\infty\,.$$

Obviously, $(n-1)/\alpha > n-\alpha$. Moreover, if $\alpha < 1-n^{-1}$ then $(n-1)/\alpha > n$, which means that $R_{\mathrm{N}}(\Omega,\lambda)$ may grow faster than $\lambda^n$ as $\lambda\to\infty$.

*Remark 4.* In a number of papers, estimates for $R_{\mathrm{D}}(\Omega,\lambda)$ were obtained in terms of the so-called upper Minkowski dimension and the corresponding Minkowski content of the boundary (see, for instance, [BC], [BL] or [FV]). Our formulae (2) and (3) are universal and imply the known estimates.

## 5 Planar domains

In the two-dimensional case it is much easier to construct partitions of a domain $\Omega$, since the intersection of $\Omega$ with any straight line consists of disjoint open intervals. This allows one to refine the above results. Throughout this section we shall be assuming that $\Omega\subset\mathbb{R}^2$.

### *5.1 The Neumann Laplacian*

Consider the domain

$$\Omega \;=\; G_\varphi \;:=\; \{(x,y)\in\mathbb{R}^2 \mid 0<x<1,-1<y<\varphi(x)\}\,, \tag{6}$$

where $\varphi:(0,1)\mapsto[0,+\infty]$ is a lower semicontinuous function such that $|G_\varphi|<\infty$ (this implies, in particular, that $\varphi$ is finite almost everywhere). Note that $\Omega$ does not have to be bounded; the results of this subsection hold for unbounded domains of the form (6).

For each fixed $s>0$, the intersection of $G_\varphi$ with the horizontal line $\{y=s\}$ coincides with a countable collection of open intervals. Let us consider the open set $E(\varphi,s)$ obtained by projecting these intervals onto the horizontal axis $\{y=0\}$,

$$E(\varphi,s) \;=\; \{x\in(0,1)\mid (x,s)\in G_\varphi\} \;=\; \bigcup_{j\in\Gamma(\varphi,s)} I_j\,,$$

where $I_j$ are the corresponding open subintervals of $(0,1)$ and $\Gamma(\varphi,s)$ is an index set. Obviously, $E(\varphi,s_2)\subset E(\varphi,s_1)$ whenever $s_2>s_1$.

It turns out that the spectral properties of the Neumann Laplacian on $G_\varphi$ are closely related to the following function, describing geometric properties of $G_\varphi$. Given $t\in\mathbb{R}_+$, let us denote

$$n(\varphi,t)=\sum_{k=1}^{+\infty}\#\Big\{j\in\Gamma(\varphi,kt)\mid \mu(I_j)<2\,\mu\big(I_j\bigcap E(\varphi,kt+t)\big)\Big\}\,,$$

where $\mu(\cdot)$ is the one dimensional measure of the corresponding set. Note that $n(\varphi,t)$ may well be $+\infty$.

Recall that the first eigenvalue of the Neumann Laplacian is equal to zero, and the corresponding eigenfunction is constant. If the rest of the spectrum is separated from 0 and lies in the interval $[\nu^2,\infty)$ then we have the so-called Poincaré inequality

$$\inf_{c\in\mathbb{R}} \|u-c\|^2_{L_2(\Omega)} \;\le\; \nu^{-2}\,\|\nabla u\|^2_{L_2(\Omega)}\,, \qquad \forall u\in W^{2,1}(\Omega)\,,$$

where $W^{2,1}(\Omega)$ is the Sobolev space.

**Theorem 3.** *The Poincaré inequality holds in $\Omega = G_\varphi$ if and only if there exists $t>0$ such that $n(\varphi,t)=0$. Moreover, there is a constant $C\ge 1$ independent of $\varphi$ such that*

$$C^{-1}(t_0+1) \;\le\; \nu^{-2} \;\le\; C\,(t_0+1),$$

*where $t_0 := \inf\{t>0 \mid n(\varphi,t)=0\}$.*

*Proof.* See Theorem 1.2 in [Ne].

**Theorem 4.** *The spectrum of Neumann Laplacian on $G_\varphi$ is discrete if and only if $n(\varphi,t)<+\infty$ for all $t>0$.*

*Proof.* See Corollary 1.4 in [Ne].

**Theorem 5.** *Let $\Psi : [1,+\infty)\mapsto(0,+\infty)$ be a function such that*

$$C^{-1}s^a \;\le\; \frac{\Psi(st)}{\Psi(t)} \;\le\; C\,s^b\,, \qquad \forall s,t\ge 1,$$

*where $a>1$, $b\ge a$ and $C\ge 1$ are some constants. Then the following two conditions are equivalent.*

*(i) There exist constants $C_1\ge 1$ and $\lambda_*>0$ such that*

$$C_1^{-1}\,\Psi(\lambda) \;\le\; R_{\mathrm{N}}(G_\varphi,\lambda) \;\le\; C_1\,\Psi(\lambda)\,, \qquad \forall\lambda\ge\lambda_*\,.$$

*(ii) There exist constants $C_2\ge 1$ and $t_*>0$ such that*

$$C_2^{-1}\,\Psi(t) \;\le\; n(\varphi,t^{-1}) \;\le\; C_2\,\Psi(t)\,, \qquad \forall t\ge t_*\,.$$

*Proof.* See Theorem 1.6 in [Ne].

## *5.2 The Dirichlet Laplacian*

M. Berry conjectured in [Be] that the Weyl formula for the Dirichlet Laplacian on a domain with rough boundary might contain a second asymptotic term depending on the fractal dimension of the boundary. This problems was investigated by a number of mathematicians and physicists and was discussed in many papers (see, for instance, [BC], [FV] and references therein). To the best of our knowledge, positive results were obtained only for some special classes of domains (such as domains with model cusps and disconnected self-similar fractals). The following theorem justifies the conjecture for planar domains of class $C$.

**Theorem 6.** *Let $\Omega$ be a planar domain of class $C$ such that*

$$|\Omega_\delta^{\mathrm{b}}| \;=\; C_1\delta^{\alpha_1}+\cdots+C_m\delta^{\alpha_m}+o(\delta^\beta)\,, \qquad \delta\to 0,$$

*where $C_j$, $\alpha_i$ and $\beta$ are real constants such that $0<\alpha_1<\alpha_2<\cdots<\alpha_m\le\beta<1$ and $\beta<(1+\alpha_1)/2$. Then*

$$R_{\mathrm{D}}(\Omega,\lambda) \;=\; \tau_{\alpha_1}C_1\lambda^{2-\alpha_1}+\cdots+\tau_{\alpha_m}C_m\lambda^{2-\alpha_m}+o(\lambda^{2-\beta})\,, \qquad \lambda\to\infty,$$

*where $\tau_{\alpha_j}$ is a constant depending only on $\alpha_j$ for each $j=1,\dots,m$.*

Recall that the interior Minkowski content of order $\alpha$ of a planar domain $\Omega$ is defined as

$$M_\alpha^{\mathrm{int}}(\Omega) \;:=\; c(\alpha)\,\lim_{\delta\to 0}\,\delta^{\alpha-2}\,|\Omega_\delta^{\mathrm{b}}| \tag{7}$$

provided that the limit exists; here $\alpha\in(0,2)$ and $c(\alpha)$ is a normalising constant. Theorem 6 with $m=1$ and $\alpha_1=\beta=\alpha$ immediately implies the following

**Corollary 1.** *If $\Omega$ is a planar domain of class $C$ and $0<M_\alpha^{\mathrm{int}}(\Omega)<+\infty$ for some $\alpha\in(1,2)$ then $\lim_{\lambda\to+\infty}R_{\mathrm{D}}(\Omega,\lambda)/\lambda^{2-\alpha}=\tau_\alpha M_\alpha^{\mathrm{int}}(\Omega)$, where $\tau_\alpha$ is a constant depending only on $\alpha$.*

The proof of Theorem 6 consists of two parts, geometric and analytic. The first part uses the technique developed in [Ne] and the following lemma about partitions of planar domains $\Omega\in C$.

**Lemma 5.** *For every planar $\Omega\in C$, there exist a finite collection of open connected disjoint subsets $\Omega_i\subset\Omega$ and a set $D$ such that*

*(i)* $\Omega\subset((\cup_i\Omega_i)\cup D)\subset\overline{\Omega}$;
*(ii)* $D$ *coincides with the union of a finite collection of closed line segments;*
*(iii) each set* $\Omega_i$ *is either a Lipschitz domain or is obtained from a domain given by* (6) *with a continuous function* $\varphi_i$ *by translation, rotation and dilation.*

The second, analytic part of the proof involves investigation of some one dimensional integral operators.

## 6 Concluding remarks and open problems

*Remark 5.* It is not clear how to obtain upper bounds for $N_{\mathrm{N}}(\Omega,\lambda)$ for general domains $\Omega$. It is not just a technical problem; for instance, the Neumann Laplacian on the relatively simple planar domain $\Omega$ obtained from the square $(0,2)\times(0,2)$ by removing the line segments $\frac{1}{n}\times(0,1)$, $n=1,2,3\ldots$, has a nonempty essential spectrum.

*Remark 6.* It may be possible to extend and/or refine our results, using a combination of our variational approach with the technique developed by V. Ivrii in [Iv].

*Remark 7.* There are strong reasons to believe that Theorem 6 cannot be extended to higher dimensions.

Finally, we would like to draw reader's attention to the following open problems.

**Problem 1.** By Lemma 2, $N_{\mathrm{N}}(\Omega,\varkappa^{-1/2}\lambda)\le\sum_j N_{\mathrm{N}}(\Omega_j,\lambda)$ for any finite family $\{\Omega_j\}$ of open sets $\Omega_j\subset\Omega$ such that $|\Omega|=|\cup_j\Omega_j|$ and $\aleph\{\Omega_j\}\le\varkappa<+\infty$. It is possible that the better estimate $N_{\mathrm{N}}(\Omega,\lambda)\le\sum_j N_{\mathrm{N}}(\Omega_j,\lambda)$ holds. This conjecture looks plausible and is equivalent to the following statement: if $\Omega_1\subset\Omega$, $\Omega_2\subset\Omega$ and $\Omega\subset\Omega_1\bigcup\Omega_2$ then $N_{\mathrm{N}}(\Omega_1,\lambda)+N_{\mathrm{N}}(\Omega_2,\lambda)\ge N_{\mathrm{N}}(\Omega,\lambda)$.

**Problem 2.** It would be interesting to know whether the converse statement to Corollary 1 is true. Namely, assume that $\Omega$ is a planar domain of class $C$ such that

$$R_{\mathrm{D}}(\Omega,\lambda) \;=\; C\lambda^{2-\alpha}+o(\lambda^{2-\alpha}), \qquad \lambda\to\infty,$$

with some constant $C$. Does this imply that the limit (7) exists and finite?

**Problem 3.** Is it possible to improve the estimate $R_{\mathrm{B}}(\Omega,\lambda)=O(\lambda^{n-1}\log\lambda)$ for Lipschitz domains? The variational methods are applicable to all domains $\Omega$ of class $C$ but do not allow one to remove the $\log\lambda$, whereas Ivrii's technique gives the best possible result $R_{\mathrm{B}}(\Omega,\lambda)=O(\lambda^{n-1})$ but works only for $\Omega$ which are "logarithmically" better than Lipschitz domains.

## References


[Be] Berry, M. V.: Some geometric aspects of wave motion: wavefront dislocations, diffraction catastrophes, diffractals. Geometry of the Laplace Operator, Proc. Sympos. Pure Math. **36**, 13–38 (1980).

[BC] Brossard, J., Carmona, R.: Can one hear the dimension of a fractal? Comm. Math. Phys. **104**, 103-122 (1986).

[BL] van den Berg, M., Lianantonakis, M.: Asymptotics for the spectrum of the Dirichlet Laplacian on horn-shaped regions. Indiana Univ. Math. J. **50**, 299–333 (2001).

[BS] Birman, M.S., Solomyak, M.Z.: The principal term of spectral asymptotics for "nonsmooth" elliptic problems. Funktsional. Anal. i Prilozhen. **4:4**, 1–13 (1970) (Russian), English transl. in Functional Anal. Appl. **4** (1971).

[FV] Fleckinger-Pellé, J., Vassiliev, D.: An example of a two-term asymptotics for the "counting function" of a fractal drum. Trans. Amer. Math. Soc. **337:1**, 99–116 (1993).

[HSS] Hempel, R., Seco, L., Simon, B.: The essential spectrum of Neumann Laplacians on some bounded singular domains. J. Funct. Anal. **102**, 448–483 (1991).

[Iv] Ivrii, V.: Sharp spectral asymptotics for operators with irregular coefficients. II. Domains with boundaries and degenerations. Comm. Partial Differential Equations **28**, 103–128 (2003).

[M1] Maz'ya, V.: On Neumann's problem for domains with irregular boundaries. Siberian Math. J. **9**, 990–1012 (1968).

[M2] Maz'ya, V.: Sobolev spaces. Leningrad University, Leningrad (1985). English translation in Springer Series in Soviet Mathematics, Springer-Verlag, Berlin (1985).

[Ne] Netrusov, Y.: Sharp remainder estimates in the Weyl formula for the Neumann Laplacian on a class of planar regions. J. Funct. Anal. **250**, 21–41 (2007).

[NS] Netrusov, Y., Safarov, Y.: Weyl asymptotic formula for the Laplacian on domains with rough boundaries. Comm. Math. Physics, **253**, 481–509 (2005).

[Sa] Safarov, Y.: Fourier Tauberian Theorems and applications. J. Funct. Anal. **185**, 111–128 (2001).

[St] Stein, E.: Singular integrals and differentiability properties of functions. Princeton University Press, Princeton (1970).