BIRKHOFF'S THEOREM FOR A FAMILY OF PROBABILITY SPACES

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To the memory of O.A. Ladyzhenskaya

Abstract. We extend Birkhoff's theorem on doubly stochastic matrices to some countable families of discrete probability spaces with nonempty intersections.

A (possibly infinite) square matrix $\{w_{ij}\}_{i,j=1,2,\dots}$ with nonnegative entries w_{ij} is said to be *doubly stochastic* if its row and column sums are equal to one. The matrix $\{w_{ij}\}_{i,j=1,2,\dots}$ can be identified with a function w on the direct product of two discrete spaces $X=\{x_1,x_2,\dots\}$ and $Y=\{y_1,y_2,\dots\}$, such that $w(x_i,y_j)=w_{ij}$, $\forall i,j=1,2,\dots$. Under this identification, the matrix $\{w_{ij}\}_{i,j=1,2,\dots}$ is doubly stochastic if and only if the restriction of w to each subset $\{x_i\}\times Y$ or $X\times\{y_j\}$ is the density of a probability measure.

According to Birkhoff's theorem [Bi1],

(I) the extreme points of the convex set of doubly stochastic matrices are permutation matrices,

(II) the set of doubly stochastic matrices coincides with the closed convex hull of the set of permutation matrices.

Obviously, if (I) holds then (II) follows from

(II′) the set of doubly stochastic matrices coincides with the closed convex hull of the set of its extreme points.

Many proofs of Birkhoff's theorem are known for finite matrices (see, for example, [An] [BR] or [Ro]). The set of finite doubly stochastic matrices $\{w_{ij}\}_{i,j=1,2,\dots,n}$ is compact. Therefore (II′) is a particular case of the Krein–Milman theorem.

The problem of extending (I) and (II) to infinite matrices is known as Birkhoff's problem 111 [Bi2]. It was considered in [Is], [RP], [Ke] and [Sa]. In the infinite case (I) remains true but the validity of (II) depends on the choice of topology. The set of doubly stochastic matrices is not compact in any natural locally convex topology on the liner space of infinite matrices (see Section 3). Therefore the Krein–Milman theorem is not applicable and one has to prove (II) separately.

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Birkhoff's theorem has been generalized in many directions. In particular,

- in [Ho], [Ka], [LLL], [ST] and [Ti] the authors considered various subsets of the set of doubly stochastic matrices.
- The papers [CLMST], [Gr], [Le] and [Mu] dealt with classes of matrices with fixed, but not necessarily equal to one, row and column sums.
- A measure μ on the direct product of two unit intervals is said to be doubly stochastic if $\mu(A \times X) = \mu(X \times A) = |A|$ for every Borel set $A \subseteq X$, where $|A|$ denotes the Lebesgue measure of A. The doubly stochastic measures were studied in [BS], [Do], [Fe], [Li], [KST] and [Vi]. In [Do] and [Li] the authors independently obtained a continuous analogue of the first statement of Birkhoff's theorem.
- In [LMST] the authors considered nonnegative "hypermatrices", that is, nonnegative functions defined on the direct product of several discrete spaces.

The aim of this paper is to obtain an analogue of Birkhoff's theorem for a countable family of probability spaces with nonempty intersections. We consider a family Γ of countable sets $\Omega_1, \Omega_2, \dots$ and the convex set $\mathcal{S}(\Gamma)$ of nonnegative functions defined on the union $\Omega := \cup_k \Omega_k$ whose restrictions $w|_{\Omega_k}$ are densities of probability measures on the sets Ω_k. Each function from $\mathcal{S}(\Gamma)$ can be identified with a family of probability measures on the sets Ω_k which coincide on the intersections $\Omega_i \cap \Omega_j$.

Unlike in the above mentioned papers, we do not assume that the set Ω is a direct product and the sets Ω_k are its fibres. If Ω is infinite, we also consider convex subsets $\mathcal{S}(\Gamma, \mathcal{W}) \subset \mathcal{S}(\Gamma)$ which consist of functions satisfying certain decay conditions at infinity. Such conditions often appear in applications (see, for example, [Sa]).

The idea to consider several probability spaces with nonempty intersections is quite natural. This model seems to be a more adequate reflection of reality than the classical scheme with one probability space, where one implicitly assumes that all events lying outside its scope either have probability zero or are totally unrelated to the space under consideration.

The sets $\mathcal{S}(\Gamma, \mathcal{W})$ themselves and their extreme points are determined by the layout of the sets Ω_k. In Section 2 we join every two elements of Ω lying in the same set Ω_k by an edge and formulate our results in terms of the obtained graph G. Theorem 2.11 gives a complete description of the set of extreme points under the assumption that the multiplicity $\varkappa(g)$ of the covering $\{\Omega_k\}_{k=1,2,\dots}$ does not exceed two for each $g \in \Omega$ (in other words, this means that each point $g \in \Omega$ belongs to at most two distinct sets Ω_k). If $\varkappa(g) \leqslant 2$ then each function lying in the set of extreme points may take only the values 0, $\frac{1}{2}$, 1, and the subgraph associated with its support consists of isolated vertices and isolated odd primitive cycles (see Definition 2.1).

If Ω is the direct product of two discrete spaces X and Y, and Γ is the family of all sets of the form $\{x_i\} \times Y$ or $X \times \{y_j\}$, then $\varkappa \equiv 2$ and the corresponding graph G does not have odd primitive cycles. Therefore (I) is a special case of Theorem 2.11. The graphs generated by multidimensional hypermatrices contain odd primitive cycles. Possibly, this explains the presence of the non-standard extreme points discussed in [LMST]. Even primitive cycles were considered in [Gr], [Le] and [Mu]. Theorem 2.11 shows that, under the condition $\varkappa(g) \leqslant 2$, this cycles do not affect the structure of the set $\mathcal{S}(\Gamma)$.

The main result of Section 3 is Theorem 3.3, where we prove (II′) under certain conditions on topology and the family G. These conditions do not include any assumptions about extreme points. Moreover, in Section 3 we do not use any results or definitions from Section 2. Therefore this section can be read separately.

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1. Notation and definitions

1.1. **Notation.** We shall be using the following standard notation.

- $\mathbb{N}$ is the set of positive integers.
- $\{a_1, a_2, \dots\}$ is the set with elements $a_1, a_2, \dots$.
- $\# A$ denotes the number of elements of a set A.
- $\operatorname{supp} w$ denotes the support of a function w.

If A is a subset of a real linear space $\mathcal{W}$ then

- $\operatorname{ex} A$ is the set of extreme points of the set A,
- $\operatorname{conv} A$ is the convex hull of A.

If $\mathcal{W}$ is equipped with a topology $\mathfrak{T}$ then

- $\overline{\operatorname{conv}}\, A$ is the closure of $\operatorname{conv} A$.

Let $\Gamma = \{\Omega_1, \Omega_2, \dots\}$ be a family of countable sets Ω_k which may have nonempty intersection. Denote $\Omega := \cup_k \Omega_k$, and let

- $\mathcal{S}(\Gamma)$ be the convex set of nonnegative functions w on Ω such that $\sum_{g\in\Omega_k} w(g) = 1$ for all $\Omega_k \in \Gamma$;
- $\mathcal{S}^0(\Gamma)$ be the convex set of nonnegative functions w on Ω such that $\sum_{g\in\Omega_k} w(g) \leqslant 1$ for all $\Omega_k \in \Gamma$;
- $\mathcal{P}(\Gamma)$ be the set of functions $w \in \mathcal{S}(\Gamma)$ taking only the values 0 and 1;
- $\mathcal{P}^0(\Gamma)$ be the set of functions $w \in \mathcal{S}^0(\Gamma)$ taking only the values 0 and 1.

If $w \in \mathcal{S}(\Gamma)$ then the restrictions $w|_{\Omega_k}$ are densities of probability measures on Ω_k such that

$$\mu_i|_{\Omega_i\cap\Omega_j} = \mu_j|_{\Omega_i\cap\Omega_j}\,, \qquad \forall i,j = 1,2,\dots. \tag{1.1}$$

The other way round, every family of probability measures μ_k on Ω_k satisfying (1.1) generates a function $w \in \mathcal{S}(\Gamma)$.

If $\Omega_k \subseteq \Omega_j$ then all functions $w \in \mathcal{S}(\Gamma)$ are identically equal to zero on $\Omega_j \setminus \Omega_k$. In this case we can remove the set Ω_j from Γ and all elements

$g \in \Omega_j \setminus \Omega_k$ from Ω without changing the structure of the sets $\mathcal{S}(\Gamma)$ and $\mathcal{P}(\Gamma)$. Therefore, without loss of generality, we shall always be assuming that

$$\Omega_k \setminus \Omega_j \neq \varnothing\,, \qquad \forall j \neq k\,. \tag{1.2}$$

Given $g \in \Omega$ and a subset $\tilde\Omega \subseteq \Omega$, let us define

- $\Gamma(g) := \{\Omega_k \in \Gamma \,:\, g \in \Omega_k\}$, $\Gamma(\tilde\Omega) := \cup_{g\in\tilde\Omega}\Gamma(g)$,
- $\varkappa(g) := \#\,\Gamma(g)$ and $\varkappa(\tilde\Omega) := \sup_{g\in\tilde\Omega}\Gamma(g)$.

Further on, we shall be assuming that

(a) $\#\,\Gamma(g) < \infty$ all $g \in \Omega$.

We shall also need the following technical assumption:

($\mathbf{a}_1$) if $g \neq \tilde g$ then $\Gamma(g) \neq \Gamma(\tilde g)$ for all $g, \tilde g \in \tilde\Omega$.

In the following two examples $\varkappa(g) = 2$ for all $g \in \Omega$ and holds **($\mathbf{a}_1$)** for the whole set $\Omega = \cup_k \Omega_k$.

Example 1.1. Suppose that one can split the family Γ into the union of two subfamilies $\Gamma^+ = \{\Omega_1^+, \Omega_2^+, \dots\}$ and $\Gamma^- = \{\Omega_1^-, \Omega_2^-, \dots\}$ in such a way that $\Omega_i^+ \cap \Omega_j^+ = \varnothing$, $\Omega_i^- \cap \Omega_j^- = \varnothing$, $\Omega = \cup_{i,j}(\Omega_i^+ \cap \Omega_j^-)$ and $\#\,(\Omega_i^+ \cap \Omega_j^-) = 1$ for all $i, j = 1, 2, \dots$ Then Ω_i^+ and Ω_j^- may be considered as rows and columns of an $m_+ \times m_-$-matrix, where $m^\pm := \#\,\Gamma^\pm$. Under this identification, $\mathcal{S}(\Gamma)$ is the set of doubly stochastic matrices and $\mathcal{P}(\Gamma)$ is the set of permutation matrices.

Example 1.2. Let $\Omega = \{g_1, g_2, \dots, g_n\}$, $\Omega_n := \{g_n, g_1\}$ and $\Omega_k := \{g_k, g_{k+1}\}$ for $k = 1, 2, \dots, n-1$. If n is odd then $\mathcal{P}(\Gamma) = \varnothing$ and $\mathcal{S}(\Gamma)$ consists of one function w identically equal to $\frac{1}{2}$.

1.2. **The set $\mathcal{S}(\Gamma)$.** It may well happen that $\mathcal{S}(\Gamma) = \varnothing$. Indeed,

$$\#\,\Gamma \;=\; \sum_{g\in\Omega} \varkappa(g)\, w(g) \;\leqslant\; \varkappa(\Omega) \sum_{g\in\Omega} w(g)\,, \qquad \forall w \in \mathcal{S}(\Gamma)\,. \tag{1.3}$$

Therefore $\mathcal{S}(\Gamma) = \varnothing$ for all finite non-square matrices (in which the number of rows is not equal to the number of columns). The equality (1.3) also implies the following more general

Lemma 1.3. *Assume that Ω coincides with the union of m sets $\Omega_k \in \Gamma$. If $\#\,\Gamma > m\,\varkappa(\Omega)$ then $\mathcal{S}(\Gamma) = \varnothing$.*

Proof. Under the conditions of lemma, $\sum_{g\in\Omega} w(g) \leqslant m$ for all $w \in \mathcal{S}(\Gamma)$. If $\mathcal{S}(\Gamma) \neq \varnothing$ then $\#\,\Gamma \leqslant m\,\varkappa(\Omega)$ by virtue of (1.3). $\square$

In particular, $\mathcal{S}(\Gamma) = \varnothing$ whenever $\#\,\Gamma = \infty$, $\varkappa(\Omega) < \infty$ and Ω coincides with the union of a finite collection of the sets $\Omega_k \in \Gamma$. The following lemma shows that this remains true under the less restrictive condition **(a)**.

Lemma 1.4. *Let $n \in \mathbb{N}$ and w be a nonnegative function on the union $G_n := \cup_{k \leqslant n} \Omega_k$ such that*

$$\sum_{g \in \Omega_k} w(g) = 1 \text{ for all } k \leqslant n \quad \text{and} \quad \sum_{g \in \Omega_k \cap G_n} w(g) \leqslant 1 \text{ for all } k > n. \tag{1.4}$$

If $\# \Gamma = \infty$ and the condition **(a)** *is fulfilled then*

$$\sum_{g \in \Omega_j \cap G_n} w(g) \ \to \ 0, \qquad j \to \infty. \tag{1.5}$$

Proof. Since $\sum_{g \in G_n} w(g) \leqslant n < \infty$, for each $\varepsilon > 0$ we can find a finite subset $G_{n,\varepsilon} \subseteq G_n$ such that

$$0 \ \leq \ \sum_{g \in G_n} w(g) - \sum_{g \in G_{n,\varepsilon}} w(g\ ) \ < \varepsilon .$$

The condition **(a)** implies that $\sup\{j \ : \ \Omega_j \cap G_{n,\varepsilon}\} < \infty$. Therefore the sum in (1.5) does not exceed ε for all sufficiently large $j \in \mathbb{N}$. □

Corollary 1.5. *Let $\# \Gamma = \infty$ and the condition* **(a)** *be fulfilled. If Ω coincides with the union of a finite collection of the sets $\Omega_k \in \Gamma$ then $\mathcal{S}(\Gamma) = \varnothing$.*

1.3. **The space $\mathcal{W}$.** Let $\mathcal{W}$ be an arbitrary linear space of real-valued functions on Ω, which includes $\mathcal{P}^0(\Gamma)$ and satisfies the following condition:

(w) if $w \in \mathcal{W}$ and $|\tilde{w}| \leq |w|$ then $\tilde{w} \in \mathcal{W}$.

Since $\mathcal{P}^0(\Gamma) \subset \mathcal{W}$, the space $\mathcal{W}$ contains all functions with finite supports. Since Ω is countable, $\mathcal{W}$ can be thought of as a subspace of the linear space of infinite real sequences (or a subspace of $\mathbb{R}^m$, if $\# \Omega = m < \infty$). Let

- $\mathcal{S}(\Gamma, \mathcal{W}) := \mathcal{S}(\Gamma) \cap \mathcal{W} \quad \mathcal{S}^0(\Gamma, \mathcal{W}) := \mathcal{S}^0(\Gamma) \cap \mathcal{W}$.

Obviously, $\mathcal{P}(\Gamma) \subseteq \operatorname{ex} \mathcal{S}(\Gamma, \mathcal{W})$ and $\mathcal{P}^0(\Gamma) \subseteq \operatorname{ex} \mathcal{S}^0(\Gamma, \mathcal{W})$.

If $\mathcal{W}$ contains the linear space

$$\mathcal{W}_1 \ := \ \{ w \ : \ \sup_k \sum_{g \in \Omega_k} |w(g)| < \infty \}, \tag{1.6}$$

then $\mathcal{S}(\Gamma, \mathcal{W}) = \mathcal{S}(\Gamma)$ and $\mathcal{S}^0(\Gamma, \mathcal{W}) = \mathcal{S}^0(\Gamma)$. In particular, this is the case when $\# \Omega < \infty$. If $\# \Omega = \infty$ and $\mathcal{W}_1 \not\subset \mathcal{W}$ then the condition $w \in \mathcal{S}(\Gamma, \mathcal{W})$ imposes some restrictions on the behavior of the function $w \in \mathcal{S}(\Gamma)$ at infinity.

Remark 1.6. In [Sa] I also considered the sets $\mathcal{S}(\Gamma, \Gamma_1)$ formed by the nonnegative functions w such that

$$\sum_{g \in \Omega_k} w(g) \leqslant 1 \text{ for all } \Omega_k \in \Gamma \quad \text{and} \quad \sum_{g \in \Omega_k} w(g) = 1 \text{ for all } \Omega_k \in \Gamma_1 ,$$

where Γ_1 is a subset of Γ. Let Ω_k' be the set obtained from Ω_k by adding one new element g_k' that does not belong any other set Ω_j'. If $w \in \mathcal{S}(\Gamma, \Gamma_1)$

and

$$w'(g') := \begin{cases} w(g') & \text{for all } g' \in \cup_k \Omega_k\,, \\ 1 - \sum_{g\in\Omega_k} w(g) & \text{for } g' = g'_k\,, \end{cases}$$

then $w' \in \mathcal{S}(\Gamma')$ where $\Gamma' = \{\Omega'_1, \Omega'_2, \ldots\}$. Using this observation, one can extend results of this paper to the sets $\mathcal{S}(\Gamma, \Gamma_1)$.

2. Extreme points

2.1. **The associated graph G.** Let us join every two elements of $\Omega = \cup_k \Omega_k$ lying in the same set Ω_k by an edge and consider the graph G obtained by means of this procedure. In this section we shall identify subsets of Ω with subgraphs of G, assuming that their vertices are adjacent if and only if they are adjacent in G (in other words, the subgraphs include all edges joining their vertices). Then the sets Ω_k become complete subgraphs of the graph G. Note that Ω_k is not necessarily a maximal complete subgraph (a clique) and that there may be cliques in G which do not contain any of the sets Ω_k.

Recall that a path γ in G is s sequence $(g_1, g_2, \ldots, g_m)$ of vertices g_j such that (g_j, g_{j+1}) are distinct graph edges. A path γ is said to be *simple* if each its vertex is adjacent to at most two other vertices of γ. A path is called a cycle if $g_m = g_1$. One says that a cycle is odd (or even) if it contains an odd (or even) number of vertices.

Definition 2.1. We shall call a simple path $\gamma \in G$ *primitive* if none of the sets Ω_k contain more than two its vertices.

Note that the graph G in Example 1.2 consists of one primitive cycle with n vertices.

If a simple path γ is not primitive then

- either γ has exactly three vertices lying in a subgraph Ω_k,
- or γ contains at least two non-consecutive vertices which are adjacent in G.

If $\gamma = (g_1, g_2, \ldots, g_m)$ is a simple path in G joining the vertices g_1 and $g_m \neq g_1$, then the shortest path of the form $(g_1, g_{j_1}, g_{j_2} \ldots, g_{j_l}, g_m)$ is primitive. Therefore any two vertices lying in a connected subgraph G' can be joined by a primitive path $\gamma \in G'$. In particular, any path joining two given vertices with the minimal possible number of vertices is primitive.

Lemma 2.2. *Let $\gamma_0 = (g_1, g_2, \ldots, g_m, g_1)$ be a simple cycle. Then there is a finite collection of cycles $\{\gamma'_1, \gamma'_2, \ldots \gamma'_n\}$ such that*

(1) *each cycle γ'_i either is primitive or has exactly three vertices lying in a subgraph Ω_k;*
(2) *the set of vertices of the cycles γ'_i coincides with $\{g_1, g_2, \ldots, g_m\}$;*
(3) *γ'_i and γ'_{i+1} have exactly two common vertices and one common edge;*
(4) *if $j \geqslant 2$ then γ'_i and γ'_{i+j} have at most one common vertex.*

Proof. Consider the shortest cycle γ_1' of the form

$$\gamma_1' = (g_1, g_2, \dots g_j, g_l, g_{l+1}, \dots, g_m, g_1)$$

which includes the edge (g_1, g_2). This cycle either is primitive or has three vertices lying in a subgraph Ω_k. Denote by γ_1 the cycle $(g_j, g_{j+1}, ..., g_l, g_j)$ obtained from γ_0 by removing γ_1', and enumerate its vertices such a way that $g_l = g_1$ and $g_j = g_2$. Consecutively applying this procedure to $\gamma_0, \gamma_1, \gamma_2, \dots$, after finitely many steps we obtain a collection of cycles $\gamma_1', \gamma_2', \dots \gamma_n'$ satisfying the conditions (1)–(4). □

Lemma 2.3. *Let $\tilde{G}$ be an arbitrary subgraph of G satisfying the condition* $(\mathbf{a}_1)$. *If $\varkappa(\tilde{G}) \leqslant 2$ and $\tilde{G}$ does not contain primitive cycles then every simple cycle $\gamma_0 \in \tilde{G}$ is contained in a subgraph Ω_k.*

Proof. Let $\gamma_1', \gamma_2', \dots \gamma_n'$ be the cycles constructed in Lemma 2.2. If $\tilde{G}$ does not contain primitive cycles then each cycle γ_i' lies in a subgraph Ω_{k_i}. Now the condition (3) of Lemma 2.2, $(\mathbf{a}_1)$ and the estimate $\varkappa(\tilde{G}) \leqslant 2$ imply that $\Omega_{k_1} = \Omega_{k_2} = \dots = \Omega_{k_n}$. □

Corollary 2.4. *If a connected subgraph G' satisfies the conditions of Lemma* 2.3 *then every two its vertices are joined by a unique primitive path.*

Proof. Suppose that there are two distinct primitive paths in G' joining the same two vertices. Then G' contains a simple cycle γ formed by edges and vertices of these paths. Since no three consecutive vertices of a primitive path belong to the same subgraph Ω_k, the cycle γ is not contained in any of these subgraphs. This contradicts to Lemma 2.3. □

In the general case, the absence of primitive cycles does not imply that every two vertices of a connected subgraph G' are joined by a unique primitive path.

Example 2.5. Let $\Omega = \{g_0, g_1, \dots, g_{m+1}\}$ and $\Omega_k = \{g_0, g_k, g_{k+1}\}$, where $k = 1, \dots, m \geqslant 2$. Then G does not contain any primitive cycles but g_1 and g_{m+1} are joined by the two distinct primitive paths $(g_1, g_2, \dots, g_m, g_{m+1})$ and (g_1, g_0, g_{m+1}). In this example $\varkappa(g_0) = m$ and $\varkappa(g) \leqslant 2$ for all other vertices g. If $m > 2$ then $\varkappa(\Omega) > 2$; if $m = 2$ then $(\mathbf{a}_1)$ is not true because $\Gamma(g_0) = \Gamma(g_2) = \Gamma$.

Remark 2.6. Instead of G, one can consider the so-called intersection graph where vertices are identified with the sets Ω_k and two vertices are adjacent if and only if the intersection of the corresponding sets is not empty. However, this construction seems to be less suitable for our purposes because the intersection graph may contain several distinct edges associated with the same element $g \in \Omega$.

2.2. **Necessary conditions.** Given $w \in \mathcal{W}$, denote $G_w := \operatorname{supp} w$ and

$$\hat{w}(\tilde{G}) \ := \ \inf_{g \in \tilde{G}} \min\{w(g), 1 - w(g)\}, \qquad \forall \tilde{G} \subseteq G_w .$$

If $w \in \mathcal{S}(\Gamma)$ then

- $\hat{w}(\tilde{G}) \geqslant 0$ for all $\tilde{G} \subseteq G_w$ and
- $\hat{w}(G') > 0$ for every finite connected subgraph G' containing more than one vertex.

The following two lemmas give necessary conditions on G_w, which are fulfilled for every extreme point w of the set $\mathcal{S}(\Gamma, \mathcal{W})$.

Lemma 2.7. *If $w \in \operatorname{ex}\mathcal{S}(\Gamma, \mathcal{W})$ then the graph G_w does not contain any finite nonempty subgraphs $\tilde{G}$ satisfying the following two conditions:*

(1) *the intersection of $\tilde{G}$ with each subgraph Ω_k either is empty or contains exactly two distinct vertices;*
(2) *$\tilde{G}$ does not contain odd primitive cycles.*

Proof. Let $w \in \mathcal{S}(\Gamma, \mathcal{W})$ and $\tilde{G}$ be a finite subgraph of G_w satisfying the conditions (1) and (2). Let us consider an arbitrary connected component G' of the graph $\tilde{G}$. Since the subgraphs Ω_k are complete, the intersection $G' \cap \Omega_k$ either is empty or coincides with $\tilde{G} \cap \Omega_k$ for each $\Omega_k \in \Gamma$. Therefore the subgraph G' also satisfies the conditions (1) and (2). In particular, from (1) it follows that $w(g) \in (0,1)$ for all $g \in G'$. Since G' is finite, this implies that $\hat{w}(G') > 0$.

In view of (1), every simple path in G' is primitive. From here and the condition (2) it follows that the chromatic number of the graph G' equals two. In other words, the vertices of G' can be divided into two groups in such a way that no two vertices from one group are adjacent to each other. Let us denote $\varepsilon := \hat{w}(G')$ and define functions $w_\varepsilon^\pm$ as follows:

- $w_\varepsilon^\pm(g) := w(g)$ for all $g \notin G'$;
- $w_\varepsilon^\pm(g) := w(g) \pm \varepsilon$ if g belongs to the first group of vertices;
- $w_\varepsilon^\pm(g) := w(g) \mp \varepsilon$ if g belongs to the second group of vertices.

It follows straight from the definition that $w_\varepsilon^\pm(g) \in \mathcal{S}(\Gamma)$. Also, $w_\varepsilon^\pm \in \mathcal{W}$ because $w \in \mathcal{W}$ and $\#\operatorname{supp}(w - w_\varepsilon^\pm) < \infty$. Thus $w_\varepsilon^\pm \in \mathcal{S}(\Gamma, \mathcal{W})$ and $w = \frac{1}{2}(w_\varepsilon^+ + w_\varepsilon^-) \notin \operatorname{ex}\mathcal{S}(\Gamma, \mathcal{W})$. □

Corollary 2.8. *If $w \in \operatorname{ex}\mathcal{S}(\Gamma, \mathcal{W})$ then each connected component of the graph G_w satisfies the condition* $(\mathbf{a}_1)$.

Proof. If g_1 and g_2 lie in the same connected component of G_w and $\Gamma(g_1) = \Gamma(g_2)$ then the subgraph $\tilde{G} := \{g_1, g_2\}$ satisfies the conditions (1) and (2) of Lemma 2.7. □

Lemma 2.9. *If $\varkappa(\Omega) \le 2$ and $w \in \operatorname{ex}\mathcal{S}(\Gamma, \mathcal{W})$ then each connected component of the graph G_w either consists of one vertex or contains a primitive cycle.*

Proof. Let $w \in \mathcal{S}(\Gamma, \mathcal{W})$. Suppose that G_w has a connected component G' that includes at least one edge and does not contain primitive cycles. Then, in view of Corollaries 2.4 and 2.8, every two vertices of G' are joined by a unique primitive path. Let us fix $g_0 \in G'$ and denote by $\mathcal{G}_n$ the set of vertices in G' which are joined with g_0 by primitive paths with n edges. Since the subgraphs Ω_k are complete, for every $k = 1, 2, \ldots$ there exists $n \geqslant 0$ such that $\Omega_k \subseteq \mathcal{G}_n \cup \mathcal{G}_{n+1}$. Moreover, if $\Omega_k \subseteq \mathcal{G}_n \cup \mathcal{G}_{n+1}$ then the intersection $\Omega_k \cap \mathcal{G}_n$ contains exactly one vertex, which we shall denote $g_{k,n}$. Indeed, if $\Omega_k \cap \mathcal{G}_n$ contained another vertex $g'_{k,n}$ then, joining g_0 with $g_{k,n}$ and $g'_{k,n}$ by primitive paths and adding the edge $(g'_{k,n}, g_{k,n})$, we would obtain a simple cycle not lying in any of the sets Ω_k. This would contradict to Lemma 2.3.

Since $\# G' > 1$, we have $w(g_0) \in (0,1)$. Let us denote

$$\varepsilon_0 \ := \ \min \left\{ \frac{1}{2} \, , \ \frac{1 - w(g_0)}{2\, w(g_0)} \right\} , \tag{2.1}$$

fix an arbitrary $\varepsilon \in (0, \varepsilon_0)$ and define

$$\varepsilon_{k,0} := \varepsilon \, , \qquad \varepsilon_{k,n+1} \ := \ \varepsilon_{k,n}\, w(g_{k,n})\, (1 - w(g_{k,n}))^{-1} \, , \qquad n = 1, 2, \ldots$$

Consider the sequences of functions $w^+_{\varepsilon,n}$ and $w^-_{\varepsilon,n}$ defined as follows:

- $w^{\pm}_{\varepsilon,0}(g_0) := (1 \pm \varepsilon)\, w(g_0)$ and $w^{\pm}_{\varepsilon,0}(g) := w(g)$ for all $g \neq g_0$;
- $w^{\pm}_{\varepsilon,n+1}(g) := w^{\pm}_{\varepsilon,n}(g)$ for all $g \in \cup_{j \leqslant n} \mathcal{G}_j$;
- $w^{\pm}_{\varepsilon,n+1}(g) := w(g)$ for all $g \notin \cup_{j \leqslant n+1} \mathcal{G}_j$;
- if $\Omega_k \subseteq \mathcal{G}_n \cup \mathcal{G}_{n+1}$ then $w^{\pm}_{\varepsilon,n}(g_{k,n}) := (1 \pm \varepsilon_{k,n})\, w(g_{k,n})$ and $w^{\pm}_{\varepsilon,n+1}(g) := (1 \mp \varepsilon_{k,n+1}) w(g)$ for all $g \in \Omega_k \cap \mathcal{G}_{n+1}$.

Obviously, $w(g) = \frac{1}{2}(w^+_{\varepsilon,n}(g) + w^-_{\varepsilon,n}(g))$ for all $g \in \Omega$. Since $w \in \mathcal{S}(\Gamma)$, we have

$$\frac{w(g_{k,n})}{1 - w(g_{k,n})} \ \leqslant \ \frac{1 - w(g)}{w(g)} \, , \qquad \forall g \in \Omega_k \cap \mathcal{G}_{n+1} \, .$$

Using these inequalities, one can easily prove that

$$\varepsilon_{k,n} \ \leqslant \ \min \left\{ \frac{1}{2} \, , \ \frac{1 - w(g_{k,n})}{2\, w(g_{k,n})} \right\} , \qquad k = 1, 2, \ldots \tag{2.2}$$

The estimate (2.2) and identity $w = \frac{1}{2}(w^+_\varepsilon + w^-_\varepsilon)$ imply that

$$0 \ \leqslant \ w^{\pm}_{\varepsilon,n}(g) \ \leqslant \ w(g) \, , \qquad \forall g \in G' \, , \quad \forall n = 1, 2, \ldots \tag{2.3}$$

If $\Omega_k \subseteq \mathcal{G}_n \cup \mathcal{G}_{n+1}$ then

$$\sum_{g \in \Omega_k} w^{\pm}_{\varepsilon,n}(g) = (1 \pm \varepsilon_{k,n}) w(g_{k,n}) \ + \ (1 \mp \varepsilon_{k,n+1})(1 - w(g_{k,n})) \ = \ 1. \tag{2.4}$$

Let $w^{\pm}_\varepsilon(g) := \lim_{n \to \infty} w^{\pm}_{\varepsilon,n}(g)$. The condition **(w)** and inequalities (2.3) imply that $w^{\pm}_\varepsilon \in \mathcal{W}$. By (2.4), we have $w^{\pm}_\varepsilon \in \mathcal{S}(\Gamma)$. Therefore $w^{\pm}_\varepsilon \in \mathcal{S}(\Gamma, \mathcal{W})$ and $w = \frac{1}{2}(w^+_\varepsilon + w^-_\varepsilon)$ is not an extreme point of the set $\mathcal{S}(\Gamma, \mathcal{W})$. □

Remark 2.10. If $w \in \mathcal{P}(\Gamma)$ then $\#(\Omega_k \cap G_w) = 1$ for all $\Omega_k \in \Gamma$, which implies that the subgraph G_w does not have any edges. Conversely, every edgeless graph containing one element of each set Ω_k is the support of a function from $\mathcal{P}(\Gamma)$. If $w_1, w_2 \in \mathcal{P}(\Gamma)$ then the subgraph $\tilde{G} := \operatorname{supp}(w_1 - w_2)$

satisfies the conditions (1) and (2) of Lemma 2.7. The converse is not always true; a subgraph satisfying (1) and (2) may not coincide with the support of the difference $w_1 - w_2$ with $w_1, w_2 \in \mathcal{P}(\Gamma)$.

For example, let $\Omega = \{g_1, g_2, \dots, g_5\}$, $\Omega_1 = \{g_2, g_3\}$, $\Omega_2 = \{g_1, g_3\}$, $\Omega_3 = \{g_1, g_2\}$ and $\Omega_4 = \{g_4, g_5\}$. Then the complete subgraph Ω_4 satisfies the conditions (1) and (2). However, $\mathcal{S}(\Gamma)$ consists of functions w such that $w(g_1) = w(g_2) = w(g_3) = \frac{1}{2}$ and $1 - w(g_5) = w(g_4) \in [0,1]$. Therefore $\mathcal{P}(\Gamma) = \varnothing$.

2.3. **Necessary and sufficient conditions.** The following theorem is the main result of this section.

Theorem 2.11. *Assume that $\varkappa(\Omega) \leqslant 2$. Then $w \in \operatorname{ex}\mathcal{S}(\Gamma, \mathcal{W})$ if and only if $G_w \cap \Omega_k \neq \varnothing$ for all $\Omega_k \in \Gamma$ and w satisfies the following two conditions:*

(1) *each connected component of the graph G_w*
  (1_1) *either consists of one isolated vertex g,*
  (1_2) *or coincides with an odd primitive cycle γ,*

(2) *$w(g) = 1$ in the case (1_1), and $w|_\gamma \equiv \frac{1}{2}$ in the case (1_2).*

Proof. If (1) holds and $G_w \cap \Omega_k \neq \varnothing$ then

($1'$) for each $\Omega_k \in \Gamma$ the intersection $\Omega_k \cap G_w$ consists of
  ($1'_1$) either one isolated vertex $g \in G_w$,
  ($1'_2$) or two consecutive vertices of a primitive cycle $\gamma \in G_w$.

Indeed, since the graph Ω_k is complete, the intersection $\Omega_k \cap G_w$ lies in a connected component G' of the graph G_w. If G' is an isolated vertex then ($1'_1$) is true. If G' coincides with a primitive cycle then the intersection $\Omega_k \cap G_w$ cannot contain more than two vertices because the cycle is primitive. On the other hand, $\Omega_k \cap G_w$ cannot consists of one vertex because $\varkappa(\Omega) \leqslant 2$. Therefore ($1'_2$) holds.

1. Let $G_w \cap \Omega_k \neq \varnothing$ for all $\Omega_k \in \Gamma$ and the conditions (1) and (2) be fulfilled.

The set of vertices of an odd primitive cycle can be represented as the union of three disjoint subsets such that $\Gamma(g) \cap \Gamma(g') = \varnothing$ for every pair of elements g, g' lying in the same subset. Therefore w coincides with $\frac{1}{2}(w_1 + w_2 + w_3)$ where $w_i \in \mathcal{P}^0(\Gamma) \subset \mathcal{W}$, $i = 1, 2, 3$. This observation and the conditions ($1'$), (2) imply that $w \in \mathcal{S}(\Gamma, \mathcal{W})$.

Assume that $w_\pm \in \mathcal{S}(\Gamma, \mathcal{W})$ and $w = \frac{1}{2}(w_+ + w_-)$. Then $\operatorname{supp} w_\pm \subseteq G_w$ and $w_\pm(g) = 1$ at all isolated vertices $g \in G_w$. If γ is a cycle in G_w then $w_\pm(g) + w_\pm(g') = 1$ for each pair of consecutive vertices $g, g' \in \gamma$. Since γ is odd, these equalities imply that $w_\pm(g) = \frac{1}{2}$ at all vertices $g \in \gamma$. Thus $w_\pm = w$ and, consequently, $w \in \operatorname{ex}\mathcal{S}(\Gamma, \mathcal{W})$.

2. Let $w \in \mathcal{S}(\Gamma, \mathcal{W})$ and (1) be fulfilled. Then $G_w \cap \Omega_k \neq \varnothing$ for all $\Omega_k \in \Gamma$. In the case ($1'_1$) $w(g) = 1$; in the case ($1'_2$) $w(g) = \frac{1}{2}$ for all $g \in \gamma$ because the cycle γ is odd. Thus (2) follows from the inclusion $w \in \mathcal{S}(\Gamma, \mathcal{W})$ and (1).

3. It remains to show that (1) holds for all $w \in \operatorname{ex}\mathcal{S}(\Gamma,\mathcal{W})$. Assume that $w \in \mathcal{S}(\Gamma,\mathcal{W})$ and consider an arbitrary connected component G' of the graph G_w containing more than one vertex. We are going to prove that $w \notin \operatorname{ex}\mathcal{S}(\Gamma,\mathcal{W})$ unless G' coincides with an odd primitive cycle. In view of Corollary 2.8 and Lemma 2.9, we can assume without loss of generality that G' satisfies $(\mathbf{a}_1)$ and contains at least one primitive cycle.

4. If there is an even primitive cycle $\gamma \subset G'$ then the estimate $\varkappa(\Omega) \leqslant 2$ implies that the subgraph $\tilde G = \gamma$ satisfies the conditions of Lemma 2.7 and, consequently, $w \notin \operatorname{ex}\mathcal{S}(\Gamma,\mathcal{W})$. Therefore we shall be assuming that G' does not contain even primitive cycles.

5. Let $\gamma = (g_1, g_2, \dots, g_l, g_1)$ be an odd primitive cycle in G'. Since G_w satisfies $(\mathbf{a}_1)$, for each pair of consecutive vertices $g_i, g_{i+1} \in \gamma$ there exists a unique set Ω_{k_i} containing these vertices (we take $g_{l+1} := g_1$). Denote by G_γ the graph generated by the family of sets $\Gamma(\gamma) = \{\Omega_{k_1}, \dots, \Omega_{k_l}\}$.

Let $G' \setminus \gamma \neq \varnothing$. Since the graph G' is connected, the estimate $\varkappa(\Omega) \leqslant 2$ implies that at least one of the intersections $\Omega_{k_i} \cap G'$ contains a vertex that does not belong to γ. Assume, for the sake of definiteness, that $\Omega_{k_1} \cap G'$ contains a vertex $g_0 \notin \gamma$.

5(a). If g_0 belongs to a set Ω_{k_j} with $j > 1$ then one of the primitive cycles $(g_0, g_2, \dots, g_j, g_0)$ and $(g_0, g_{j+1}, \dots, g_l, g_1, g_0)$ is even. If there exists a primitive path $(g_0, \tilde g_1, \dots, \tilde g_m)$ in G' such that $\tilde g_m \in \Omega_{k_j}$ and $\tilde g_i \notin G_\gamma$ for all $i = 1, \dots, m-1$ then

- in the case $j > 1$ either $(g_0, \tilde g_1, \dots, \tilde g_m, g_{j+1}, g_{j+2}, \dots, g_l, g_1, g_0)$ or $(g_0, \tilde g_1, \dots, \tilde g_m, g_j, g_{j-1}, \dots, g_2, g_0)$ is an even primitive cycle;
- in the case $j = 1$ one of the primitive cycles $(g_0, \tilde g_1, \dots, \tilde g_m, g_0)$ and $(g_0, \tilde g_1, \dots, \tilde g_m, g_2, \dots, g_l, g_1, g_0)$ is even.

Therefore the absence of even primitive cycles implies the following condition:

(5a) if the first edge of the path $(g_0, \tilde g_1, \dots, \tilde g_m)$ is not contained in the subgraph Ω_{k_1} then none of its vertices $\tilde g_i$ belongs to G_γ.

Denote by G'' the subgraph of G' formed by the vertex g_0 and all the vertices g which are joined with g_0 by such paths. By (5a), we have $G_\gamma \cap G'' = \{g_0\}$. If $\Omega_k \notin \Gamma(\gamma)$ and $\Omega_k \cap G'' \neq \varnothing$ then $\Omega_k \subseteq G''$ because the subgraph Ω_k is complete. Therefore either $G'' = \{g_0\}$ or G'' coincides with the graph generated by the family of sets $\Gamma'' = \{\Omega_{n_1}, \Omega_{n_2}, \dots\} \subset \Gamma$ such that $\Omega_{n_i} \notin \Gamma(\gamma)$ and $\Omega_{n_i} \cap G'' \neq \varnothing$.

5(b). Assume that $G'' = \{g_0\}$. Then g_0 belongs only to the set Ω_{k_1}. Let

$$\varepsilon_1 := \hat w(\{g_0, g_1, \dots, g_l\}) \tag{2.5}$$

and $\varepsilon \in (0, \varepsilon_1)$. Consider the functions w_ε^+ and w_ε^- defined as follows:

- $w_\varepsilon^\pm(g) = w(g)$ for all $g \notin \{g_0, g_1, \dots, g_l\}$;
- $w_\varepsilon^\pm(g_0) = w(g_0) \pm \varepsilon$;
- $w_\varepsilon^\pm(g_1) = w(g_1) \mp \varepsilon/2$;

- $w_\varepsilon^\pm(g_i) = w(g_1) \pm (-1)^{i-1}\,\varepsilon/2$ for all $i = 2,\dots,l$.

Since $\varkappa(\Omega) \leqslant 2$, the vertex g_1 belongs only to the sets Ω_{k_1} and Ω_{k_l}, and each vertex g_i with $i \geqslant 2$ belongs only to the sets $\Omega_{k_{i-1}}$ and Ω_{k_i}. Therefore $w_\varepsilon^\pm \in \mathcal{S}(\Gamma,\mathcal{W})$ and $w = \frac{1}{2}\,(w_\varepsilon^+ + w_\varepsilon^-) \notin \operatorname{ex}\mathcal{S}(\Gamma,\mathcal{W})$.

5(c). Assume that G'' has more than one vertex and does not contain primitive cycles. Let $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ where ε_0 and ε_1 are defined in (2.1) and (2.5). Consider the functions w_ε^+ and w_ε^-, defined as follows:

- if $g \notin \gamma$ and $g \notin G''$ then $w_\varepsilon^\pm(g) := w(g)$;
- if $g \in \gamma$ then $w_\varepsilon^\pm(g)$ is defined as in the part 5(b);
- if $g \in G''$ then $w_\varepsilon^\pm(g)$ is defined as in the proof of Lemma 2.9.

Since $G_\gamma \cap G'' = \{g_0\}$, the inequalities $\varkappa(\Omega) \leqslant 2$ and (2.3) imply that $w_\varepsilon^\pm \in \mathcal{S}(\Gamma,\mathcal{W})$. Thus $w = \frac{1}{2}\,(w_\varepsilon^+ + w_\varepsilon^-) \notin \operatorname{ex}\mathcal{S}(\Gamma,\mathcal{W})$.

5(d). Finally, let us assume that G'' contains an odd primitive cycle γ'. In view of (5a), g_0 does not belong to γ'. Let us consider the shortest primitive path $\gamma'' = (g_0, g_1'', \dots, g_m'', g')$ joining g_0 with γ', and enumerate vertices of the cycle $\gamma' = (g_1', g_2', \dots, g_n', g_1')$ in such a way that $g_1' = g'$. Since $\varkappa(\Omega) \leqslant 2$, either the three vertices g_m'', g_n', g_1' or the three vertices g_m'', g_1', g_2' belong to the same set Ω_k. Assume, for the sake of definiteness, that the latter is true.

Let

$$\varepsilon_2 \;:=\; \hat{w}\{g_0, g_1, \dots, g_l, g_1', \dots, g_n', g_1'', \dots, g_m''\}\,, \qquad \varepsilon \in (0, \varepsilon_2)\,,$$

and $w_\varepsilon^\pm$ be the functions defined as follows:

- if $g \notin \gamma \cup \gamma' \cup \gamma''$ then $w_\varepsilon^\pm(g) := w(g)$;
- if $g \in \gamma$ or $g = g_0$ then $w_\varepsilon^\pm(g)$ is defined as in the part 6(b);
- $w_\varepsilon^\pm(g_i'') = w(g_i'') \pm (-1)^i\,\varepsilon$ for all $i = 1,\dots,m$;
- $w_\varepsilon^\pm(g_1') = w(g_1') \pm (-1)^{m+1}\,\varepsilon/2$;
- $w_\varepsilon^\pm(g_i') = w(g_i') \pm (-1)^{m+i-1}\,\varepsilon/2$ for all $i = 2,\dots,n$.

Since $\varkappa(\Omega) \leqslant 2$, we have $w_\varepsilon^\pm \in \mathcal{S}(\Gamma,\mathcal{W})$. Therefore $w = \frac{1}{2}\,(w_\varepsilon^+ + w_\varepsilon^-)$ is not an extreme point of $\mathcal{S}(\Gamma,\mathcal{W})$. □

Theorem 2.11 immediately implies

Corollary 2.12. *Let $\varkappa(\Omega) \leqslant 2$. Then*

(1) $w(g) \in \{0, \frac{1}{2}, 1\}$ *for all* $w \in \operatorname{ex}\mathcal{S}(\Gamma,\mathcal{W})$ *and* $g \in \Omega$*;*

(2) $\operatorname{ex}\mathcal{S}(\Gamma,\mathcal{W}) = \mathcal{P}(\Gamma)$ *whenever G does not contain odd primitive cycles.*

Remark 2.13. If $\varkappa(\Omega) \leqslant 2$ and G does not contain odd primitive cycles then one can split Γ into the union of two disjoint subsets $\Gamma^+ = \{\Omega_1^+, \Omega_2^+, \dots\}$ and $\Gamma^- = \{\Omega_1^-, \Omega_2^-, \dots\}$ in such a way that $\Omega_i^+ \cap \Omega_j^+ = \varnothing$ and $\Omega_i^- \cap \Omega_j^- = \varnothing$ for all $i, j = 1, 2, \dots$ (Ω_i and Ω_j are included into the same set $\Gamma^\pm$ if every primitive path joining Ω_i and Ω_j has an odd number of edges). If, in addition, $\Omega = \cup_{i,j}(\Omega_i^+ \cap \Omega_j^-)$ and $\#\,(\Omega_i^+ \cap \Omega_j^-) = 1$ for all $i, j = 1, 2, \dots$ then $\mathcal{S}(\Gamma)$ can be thought of as a set of doubly stochastic matrices (see Example 1.1).

3. Closed convex hull of extreme points

3.1. **Topologies on $\mathcal{W}'$.** Let $\mathcal{W}'$ be the space of real-valued functions w' on Ω such that $\sum_{g\in\Omega} |w(g)\, w'(g)| < \infty$ for all $w \in \mathcal{W}$. Further on we shall identify functions $w' \in \mathcal{W}'$ with the corresponding linear functionals

$$w \;\to\; \langle w, w'\rangle \;:=\; \sum_{g\in\Omega} w(g)\, w'(g)$$

on the space $\mathcal{W}$.

Let $\mathfrak{T}$ be a locally convex topology on $\mathcal{W}$ satisfying the following conditions:

($\mathbf{w}_1$) the topological dual $\mathcal{W}^*$ is a subspace of $\mathcal{W}'$;
($\mathbf{w}_2$) $\mathcal{W}^*$ contains all the functionals $w \to \sum_{g\in\Omega_k} w(g)$, $k = 1, 2, \ldots$

From the condition ($\mathbf{w}_2$) it follows that $\overline{\mathrm{conv}}\, \mathrm{ex}\, \mathcal{S}(\Gamma, \mathcal{W}) \subseteq \mathcal{S}(\Gamma, \mathcal{W})$ and $\overline{\mathrm{conv}}\, \mathrm{ex}\, \mathcal{S}^0(\Gamma, \mathcal{W}) \subseteq \mathcal{S}^0(\Gamma, \mathcal{W})$.

Denote by $\mathfrak{T}_0$ the topology of pointwise convergence on $\mathcal{W}$. If ($\mathbf{w}_1$) holds and $\mathcal{W}^*$ consists of functions with finite supports then $\mathfrak{T} = \mathfrak{T}_0$. The Tikhonov theorem and Fatou lemma imply that the set $\mathcal{S}^0(\Gamma)$ is $\mathfrak{T}_0$-compact. Therefore, by the Krein–Milman theorem, $\mathcal{S}^0(\Gamma)$ coincides with the $\mathfrak{T}_0$-closure of the set $\mathrm{conv}\, \mathrm{ex}\, \mathcal{S}^0(\Gamma)$.

If $\#\Omega < \infty$ then $\dim \mathcal{W} < \infty$, $\mathfrak{T} = \mathfrak{T}_0$, the set $\mathcal{S}(\Gamma, \mathcal{W})$ is a compact convex polytope and, consequently, $\mathcal{S}(\Gamma, \mathcal{W}) = \mathrm{conv}\, \mathrm{ex}\, \mathcal{S}(\Gamma, \mathcal{W})$. If at least one of the sets Ω_k is infinite then it may well happen that the set $\mathcal{S}(\Gamma, \mathcal{W})$ is not compact in any locally convex topology on $\mathcal{W}$. Indeed, if the linear functional $w \to \sum_{g\in\Omega_k} w(g)$ is not continuous then, as a rule, the set $\mathcal{S}(\Gamma, \mathcal{W})$ is not closed (there are exceptions, for instance when $\mathcal{S}(\Gamma, \mathcal{W}) = \varnothing$, but such exotic examples hardly deserve serious consideration). On the other hand, if the functional $w \to \sum_{g\in\Omega_k} w(g)$ is continuous and $G_k = \{g_1, g_2, \ldots\}$ then $\mathcal{S}(\Gamma, \mathcal{W})$ may be compact only under the very restrictive assumption that

$$\sup_{w\in\mathcal{S}(\Gamma,\mathcal{W})} \sum_{i>j} |w(g_i)| \underset{j\to\infty}{\to} 0$$

(this follows from Theorem 1.2 in [Sa] which is proved in the same way as Theorem 2.4 in the second chapter of [Ru]).

In all known to us examples either $\overline{\mathrm{conv}}\, \mathrm{ex}\, \mathcal{S}(\Gamma, \mathcal{W}) = \mathcal{S}(\Gamma, \mathcal{W})$ or at least one of the conditions ($\mathbf{w}_1$) and ($\mathbf{w}_2$) is not satisfied (see, for instance, [Is] or Remark 3.7 in the end of the section). It is quite possible that these conditions are sufficient. However, we can prove that $\overline{\mathrm{conv}}\, \mathrm{ex}\, \mathcal{S}(\Gamma, \mathcal{W}) = \mathcal{S}(\Gamma, \mathcal{W})$ only under the following additional assumption:

($\mathbf{a}_2$) there exists $m \in \mathbb{N}$ such that
  ($\mathbf{a}_{21}$) either $\Omega = \cup_{j=1}^{m} \Omega_j$,
  ($\mathbf{a}_{22}$) or none of the sets $\Omega_k \in \{\Omega_{m+1}, \Omega_{m+2}, \ldots\}$ lies in the union of a finite collection of other sets $\Omega_j \in \Gamma$.

In particular, the condition $(\mathbf{a}_2)$ is satisfies if $\#\,\Gamma < \infty$, or if the number of finite sets Ω_k is finite and $\#\,(\Omega_i \cap \Omega_j) < \infty$ for all $i \neq j$.

3.2. **An extension lemma.** Let $G_n := \cup_{k \leqslant n} \Omega_k$, and let

- T_0 be the operator of extension by zero from G_n to Ω,
- $\mathcal{S}_n^0(\Gamma, \mathcal{W})$ be the convex set of nonnegative functions w on G_n satisfying (1.4) and such that $T_0 w \in \mathcal{W}$.

The role of the condition $(\mathbf{a}_{22})$ is clarified in the following lemma.

Lemma 3.1. *If $\#\,\Gamma = \infty$ and the conditions $(\mathbf{w})$, $(\mathbf{a})$, $(\mathbf{a}_{22})$ are fulfilled then for each $n > m$ there exists a (nonlinear) extension operator T_n from G_n to Ω such that*

(1) $T_n : \mathcal{S}_n^0(\Gamma, \mathcal{W}) \to \mathcal{S}(\Gamma, \mathcal{W})$,
(2) $T_n : \operatorname{ex} \mathcal{S}_n^0(\Gamma, \mathcal{W}) \to \operatorname{ex} \mathcal{S}(\Gamma, \mathcal{W})$,
(3) $\sup\limits_{w \in \mathcal{S}_n^0(\Gamma, \mathcal{W})} \langle T_0 w - T_n w, w' \rangle \underset{n \to \infty}{\to} 0$ *for all* $w' \in \mathcal{W}'$.

Proof.
1. Consider an arbitrary function $w \in \mathcal{S}_n^0(\Gamma, \mathcal{W})$. Let Γ_1 be the family of all sets $\Omega_k \in \Gamma$ such that $\sum_{g \in \Omega_k \cap G_n} w(g) = 1$, $\mathcal{G}_1$ be the union of these sets, and w_1 be the extension of w by zero to $\mathcal{G}_1$. By Lemma1.4,

$$\delta_j(w_1) \;:=\; \sum_{g \in \Omega_j \cap \mathcal{G}_1} w_1(g) \underset{j \to \infty}{\to} 0.$$

In particular, this implies that $\#\,\Gamma_1 < \infty$.

Let $k_1 := \min\{k \;:\, \Omega_k \notin \Gamma_1\}$, Γ_2 be the family of sets obtained from Γ_1 by adding the set Ω_{k_1}, and $\mathcal{G}_2$ be the union of sets $\Omega_k \in \Gamma_2$. In view of the condition $(\mathbf{a}_{22})$, we have $\Omega_{k_1} \setminus G_j \neq \varnothing$ for all $j > k_1$. Let us choose an index j such that $\delta_i(w_1) < \delta_{k_1}(w_1)$ for all $i > j$ and fix an arbitrary element $g_1 \in \Omega_{k_1} \setminus G_j$. Let w_2 be the function on $\mathcal{G}_2$ defined by the equalities $w_2|_{\mathcal{G}_1} := w_1|_{\mathcal{G}_1}$, $w_2(g_1) := 1 - \delta_{k_1}(w_1)$ and $w_2(g') := 0$ at all other vertices g'. Then $\sum_{g \in \Omega_k} w_2(g) = 1$ for all $\Omega_k \in \Gamma_2$, $\sum_{g \in \Omega_k \cap \mathcal{G}_2} w_2(g) < 1$ for all $\Omega_k \in \Gamma \setminus \Gamma_2$ and, by Lemma 1.4,

$$\delta_j(w_2) \;:=\; \sum_{g \in \Omega_j \cap \mathcal{G}_2} w_2(g) \underset{j \to \infty}{\to} 0.$$

Let $k_2 := \min\{k \;:\, \Omega_k \notin \Gamma_2\}$, $\Gamma_3 := \Gamma_2 \cup \{\Omega_{k_2}\}$ and $\mathcal{G}_3$ be the union of sets $\Omega_k \in \Gamma_3$. Let us define $\Gamma_2' := \Gamma_1 \cup \Gamma(g_1)$, choose $j \in \mathbb{N}$ so large that $\delta_i(w_2) < \delta_{k_2}(w_2)$ for all $i > j$ and fix an element $g_2 \in \Omega_{k_2} \setminus G_j$ such that $\Gamma(g_2) \cap \Gamma_2' = \varnothing$ (in view of $(\mathbf{a})$ and $(\mathbf{a}_{22})$ such an element does exist). Let w_3 be the function on $\mathcal{G}_3$ defined by the equalities $w_3|_{\mathcal{G}_2} := w_2|_{\mathcal{G}_2}$, $w_3(g_2) := 1 - \delta_{k_2}(w_2)$ and $w_3(g') := 0$ at all other vertices g'. Then

$$\delta_j(w_3) \;:=\; \sum_{g \in \Omega_j \cap \mathcal{G}_3} w_3(g) \underset{j \to \infty}{\to} 0.$$

Now, let $k_3 := \min\{k \ : \Omega_k \notin \Gamma_3\}$ and $\mathcal{G}_4$ be the union of all sets $\Omega_k \in \Gamma_4 := \Gamma_3 \cup \{\Omega_{k_3}\}$. Let us define $\Gamma_3' := \Gamma_2' \cup \Gamma(g_2)$, choose $j \in \mathbb{N}$ such that $\delta_i(w_3) < \delta_{k_3}(w_3)$ for all $i > j$, fix an element $g_3 \in \Omega_{k_3} \setminus G_j$ such that $\Gamma(g_3) \cap \Gamma_3' = \varnothing$, and consider the function w_4 on $\mathcal{G}_4$ defined by the equalities $w_4|_{\mathcal{G}_3} = w_3|_{\mathcal{G}_3}$, $w_4(g_3) = 1 - \delta_{k_3}(w_3)$ and $w_4(g') = 0$ at all other vertices g'.

Iterating this procedure, we obtain sequences of families $\Gamma_j \subset \Gamma$ of sets Ω_k, their unions $\mathcal{G}_j \subset \Omega$, elements $g_j \in \mathcal{G}_{j+1} \setminus \mathcal{G}_j$ and functions w_j on the sets $\mathcal{G}_j$ such that

(c$_1$) $\Gamma_j \subset \Gamma_{j+1}$ and $\cup_{j=1}^{\infty} \Gamma_j = \Gamma$;
(c$_2$) $w_{j+1}|_{\mathcal{G}_j} = w_j|_{\mathcal{G}_j}$ and $\operatorname{supp}(w_{j+1} - w_j) = \{g_j\} \subset \Omega_{k_j}$;
(c$_3$) $\sum_{g \in \Omega_k \cap \mathcal{G}_j} w_j(g) \leqslant 1$ for all $\Omega_k \in \Gamma$ and $\sum_{g \in \Omega_k} w_j(g) = 1$ for all $\Omega_k \in \Gamma_j$;
(c$_4$) for each $j \in \mathbb{N}$ there exists at most one index $i < j$ such that $\Gamma(g_j) \cap \Gamma(g_i) \neq \varnothing$.

Let T_n be the operator defined by the equality $T_n w := \lim_{j \to \infty} T_0 w_j$ where the limit is taken in the topology of pointwise convergence $\mathfrak{T}_0$. Since $0 \leqslant T_n w \leqslant 1$, from (c$_1$)–(c$_3$) it follows that $T_n w \in \mathcal{S}(\Gamma)$.

The condition (c$_2$) also implies that

(c$_2'$) $\operatorname{supp}(T_n w - T_0 w) = \gamma$ where $\gamma := \{g_1, g_2, \ldots\}$.

Let us define subsets γ' and γ'' of γ as follows:

- $g_1 \in \gamma'$ and $g_j \in \gamma'$ whenever $\Gamma(g_j) \cap \Gamma(g_i) = \varnothing$ for all $i < j$;
- if $\Gamma(g_j) \cap \Gamma(g_i) \neq \varnothing$ for some $i < j$ then
  - $g_j \in \gamma'$ in the case where $g_i \in \gamma''$,
  - $g_j \in \gamma''$ in the case where $g_i \in \gamma'$.

In view of (c$_4$), $\gamma = \gamma' \cup \gamma''$, $\gamma' \cap \gamma'' = \varnothing$ and $\Gamma(g') \cap \Gamma(g'') = \varnothing$ for every pair of distinct elements $g', g'' \in \gamma'$ and every pair of distinct elements $g', g'' \in \gamma''$. Therefore the characteristic functions χ' and χ'' of the sets γ' and γ'' belong to $\mathcal{P}^0(\Gamma)$. Since $T_n w - T_0 w \leqslant 1$, we have $T_n w - T_0 w \leqslant \chi' + \chi''$. This estimate and the condition **(w)** imply that $T_n w \in \mathcal{W}$. Thus $T_n w \in \mathcal{S}(\Gamma, \mathcal{W})$, that is, T_n satisfies (1).

2. Suppose that $w \in \operatorname{ex} \mathcal{S}_n^0(\Gamma, \mathcal{W})$ and $T_n w \notin \operatorname{ex} \mathcal{S}(\Gamma, \mathcal{W})$. Then there exists a function $\tilde{w} \not\equiv 0$ on the set Ω such that $T_n w \pm \tilde{w} \in \mathcal{S}(\Gamma, \mathcal{W})$. The condition (c$_2'$) and the inclusions $T_n w \in \mathcal{S}(\Gamma)$ and $T_n w \pm \tilde{w} \in \mathcal{S}(\Gamma)$ imply that

$$\operatorname{supp} \tilde{w} \;\subseteq\; \operatorname{supp} T_n w \;\subseteq\; G_n \cup \{g_1, g_2, \ldots\}$$

and $\sum_{g \in \Omega_j \cap G_n} |\tilde{w}(g)| \leqslant 1$ for all $\Omega_k \in \Gamma$.

Denote by $w_\varepsilon^{\pm}$ the restrictions of the nonnegative functions $T_n w \pm \varepsilon \tilde{w}$ to the set G_n. By Lemma 1.4,

$$\delta \;:=\; \sup_{\Omega_j \notin \Gamma_1} \sum_{g \in \Omega_j \cap \mathcal{G}_1} w(g) \;<\; 1\,.$$

Therefore $w_\varepsilon^\pm \in \mathcal{S}_n^0(\Gamma,\mathcal{W})$ for all $\varepsilon < 1-\delta$. Since $w_\varepsilon^+ + w_\varepsilon^- = w$, from here and the inclusion $w \in \operatorname{ex}\mathcal{S}_n^0(\Gamma,\mathcal{W})$ it follows that $\tilde w|_{\mathcal{G}_1} \equiv 0$. Now, using (c$_2$) and the equalities

$$w_{j+1}(g_j) \;=\; 1-\delta_{k_j}(w_j) \;=\; 1-\sum_{g\in\Omega_{k_j}\setminus\{g_j\}} w_j(g)\,,$$

by induction in j we obtain $\tilde w(g_j) = 0$ for all g_j. Thus $\tilde w \equiv 0$. This contradiction proves (2).

3. Let $w' \in \mathcal{W}'$. Suppose that $\sup\limits_{w\in\mathcal{S}_n^0(\Gamma,\mathcal{W})} \langle T_0 w - T_n w, w'\rangle$ does not converge to zero. Then there exist $\delta > 0$ and a sequence of functions $w_n \in \mathcal{S}_n^0(\Gamma,\mathcal{W})$ such that $\langle T_0 w_n - T_n w_n, w'\rangle \geqslant \delta$. If $\tilde w_n := T_n w_n - T_0 w_n$ then $\tilde w_n \in \mathcal{S}^0(\Gamma,\mathcal{W})$, $\operatorname{supp}\tilde w_n = \{g_1^n, g_2^n, \dots\} \subset \Omega\setminus G_n$ and

$$\sum_{j=1}^\infty |\tilde w_n(g_j^n)\, w'(g_j^n)| \;\geqslant\; \langle T_n w_n - T_0 w_n, w'\rangle \;\geqslant\; \delta\,,$$

where $\{g_1^n, g_2^n, \dots\}$ are the sets of vertices associated with functions w_n (see the first part of the proof).

Let us consider arbitrary finite subsets $H_n \subset \{g_1^n, g_2^n, \dots\}$ such that

$$\sum_{g\in H_n} |\tilde w_n(g) w'(g)| \;\geqslant\; \delta/2$$

Since $H_n \cap G_n = \varnothing$, **(a)** implies that $\Gamma(H_n)\cap\Gamma(H_{n+j}) = \varnothing$ for all sufficiently large $j \in \mathbb{N}$. Therefore we can choose a subsequence $\{H_{n_i}\}_{i=1,2,\dots}$ of the sequence $\{H_n\}_{n=1,2,\dots}$ in such a way that

$$\Gamma(H_{n_i})\cap\Gamma(H_{n_j}) = \varnothing\,, \qquad \forall n_i \neq n_j\,. \tag{3.1}$$

Let $\tilde w(g) := 0$ for all $g \notin \cup_{i=1}^\infty H_{n_i}$ and $\tilde w(g) := w_{n_i}(g)$ for all $g \in H_{n_i}$. We have shown in the first part of the proof that the function $\tilde w$ is estimated on every set H_{n_i} by the sum of two functions from $\mathcal{P}^0(\Gamma)$. From here and (3.1) it follows that $\tilde w$ is estimated by the sum of two functions from $\mathcal{P}^0(\Gamma)$ on the whole set Ω. Therefore $\tilde w \in \mathcal{W}$. On the other hand, $\sum_{g\in\Omega} |\tilde w(g)\, w'(g)| = \infty$ which contradicts to the condition $w' \in \mathcal{W}'$. This proves (3). $\square$

Remark 3.2. From our definition of the operator T_n it is clear that $T_n w \in \mathcal{P}(\Gamma)$ whenever w takes only the values 0 and 1. This observation can be used for constructing functions $w \in \mathcal{P}(\Gamma)$.

3.3. **Closed convex hull.** Lemma 3.1 allows us to prove the following

Theorem 3.3. *If the conditions* **(a)**, **(w)** *and* **(a$_2$)** *are fulfilled then*

$$\mathcal{S}(\Gamma,\mathcal{W}) \;=\; \overline{\operatorname{conv}}\operatorname{ex}\mathcal{S}(\Gamma,\mathcal{W}) \tag{3.2}$$

in any topology $\mathfrak{T}$ *satisfying the conditions* **(w$_1$)** *and* **(w$_2$)**.

Proof. Let us fix an arbitrary function $w \in \mathcal{S}(\Gamma,\mathcal{W})$. In view of $(\mathbf{w}_2)$, it is sufficient to prove that $w \in \overline{\mathrm{conv}}\,\mathrm{ex}\,\mathcal{S}(\Gamma,\mathcal{W})$. Recall that, by the separation theorem, $\overline{\mathrm{conv}}\,\mathrm{ex}\,\mathcal{S}(\Gamma,\mathcal{W})$ coincides with the weak closure of the convex set $\mathrm{conv}\,\mathrm{ex}\,\mathcal{S}(\Gamma,\mathcal{W})$.

1. Assume that there exist $g_1, g_2, \ldots \in \mathrm{supp}\, w$ such that $\Gamma(g_1) = \Gamma(g_j)$ for all $j \geqslant 2$. Let

- $w_i(g_i) := \sum_{j\geqslant 1} w(g_j)$,
- $w_i(g_j) := 0$ for all $j \neq i$,
- $w_i(g) = w(g)$ for all $g \notin \{g_1, g_2, \ldots\}$.

Then $w_i \in \mathcal{S}(\Gamma,\mathcal{W})$ and, in view of $(\mathbf{w}_1)$, there exists a sequence of finite convex linear combinations of the functions w_i which is weakly convergent to w. Since the set of all finite intersections of the sets Ω_k is countable, this implies that w is contained in the weak sequential closure of the set of all functions $\tilde{w} \in \mathcal{S}(\Gamma,\mathcal{W})$ whose supports satisfy the condition $(\mathbf{a}_1)$. Therefore we shall be assuming without loss of generality that $\mathrm{supp}\, w$ satisfies $(\mathbf{a}_1)$.

2. If $\#\Gamma < \infty$ then, by $(\mathbf{a}_1)$, $\#\,\mathrm{supp}\, w < \infty$. In this case w belongs to $\mathrm{conv}\,\mathrm{ex}\,\mathcal{S}(\Gamma,\mathcal{W})$. If $\#\Gamma = \infty$ and $(\mathbf{a}_{21})$ holds then $\mathcal{S}(\Gamma) = \overline{\mathrm{conv}}\,\mathrm{ex}\,\mathcal{S}(\Gamma) = \varnothing$ (see Corollary 1.5). Further on we shall be assuming that $\#\Gamma = \infty$ and Γ satisfies $(\mathbf{a}_{22})$.

3. Assume that $w_n^\star \in \mathcal{S}_n(\Gamma)$ and $\mathrm{supp}\, w_n^\star$ satisfies the condition $(\mathbf{a}_1)$. Let $\Omega_k \cap \mathrm{supp}\, w_n^\star = \{g_1^k, g_2^k, \ldots\}$ and $\Gamma_{i,k}^{(n)} := \Gamma(g_i^k) \cap \{\Omega_{n+1}, \Omega_{n+2}, \ldots\}$ where $k = 1, 2, \ldots, n$. From $(\mathbf{a}_1)$, $(\mathbf{a})$ and (1.5) it follows that

$$\sup_{\Omega_j \in \Gamma_{i,k}^{(n)}} \sum_{g \in \Omega_j \cap G_n} w_n^\star(g) \underset{i\to\infty}{\to} 0$$

for all $k = 1, 2, \ldots, n$. Also, if $\#\Omega_k = \infty$ then

$$v_n(g_i^k) := \sum_{j>i} w_n^\star(g_j^k) \underset{i\to\infty}{\to} 0$$

and $\Gamma(g_i^k) \cap \{\Omega_1, \Omega_2, \ldots, \Omega_n\} = \{\Omega_k\}$ whenever i is sufficiently large. Therefore for all sufficiently large $i \in \mathbb{N}$ the functions $w_{n,i}^\star$ defined by the equalities

- $w_{n,i}^\star(g_j^k) := w_n^\star(g_j^k)$ for all $j = 1, \ldots, i-1$,
- $w_{n,i}^\star(g_i^k) := v_n(g_i^k)$,
- $w_{n,i}^\star(g_j^k) = 0$ for all $j > i$,

belong to $\mathcal{S}_n(\Gamma)$. Each of these functions has a compact support and therefore coincides with a finite convex linear combination of functions $w_{n,i}^j$ from $\mathrm{conv}\,\mathrm{ex}\,\mathcal{S}_n(\Gamma)$. By $(\mathbf{w}_1)$, the sequence $\{T_0 w_{n,i}^\star\}_{i=1,2,\ldots}$ weakly converges to $T_0 w_n^\star$. Thus $T_0 w_n^\star$ is contained in the weak sequential closure of the set $T_0(\mathrm{conv}\,\mathrm{ex}\,\mathcal{S}_n(\Gamma))$.

4. Let $w_n^\star := w|_{G_n}$. By $(\mathbf{w}_1)$, we have $\langle w - T_0 w_n^\star, w' \rangle \underset{n\to\infty}{\to} 0$ for all $w' \in \mathcal{W}'$.

Since $\operatorname{supp} w$ satisfies $(\mathbf{a}_1)$, the same is true for $\operatorname{supp} w_n^\star$. Let $w_{n,i}^\star := \sum_j \alpha_j\, w_{n,i}^j$ be the finite convex linear combinations of functions $w_{n,i}^j \in \operatorname{ex}\mathcal{S}_n(\Gamma)$ introduced in the previous part of the proof. Then $\langle T_0 w_n^\star - T_0 w_{n,i}, w'\rangle \underset{i\to\infty}{\to} 0$ for each function $w' \in \mathcal{W}'$ and for each $n \in \mathbb{N}$.

Define $w_{n,i} := \sum_j \alpha_j\, T_n w_{n,i}^j$. The conditions (2) and (3) of Lemma 3.1 imply that $w_{n,i} \in \operatorname{conv}\operatorname{ex}\mathcal{S}(\Gamma,\mathcal{W})$ and

$$\begin{aligned}\langle T_0 w_{n,i}^\star - w_{n,i}, w'\rangle \;&=\; \sum_j \alpha_j\, \langle T_0 w_{n,i}^j - T_n w_{n,i}^j, w'\rangle\\ &\leqslant \sup_{w\in\mathcal{S}_n^0(\Gamma,\mathcal{W})} \langle T_0 w - T_n w, w'\rangle \underset{n\to\infty}{\to} 0\,, \qquad \forall w' \in \mathcal{W}'\,.\end{aligned}$$

From the above, it follows that for each function $w' \in \mathcal{W}'$, choosing a sufficiently large n and then a sufficiently large i, we can make the sum on the right hand side of the inequality

$$\begin{aligned}&|\langle w - w_{n,i}, w'\rangle| \leqslant |\langle w - T_0 w_n^\star, w'\rangle| + |\langle T_0 w_n^\star - T_0 w_{n,i}^\star, w'\rangle| + |\langle T_0 w_{n,i}^\star - w_{n,i}, w'\rangle|\\ &\quad\leqslant\; |\langle w - T_0 w_n^\star, w'\rangle| + |\langle T_0 w_n^\star - T_0 w_{n,i}^\star, w'\rangle| + \sup_{w\in\mathcal{S}_n^0(\Gamma,\mathcal{W})} \langle T_0 w - T_n w, w'\rangle\end{aligned}$$

arbitrarily small. By the separation theorem, this implies (3.2). □

Remark 3.4. If we increase the dual space $\mathcal{W}^*$ then the topology on $\mathcal{W}$ becomes finer. Therefore, without loss of generality, in Theorem 3.3 one can replace $(\mathbf{w}_1)$ with the stronger condition

$(\mathbf{w}_1')$ $\mathcal{W}^* = \mathcal{W}'$.

One can also assume that $\mathfrak{T}$ is the strongest topology satisfying $(\mathbf{w}_1')$ (which is called the Mackey topology). Finally, since $\mathcal{S}(\Gamma) \subset \mathcal{W}_1$, we can always assume that $\mathcal{W} \subseteq \mathcal{W}_1$ because a reduction of $\mathcal{W}$ increases the space $\mathcal{W}'$. If we take a smaller space $\mathcal{W} \subset \mathcal{W}_1$ (for example, one may wish to consider functions whose restrictions to Ω_k belong to a weighted space l^p) then Theorem 3.3 gives a stronger result which is valid for a narrower class of functions $w \in \mathcal{S}(\Gamma,\mathcal{W})$.

Remark 3.5. The topological space $(\mathcal{W},\mathfrak{T})$ may be incomplete. However, if $\mathcal{W} = \mathcal{W}''$ and $\mathcal{W}^* = \mathcal{W}'$ then $\mathcal{W}$ is complete in the Mackey topology and is weakly sequentially complete (see, for example, Section 30.5 in [K]).

Example 3.6. Let us consider the coarsest topology on $\mathcal{W} = \mathcal{W}_1$ satisfying $(\mathbf{w}_1)$ and $(\mathbf{w}_2)$, with respect to which all the functionals $w \to w(g)$ are continuous. This topology is generated by the seminorms $p_k(w) := \sum_{g\in\Omega_k} |w(g)|$ and $p_g(w) := |w(g)|$ and, consequently, is metrizable. Therefore Theorem 3.3 implies that, under the conditions $(\mathbf{a})$ and $(\mathbf{a}_2)$, for every function $w \in \mathcal{S}(\Gamma)$ there exists a sequence of functions $w_n \in \operatorname{conv}\operatorname{ex}\mathcal{S}(\Gamma)$ such that

$$|w(g) - w_n(g)| \underset{n\to\infty}{\to} 0 \quad\text{and}\quad \sum_{g\in\Omega_j} |w(g) - w_n(g)| \underset{n\to\infty}{\to} 0$$

for all $g \in \Omega$ and $k = 1, 2, \dots$ In [Ke] the equality (3.2) was proved for this coarsest topology on the space of infinite matrices.

A topology $\mathfrak{T}$ satisfying the conditions $(\mathbf{w}_1)$ and $(\mathbf{w}_2)$ (in particular, the Mackey topology), may well be non-metrizable. Therefore, in the general case, (3.2) does not imply the existence of a sequence of convex linear combinations $w_n \in \operatorname{conv} \operatorname{ex} \mathcal{S}(\Gamma, \mathcal{W})$ convergent to a given function $w \in \mathcal{S}(\Gamma, \mathcal{W})$. It is possible that Theorem 3.3 can be improved in this direction (note that in the parts 1 and 3 of the proof we spoke about sequential closures).

Remark 3.7. It seems to be natural to consider the closure of $\operatorname{conv} \operatorname{ex} \mathcal{S}(\Gamma)$ with respect to the norm $\|w\|_{\mathcal{S}} := \sup_j \sum_{g \in \Omega_j} |w(g)|$ on the space $\mathcal{W}_1$. However, this closure does not always coincide with $\mathcal{S}(\Gamma)$.

Indeed, let Γ be an infinite family of mutually disjoint sets Ω_k and $\# \Omega_k = k$. Then $\operatorname{ex} \mathcal{S}(\Gamma) = \mathcal{P}(\Gamma)$ and $\mathcal{S}(\Gamma)$ contains the function w_0 defined by the equalities $w_0|_{\Omega_k} \equiv k^{-1}$. On the other hand,

$$\sup_j \# \{g \in \Omega_j \,:\, w(g) \neq 0\} < \infty\,, \qquad \forall w \in \operatorname{conv} \mathcal{P}(\Gamma)\,. \tag{3.3}$$

Therefore $\|w_0 - w\|_{\mathcal{S}} = 1$ for all $w \in \operatorname{conv} \mathcal{P}(\Gamma)$.

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