ASYMPTOTIC ESTIMATES OF THE DIFFERENCE BETWEEN THE DIRICHLET AND NEUMANN COUNTING FUNCTIONS

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To the memory of M. S. Birman

Abstract. Dirichlet and Neumann problems for the Laplace operator in a bounded domain in Euclidean space are considered. Some estimates of the difference $N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda)$ of counting functions are discussed.

Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open bounded set in the Euclidean space of the dimension $d \geqslant 2$, and let Δ_{D} and Δ_{N} be the Dirichlet and Neumann Laplacians on Ω. Recall that $-\Delta_{\mathrm{D}}$ and $-\Delta_{\mathrm{N}}$ are defined as the nonnegative self-adjoint operators generated by the Dirichlet form $\int_\Omega |\nabla u|^2\,\mathrm{d}x$ with domains $W_{2,0}^1(\Omega)$ and $W_2^1(\Omega)$ respectively. Here $W_2^1(\Omega)$ is the Sobolev space and $W_{2,0}^1(\Omega)$ is the W_2^1-closure of $C_0^\infty(\Omega)$. Note that the Dirichlet and Neumann Laplacian are well-defined even if the boundary $\partial\Omega$ is not smooth.

Further on we shall be using the subscript B instead of D or N whenever the definition refers to or the result is valid both for the Dirichlet and Neumann Laplacians.

Let $N_{\mathrm{B}}(\lambda)$ be the counting functions of the operators $-\Delta_{\mathrm{B}}$. By definition,

$$N_{\mathrm{B}}(\lambda) \;:=\; \operatorname{rank} P_\lambda(-\Delta_{\mathrm{B}})$$

where $P_\lambda(-\Delta_{\mathrm{B}})$ denotes the spectral projection of $-\Delta_{\mathrm{B}}$ corresponding to the interval $[0,\lambda^2)$. Since the embedding $W_{2,0}^1(\Omega) \subset L_2(\Omega)$ is compact, $N_{\mathrm{D}}(\lambda)$ is finite for each λ. The Neumann counting function $N_{\mathrm{N}}(\lambda)$ may well be infinite. For instance, if Ω consists of infinitely many connected components then $\dim\ker(-\Delta_{\mathrm{N}}) = \infty$ and $N_{\mathrm{N}}(\lambda) = \infty$ for all $\lambda > 0$.

The aim of this paper is to obtain estimates of the difference $N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda)$ for large values of λ. If the boundary $\partial\Omega$ is sufficiently smooth and the billiard flow in Ω satisfies the non-periodicity condition (see the Section 1) then, by the Weyl asymptotic formula,

$$N_{\mathrm{D}}(\lambda) \;=\; \varkappa_d\,|\Omega|_d\,\lambda^d - \frac{1}{4}\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1} + o(\lambda^{d-1})\,, \qquad \lambda\to\infty\,, \tag{0.1}$$

$$N_{\mathrm{N}}(\lambda) \;=\; \varkappa_d\,|\Omega|_d\,\lambda^d + \frac{1}{4}\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1} + o(\lambda^{d-1})\,, \qquad \lambda\to\infty\,. \tag{0.2}$$

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Here and further on $|\cdot|_n$ stands for the n-dimensional volume, $\varkappa_n := (2\pi)^{-n}\omega_n$, and ω_n is the volume of the n-dimensional unit ball.

The above asymptotic formulae imply that

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \;=\; \frac{1}{2}\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1} + o(\lambda^{d-1})\,, \qquad \lambda\to\infty\,.$$

However, (0.1) and (0.2) may fail for the following two reasons.

(a) The boundary $\partial\Omega$ is not smooth enough. Indeed, there are many examples showing that the two-term Weyl asymptotic formulae do not hold for general domains with irregular boundaries.

(b) The boundary $\partial\Omega$ is smooth but the non-periodicity condition is not fulfilled. It is not known whether such domains exist (see Remark 1.3), so it is possible that this obstacle is only of technical nature.

In this paper we show that

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \;\leqslant\; C_\Omega\,\lambda^{(d-1)/\alpha}\,, \quad \text{ if } \partial\Omega\in C^\alpha\,, \quad \alpha\in(0,1)\,,$$

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \;\leqslant\; C_\Omega\,\lambda^{d-1}\ln\lambda\,, \quad \text{ if } \partial\Omega \text{ is Lipschitz}\,,$$

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \;\geqslant\; C_\Omega\,\lambda^{d-4}|\Omega_{\lambda^{-1}}|_d^{-1} \quad \text{ for an arbitrary domain } \Omega\,,$$

where $\Omega_\varepsilon \;:=\; \{x\in\Omega\,:\,\operatorname{dist}(x,\partial\Omega)\leqslant\varepsilon\}$. The last estimate implies, in particular, that

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda)\to\infty \quad \text{ as } \lambda\to\infty \quad \text{ for all } d\geqslant 4.$$

For domains with smooth boundaries we obtain the following more precise estimate,

$$\frac{1}{4}\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1}+o(\lambda^{d-1}) \leqslant N_{\mathrm{N}}(\lambda)-N_{\mathrm{D}}(\lambda) \leqslant \frac{3}{4}\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1}+o(\lambda^{d-1})\,, \qquad \lambda\to\infty\,,$$

which holds without the non-periodicity condition.

Throughout the paper C_d and C_Ω denote various constants, depending either only on the dimension d or on the domain Ω, whose precise values are not important for our purposes.

1. Weyl type asymptotic formulae

Let us assume that the boundary $\partial\Omega$ is sufficiently smooth and denote by n_x the unit normal vector to the boundary at the point x directed inside Ω. Let $S\,\overline{\Omega}$ be the unit tangent bundle over $\overline{\Omega}$,

$$S\,\overline{\Omega} \;:=\; \{(x,\xi)\,:\,x\in\overline{\Omega},\ \xi\in\mathbb{R}^d,\ |\xi|=1\}\,,$$

and let

$$S_+\overline{\Omega} \;:=\; \{(x,\xi)\in S\,\overline{\Omega}\,:\,\text{either } x\in\Omega,\ \text{or } x\in\partial\Omega \text{ and } \xi\cdot n_x>0\}\,,$$

If $(x,\xi)\in S_+\overline{\Omega}$, let us consider the trajectory

$$(x(t),\xi(t)) \;:= (x+t\xi,\xi) \;=\; (x(t),\dot{x}(t))\,, \qquad t>0\,.$$

Clearly, $(x(t),\xi(t))\in S_+\overline{\Omega}$ for all sufficiently small t. When the trajectory $x(t)$ hits the boundary at a time t_0, we reflect it according to the standard law of geometric optics and define $\xi(t)$ to be the unit tangent vector of the reflected trajectory for $t>t_0$. Continuing the procedure, we obtain a trajectory $(x(t),\xi(t))$ in $S_+\overline{\Omega}$ composed of line segments of the form $(x+t\xi,\xi)$, which is called a billiard trajectory. The family of maps $\Phi_t : S_+\overline{\Omega}\mapsto S_+\overline{\Omega}$

defined by $\Phi_t(x,\xi) := (x(t),\xi(t))$ is said to be the billiard flow. It is known that Φ_t is well defined almost everywhere for all $t \geqslant 0$ and that Φ_t preserves the standard measure $\mathrm{d}x\,\mathrm{d}\xi$ on $S_+\overline{\Omega}$. Precise definitions and proofs of these and other relevant results can be found, for instance, in [CFS, Chapter 6] or [SV].

The point $(x,\xi) \in S_+\overline{\Omega}$ is called periodic if $\Phi_T(x,\xi) = (x,\xi)$ for some $T > 0$. Let $\Pi \subset S_+\overline{\Omega}$ be the set of periodic points of the flow Φ_t. The non-periodicity condition asserts that

$$|\Pi|_{2d-1} := \int_\Pi \mathrm{d}x\,\mathrm{d}\xi = 0\,.$$

Theorem 1.1. *The Weyl asymptotic formulae* (0.1) *and* (0.2) *hold whenever the billiard flow* Φ_t *satisfies the non-periodicity condition.*

Remark 1.2. Theorem 1.1 was obtained in [Iv1] for infinitely smooth boundaries $\partial\Omega$ (see also [SV] for a detailed and relatively simple proof). Later V. Ivrii extended this result to the sets Ω whose boundaries belong to the Hölder class C^α with $\alpha > 1$ [Iv2].

Remark 1.3. It is a long standing conjecture that $|\Pi|_{2d-1} = 0$ for every domain Ω with a sufficiently smooth boundary. While no counterexamples are known, to the best of our knowledge, there are only two positive results in this direction. Namely, $|\Pi|_{2d-1} = 0$ if

(i) $\partial\Omega$ is analytic and convex (see [Va2, Section 1] or [SV, Lemma 1.3.19]);
(ii) $\partial\Omega$ is piecewise smooth and concave (see [Va1, Theorem 2]). In particular, it is true if Ω is a polyhedron.

If $|\Pi|_{2d-1} > 0$ then the asymptotic behaviour of the counting functions may depend on the properties of periodic trajectories. It turns out that there are two characteristics which affect the second asymptotic term. The first is the minimal period $T(x,\xi)$ of the periodic trajectory originating from $(x,\xi) \in \Pi$, and the second is the so-called total phase shift $q_{\mathrm{B}}(x,\xi)$ along the primitive periodic trajectory $\gamma(x,\xi) := \{\Phi_t(x,\xi)\}_{t\in[0,T(x,\xi)]}$. The phase shift depends on the boundary condition. The precise definition of the phase shift $q_{\mathrm{B}}(x,\xi)$ is given in [SV], where it is shown that the set Π and the functions $T(x,\xi)$, $q_{\mathrm{B}}(x,\xi)$ are measurable.

If $\tau \in \mathbb{R}$, let $\{\tau\}_{2\pi} = \tau + 2\pi k$ where k is the integer such that $\tau + 2\pi k \in (-\pi,\pi]$. Denote

$$Q_{\mathrm{B}}(\lambda) \;:=\; (2\pi)^{-d} \int_\Pi \frac{\{\pi - q_{\mathrm{B}}(x,\xi) - \lambda T(x,\xi)\}_{2\pi}}{T(x,\xi)}\,\mathrm{d}x\,\mathrm{d}\xi\,. \tag{1.1}$$

The following theorem was proved in [GS] (see also [SV, Section 1.7]).

Theorem 1.4. *If $\partial\Omega$ is infinitely smooth then there exists a function $h:\mathbb{R}_+\to\mathbb{R}$ such that*

$$N_{\mathrm{D}}(\lambda) \leqslant \varkappa_d\,|\Omega|_d\,\lambda^d - \frac14\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1} + Q_{\mathrm{D}}(\lambda+h(\lambda))\lambda^{d-1} + o(\lambda^{d-1})\,, \tag{1.2}$$

$$N_{\mathrm{D}}(\lambda) \geqslant \varkappa_d\,|\Omega|_d\,\lambda^d - \frac14\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1} + Q_{\mathrm{D}}(\lambda-h(\lambda))\lambda^{d-1} + o(\lambda^{d-1})\,, \tag{1.3}$$

$$N_{\mathrm{N}}(\lambda) \leqslant \varkappa_d\,|\Omega|_d\,\lambda^d + \frac14\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1} + Q_{\mathrm{N}}(\lambda+h(\lambda))\lambda^{d-1} + o(\lambda^{d-1})\,, \tag{1.4}$$

$$N_{\mathrm{N}}(\lambda) \geqslant \varkappa_d\,|\Omega|_d\,\lambda^d + \frac14\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1} + Q_{\mathrm{N}}(\lambda-h(\lambda))\lambda^{d-1} + o(\lambda^{d-1}) \tag{1.5}$$

and $h(\lambda)\to 0$ as $\lambda\to\infty$.

Note that the function Q_{B} may well be discontinuous. Therefore, generally speaking, it is impossible to remove $h(\lambda)$ from the right hand sides of the above inequalities (see the discussion in [SV, Section 1.7]).

2. Estimates for domains with smooth boundaries

Theorem 2.1. *If $\partial\Omega$ is infinitely smooth then*

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \leqslant \frac34\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1} + o(\lambda^{d-1})\,, \qquad \lambda\to\infty\,, \tag{2.1}$$

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \geqslant \frac14\,\varkappa_{d-1}\,|\partial\Omega|_{d-1}\,\lambda^{d-1} + o(\lambda^{d-1})\,, \qquad \lambda\to\infty\,. \tag{2.2}$$

Proof. Let $B_{\partial\Omega}$ be the set of pairs (x',ξ') where $x'\in\partial\Omega$ and ξ' is an $(d-1)$-dimensional tangent vector to $\partial\Omega$ at the point x' such that $|\xi'|<1$. For each $(x',\xi')\in B_{\partial\Omega}$ there exists a unique d-dimensional vector $\theta(\xi')$ such that $(x',\theta(\xi'))\in S_+\overline{\Omega}$ and $\theta(\xi')-\xi'$ is a normal to $\partial\Omega$ vector directed inside Ω. The billiard trajectories $\Phi_t(x',\theta(\xi'))$ originating from all such points cover a subset of $S_+\overline{\Omega}$ of full measure. Moreover, if we parameterize $S_+\overline{\Omega}$ by $t\in\mathbb{R}$ and $(x',\xi')\in B_{\partial\Omega}$ then $\mathrm{d}x\,\mathrm{d}\xi=\mathrm{d}t\,\mathrm{d}x'\mathrm{d}\xi'$, where $\mathrm{d}x'$ is the standard measure on $\partial\Omega$ [CFS, Chapter 6].

Let $\Pi_k\subset S_+\overline{\Omega}$ be the set of points $(x,\xi)\in\Pi$ such that the corresponding primitive periodic trajectories $\gamma(x,\xi)$ experience k reflections. Let

$$M_k=\{(x',\xi')\in B_{\partial\Omega}\ :\ (x',\theta(\xi'))\in\Pi_k\}.$$

Consider the set

$$\hat\Pi_k\ :=\ \{(t,x',\xi')\,:\,(x',\xi')\in M_k\,,\ 0\leqslant t<T(x',\theta(\xi'))\}$$

and the mapping $\Psi_k:(t,x',\xi')\mapsto(x,\xi):=\Phi_t(x',\theta(\xi'))$ from $\hat\Pi_k$ onto Π_k. Obviously, the inverse image $\Psi_k^{-1}(x,\xi)$ of every point $(x,\xi)\in\Pi_k$ consists of k distinct points of $\hat\Pi_k$. By the above,

$$\int_V \mathrm{d}t\,\mathrm{d}x'\,\mathrm{d}\xi'\ =\ \int_{\Psi_k(V)}\mathrm{d}x\,\mathrm{d}\xi$$

for any measurable set $V \subset \hat{\Pi}_k$ such that the restriction $\Psi_k|_V$ is a bijection. Therefore

$$\int_{\Pi_k} f(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \;=\; k^{-1}\int_{\hat{\Pi}_k} f(\Psi_k(t,x',\xi'))\,\mathrm{d}t\,\mathrm{d}x'\,\mathrm{d}\xi'$$

for all measurable functions $f:\Pi_k \mapsto \mathbb{R}_+$.

In particular, if $f(x,\xi)=\frac{1}{T(x,\xi)}$ then, integrating over t, we obtain

$$\int_{\Pi_k} \frac{\mathrm{d}x\,\mathrm{d}\xi}{T(x,\xi)} \;=\; k^{-1}\int_{\hat{\Pi}_k} \frac{\mathrm{d}x\,\mathrm{d}\xi}{T(\Psi_k(t,x',\xi'))}\,\mathrm{d}t\,\mathrm{d}x'\,\mathrm{d}\xi' \;=\; k^{-1}\int_{M_k}\mathrm{d}x'\,\mathrm{d}\xi'\,.$$

Since the sets Π_k are disjoint, the integral in the right hand side of (1.1) is equal to the sum of integrals over Π_k. Estimating in each of these integrals $|\{\pi - q_{\mathrm{B}}(x,\xi) - \lambda T(x,\xi)\}_{2\pi}| \leqslant \pi$ and using the above identity, we see that

$$|Q_{\mathrm{B}}(\lambda)| \;\leqslant\; (2\pi)^{-d}\,\pi\sum_{k=1}^{\infty} k^{-1}\int_{M_k}\mathrm{d}x'\,\mathrm{d}\xi'\,. \tag{2.3}$$

Clearly, $M_1=\varnothing$. One can easily show that $|M_2|_{2d-2}=0$. Finally, according to [Vo], the measure of M_3 is also zero. Thus we have

$$|Q_{\mathrm{B}}(\lambda)| \;\leqslant\; (2\pi)^{-d}\,\frac{\pi}{4}\sum_{k=4}^{\infty}\int_{M_k}\mathrm{d}x'\,\mathrm{d}\xi' \;\leqslant\; (2\pi)^{-d}\,\frac{\pi}{4}\int_{B_{\partial\Omega}}\mathrm{d}x'\,\mathrm{d}\xi' \;=\; \frac{1}{8}\,\varkappa_{d-1}\,|\partial\Omega|\,.$$

This estimate and (1.2)–(1.5) imply the theorem. □

Remark 2.2. It is plausible that Theorem 1.4 remains valid whenever $\partial\Omega\in C^\alpha$ with $\alpha>1$ (cf. Remark 1.2). If it is true then Theorem 2.1 also holds for such domains.

Remark 2.3. Theorem 1.4 can be extended to the Laplacian on a Riemannian manifold Ω with boundary. If the measure of the set of periodic trajectories lying in the interior $\Omega\setminus\partial\Omega$ is equal to zero then the same arguments as in the proof of Theorem 2.1 yield (2.3). In this case, making additional assumptions about the measure of the sets Π_k with small k, one can obtain estimates similar to (2.1) and (2.2).

3. Irregular boundaries: upper bounds

In order to estimate $N_{\mathrm{N}}-N_{\mathrm{D}}$ from above, we prove a lower bound for N_{D} and an upper bound for N_{N}. The former is a relatively simple task (see [NS, Theorem 1.8] and [Sa, (3.18)]).

Theorem 3.1. *For every open set $\Omega\subset\mathbb{R}^d$*

$$\left|N_{\mathrm{D}}(\lambda)-\varkappa_d\,|\Omega|_d\,\lambda^d\right| \;\leqslant\; C_d\,\lambda^{d-1}\int_0^\lambda |\Omega_{t^{-1}}|\,\mathrm{d}t\,,\qquad \forall\lambda>0\,, \tag{3.1}$$

where

$$\Omega_\varepsilon \;:=\; \{x\in\Omega\,:\,\operatorname{dist}(x,\partial\Omega)\leqslant\varepsilon\}\,.$$

If the boundary $\partial\Omega$ is Lipschitz then $|\Omega_\varepsilon|_d \sim \varepsilon$ as $\varepsilon \to 0$, and

$$\lambda^{d-1}\int_0^\lambda |\Omega_{t^{-1}}|\,\mathrm{d}t \;=\; O(\lambda^{d-1}\ln\lambda)\,, \quad \lambda\to\infty. \tag{3.2}$$

If $\partial\Omega \in C^\alpha$, $\alpha\in(0,1)$, then

$$\lambda^{d-1}\int_0^\lambda |\Omega_{t^{-1}}|\,\mathrm{d}t \;=\; O(\lambda^{d-\alpha})\,, \quad \lambda\to\infty \tag{3.3}$$

(see [NS, Lemma 4.4]).

The problem of estimating N_{N} for irregular domains is much more difficult. To the best of our knowledge, the most general results in this direction were obtained in [NS] and [N]. The statements of these results are not simple; it turns out that the asymptotic behaviour of $N_{\mathrm{N}}(\lambda)$ as $\lambda\to\infty$ depends on subtle properties of the boundary. The following theorem is contained in [NS].

Theorem 3.2. *If the boundary $\partial\Omega$ is Lipschitz then*

$$N_{\mathrm{N}}(\lambda) \;=\; \varkappa_d\,|\Omega|_d\,\lambda^d + O(\lambda^{d-1}\ln\lambda)\,, \qquad \lambda\to\infty\,. \tag{3.4}$$

If $\partial\Omega\in C^\alpha$ with $\alpha\in(0,1)$ then

$$N_{\mathrm{N}}(\lambda) \;=\; \varkappa_d\,|\Omega|_d\,\lambda^d + O(\lambda^{(d-1)/\alpha})\,, \qquad \lambda\to\infty\,. \tag{3.5}$$

Since $(d-1)/\alpha \geqslant d-\alpha$ for all $\alpha\in(0,1)$, the estimates (3.1), (3.3) and (3.5) imply that

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \;\leqslant\; C_\Omega\,\lambda^{(d-1)/\alpha}\,, \qquad \forall\lambda>1\,, \qquad \forall\alpha\in(0,1)\,, \tag{3.6}$$

whenever $\partial\Omega\in C^\alpha$. From (3.1), (3.2) and (3.4) it follows that for domains with Lipschitz boundaries

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \;\leqslant\; C_\Omega\,\lambda^{d-1}\ln\lambda\,, \qquad \forall\lambda>2\,. \tag{3.7}$$

4. Irregular boundaries: lower bounds

Theorem 4.1. *If $d\geqslant 4$ then there exists a constant $C_\Omega>0$ such that*

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \;\geqslant\; C_\Omega\,\lambda^{d-4}|\Omega_{\lambda^{-1}}|_d^{-1}\,, \qquad \forall\lambda>0\,. \tag{4.1}$$

Note that $|\Omega_\varepsilon|_d\to 0$ as $\varepsilon\to 0$ because $\bigcap_{\varepsilon>0}\Omega_\varepsilon=\varnothing$, $|\Omega_\varepsilon|_d\leqslant|\Omega|_d<\infty$ and $\Omega_\delta\subset\Omega_\varepsilon$ for all $\varepsilon>0$ and $\delta\leqslant\varepsilon$. Thus Theorem 4.1 implies that

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \underset{\lambda\to\infty}{\longrightarrow} \infty\,, \qquad \forall\, d\geqslant 4\,. \tag{4.2}$$

For Lipschitz boundaries (4.1) turns into

$$N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) \;\geqslant\; C_\Omega\,\lambda^{d-3}\,. \tag{4.3}$$

Remark 4.2. In the dimensions $d=2,3$ the inequalities (4.1) and (4.3) have no significance, since $N_{\mathrm{N}}(\lambda)-N_{\mathrm{D}}(\lambda)\;\geqslant\;1$ for all $d\geqslant 2$ (this estimate was proved in [Fr] for domains Ω with smooth boundaries and extended in [Fi] to arbitrary bounded open sets Ω).

Remark 4.3. There are reasons to believe that the difference $N_{\rm N}(\lambda)-N_{\rm D}(\lambda)$ grows faster as the boundary of Ω becomes less regular. Therefore, in view of Theorem 2.1, (4.2) is likely to be true in all dimensions $d\geqslant 2$. Moreover, it is plausible that $N_{\rm N}(\lambda)-N_{\rm D}(\lambda)\geqslant C_\Omega\,\lambda^{d-1}$ for all $d\geqslant 2$ and all bounded open domains Ω. However, we are unaware of any technique that could be used to prove these conjectures.

Remark 4.4. Recall that $W^1_{2,0}(\Omega)$ is a closed subspace of $W^1_2(\Omega)$ of infinite codimension. One could assume that the difference between counting functions of two nonnegative operators generated by the same quadratic form on two such domains always tends to infinity. However, it is not true. One can construct a positive function g on $\mathbb{R}$ and two dense subspaces D_1, D_2 of $L_2(\mathbb{R})$ such that

- D_1 is a subspace of D_2 of infinite codimension;
- $\int_{\mathbb{R}} g(t)\,|u'(t)|^2\,{\rm d}t<\infty$ for all $u\in D_2$;
- D_1 and D_2 are closed with respect to the form $\int_{\mathbb{R}} g(t)\,|u'(t)|^2\,{\rm d}t$;
- the counting functions of the operators generated by the restriction of this form to D_1 and D_2 coincide on an unbounded subset of $\mathbb{R}_+$.

Proof of Theorem 4.1. The proof proceeds in several steps.

Step 1. Denote by $L_{\rm D}(\lambda)$ the subspace of $W^1_{2,0}(\Omega)$ spanned by the eigenfunctions of the operator $-\Delta_{\rm D}$ corresponding to the eigenvalues $\lambda_k<\lambda^2$. Suppose that L is a subspace of $W^1_2(\Omega)$ such that

(4.4) $$L\cap L_{\rm D}(\lambda)=\{0\}\,,\quad -\Delta v=\lambda^2 v\quad\text{and}\quad \lambda^2\,\|v\|^2_{L_2(\Omega)}>\|\nabla v\|^2_{L_2(\Omega)}\ \text{for all}\ v\in L\setminus\{0\}\,.$$

If $u\in L_{\rm D}(\lambda)$, $v\in L$ and $u+v\neq 0$ then

$$\begin{aligned}\|\nabla(u+v)\|^2_{L_2(\Omega)} &= \int_\Omega\left(|\nabla u|^2+|\nabla v|^2+2\,{\rm Re}\,\langle\nabla u,\nabla v\rangle\right){\rm d}x\\ &< \lambda^2\int_\Omega\left(|u|^2+|v|^2\right){\rm d}x-2\,{\rm Re}\int_\Omega u\,\overline{\Delta\,v}\,{\rm d}x \;=\; \lambda^2\|u+v\|^2_{L_2(\Omega)}\,.\end{aligned}$$

Clearly, $\dim\left(L_{\rm D}(\lambda)+L\right)=N_{\rm D}(\lambda)+\dim L$. By the variational principle,

$$N_{\rm N}(\lambda)=\max\{\dim F: F\subset W^1_2(\Omega)\ \text{and}\ \|\nabla f\|^2<\lambda^2\|f\|^2\quad\text{for all}\ f\in F\setminus\{0\}\}\,.$$

This implies that

(4.5) $$N_{\rm N}(\lambda)\geqslant N_{\rm D}(\lambda)+\dim L\,.$$

In Step 5 we shall construct a subspace L satisfying (4.4) and estimate its dimension. It will consist of certain linear combinations of the functions $u_{\sigma_m,j}$ defined in the next step.

Step 2. Let $\hat\chi$ be the Fourier transform of the characteristic function χ of the set Ω. Let us fix positive constants δ and $C_{\delta,\Omega}$ such that

$$|\hat\chi(\theta)|\geqslant C_{\delta,\Omega}\qquad\text{whenever}\ \ |\theta|\leqslant\delta$$

(such constants exist because $\hat\chi$ is continuous and $\hat\chi(0)=(2\pi)^{-d/2}|\Omega|_d>0$).

Let $\xi\in\mathbb{R}^d$ and $|\xi|=\lambda$ with $\lambda>\delta/2$. Let us denote

$$S_\delta(\xi)\ :=\ \{\eta\in\mathbb{R}^d\ :\ |\eta|=|\xi|\,,\ |\eta-\xi|=\delta\}\,.$$

Clearly, the set $S_\delta(\xi)$ is a $(d-2)$-dimensional sphere whose volume does not depend on ξ. We shall write $|S_\delta(\lambda)|_{d-2} := |S_\delta(\xi)|_{d-2}$.

Let us now consider collections $\sigma_m := \{\xi_j^{(p)}\}_{j=1,2,\dots,m}^{p=1,2}$ consisting of $2m$ points $\xi_j^{(p)} \in \mathbb{R}^d$ such that $|\xi_j^{(p)}| = \lambda$ and $\xi_j^{(2)} \in S_\delta(\xi_j^{(1)})$. Such collections form a $(2d-3)m$-dimensional manifold $\Sigma_{\delta,m}(\lambda)$ whose volume is

$$|\Sigma_{\delta,m}(\lambda)|_{(2d-3)m} \;=\; \left((d-1)\,\omega_{d-1}\lambda^{d-1}\,|S_\delta(\lambda)|_{d-2}\right)^m .$$

Given a collection $\sigma_m = \{\xi_j^{(p)}\}_{j=1,2,\dots,m}^{p=1,2} \in \Sigma_{\delta,m}(\lambda)$, let us define the functions

$$u_{\sigma_m,j}(x) \;=\; (2\pi)^{-d/4}\left(e^{ix\cdot\xi_j^{(1)}} + \zeta(\xi_j^{(2)}-\xi_j^{(1)})\,e^{ix\cdot\xi_j^{(2)}}\right), \qquad j=1,\dots,m\,,$$

where $\zeta(\theta) := \hat\chi(\theta)|\hat\chi(\theta)|^{-1}$. Since

$$\lambda^2 - \xi_j^{(p)}\cdot\xi_k^{(q)} = \frac12\,|\xi_j^{(p)}-\xi_k^{(q)}|^2\,,$$

we have

$$\begin{aligned}(4.6)\quad &\left|\lambda^2\,(u_{\sigma_m,j},u_{\sigma_m,k})_{L_2(\Omega)} - (\nabla u_{\sigma_m,j},\nabla u_{\sigma_m,k})_{L_2(\Omega)}\right| \\ &\leqslant\; (2\pi)^{-d/2}\sum_{p,q=1,2}\left|\lambda^2\,(e^{ix\cdot\xi_j^{(p)}},e^{ix\cdot\xi_k^{(q)}})_{L_2(\Omega)} - (\nabla e^{ix\cdot\xi_j^{(p)}},\nabla e^{ix\cdot\xi_k^{(q)}})_{L_2(\Omega)}\right| \\ &= \frac12\sum_{p,q=1,2}|\xi_j^{(p)}-\xi_k^{(q)}|^2\,|\hat\chi(\xi_j^{(p)}-\xi_k^{(q)})|\end{aligned}$$

for all $j\ne k$, and

$$(4.7)\quad \lambda^2\,\|u_{\sigma_m,j}\|^2_{L_2(\Omega)} - \|\nabla u_{\sigma_m,j}\|^2_{L_2(\Omega)} \;=\; |\xi_j^{(1)}-\xi_j^{(2)}|^2\,|\hat\chi(\xi_j^{(1)}-\xi_j^{(2)})| \;\geqslant\; C_{\delta,\Omega}\,\delta^2\,, \qquad \forall j$$

due to the choice of δ.

Step 3. Let L'_{σ_m} be the subspace of $W_2^1(\Omega)$ spanned by all the functions $u_{\sigma_m,j}$. Obviously, we have $-\Delta u = \lambda^2 u$ for all $u\in L'_{\sigma_m}$. If all the points $\{\xi_j^{(p)}\}_{j=1,\dots,m}^{p=1,2}$ in σ_m are distinct then the functions $u_{\sigma_m,j}$ are linearly independent and $\dim L'_{\sigma_m} = m$. Denote the set of points $\sigma_m\in\Sigma_{\delta,m}(\lambda)$ such that $\dim L'_{\sigma_m} = m$ and $L'_{\sigma_m}\bigcap L_{\mathrm D}(\lambda) = \{0\}$ by $\Sigma^{(1)}_{\delta,m}(\lambda)$. This set is everywhere dense in $\Sigma_{\delta,m}(\lambda)$.

If $v\in L'_{\sigma_m}$ and $v=\sum_{j=1}^m \alpha_j\,u_{\sigma_m,j}$ then

$$(4.8)\qquad \lambda^2\,\|v\|^2_{L_2(\Omega)} - \|\nabla v\|^2_{L_2(\Omega)} = (B_{\sigma_m}\vec\alpha,\vec\alpha)_{\mathbb{C}^m}\,,$$

where $\vec\alpha$ is the column $(\alpha_1,\dots,\alpha_m)^T$ and B_{σ_m} is the Hermitian $m\times m$-matrix with the entries

$$b_{jk} \;:=\; \lambda^2\,(u_{\sigma_m,j},u_{\sigma_m,k})_{L_2(\Omega)} - (\nabla u_{\sigma_m,j},\nabla u_{\sigma_m,k})_{L_2(\Omega)}\,.$$

Let $\tilde B_{\sigma_m}$ be the diagonal matrix with entries $b_{11},\dots,b_{mm}$. By (4.7),

$$(4.9)\qquad \tilde B_{\sigma_m} \;\geqslant\; C_{\delta,\Omega}\,\delta^2 I\,.$$

The estimate (4.6) implies that

$$\|B_{\sigma_m} - \tilde{B}_{\sigma_m}\|_2^2 \;=\; \sum_{j\neq k} |b_{jk}|^2 \;\leqslant\; \sum_{j\neq k}\sum_{p,q=1,2} |\xi_j^{(p)} - \xi_k^{(q)}|^4\,|\hat\chi(\xi_j^{(p)} - \xi_k^{(q)})|^2\,,$$

where $\|\cdot\|_2$ is the Hilbert–Schmidt norm.

Step 4. If $d \geqslant 3$ then for all $\lambda > 0$ we have the inequality

$$\int_{\Sigma_{\delta,m}(\lambda)} \left(\sum_{j\neq k}\sum_{p,q=1,2} |\xi_j^{(p)} - \xi_k^{(q)}|^4\,|\hat\chi(\xi_j^{(p)} - \xi_k^{(q)})|^2\right) \mathrm{d}\xi_1^{(1)}\,\mathrm{d}\xi_1^{(2)}\ldots\mathrm{d}\xi_m^{(1)}\,\mathrm{d}\xi_m^{(2)} \leqslant\; C_d\,|\Sigma_{\delta,m}(\lambda)|_{(2d-3)m}\; m^2\,\lambda^{4-d}\,|\Omega_{\lambda^{-1}}|_d\,. \tag{4.10}$$

This is proved by straightforward calculations see §5).

Step 5. The estimate (4.10) implies that there exists a non-empty open set $\Sigma^{(2)}_{\delta,m}(\lambda) \subset \Sigma_{\delta,m}(\lambda)$ such that

$$\|B_{\sigma_m} - \tilde{B}_{\sigma_m}\|_2^2 \;<\; C_d\, m^2\,\lambda^{4-d}\,|\Omega_{\lambda^{-1}}|_d\,,\qquad \forall\sigma_m\in\Sigma^{(2)}_{\delta,m}(\lambda)\,. \tag{4.11}$$

Recall that the square of the Hilbert–Schmidt norm of a matrix coincides with the sum of the squares of its eigenvalues. The estimate (4.11) shows that the number of eigenvalues of the matrix $B_{\sigma_m} - \tilde{B}_{\sigma_m}$ lying in the interval $(-\infty, -C_{\delta,\Omega}\delta^2]$ does not exceed $C'_{\delta,\Omega}\,\delta^{-4}\,m^2\,\lambda^{4-d}\,|\Omega_{\lambda^{-1}}|_d$. This observation and (4.9) imply that B_{σ_m} has at least $m - C'_{\delta,\Omega}\,\delta^{-4}\,m^2\,\lambda^{4-d}\,|\Omega_{\lambda^{-1}}|_d$ positive eigenvalues for all $\sigma_m\in\Sigma^{(1)}_{\delta,m}(\lambda)\bigcap\Sigma^{(2)}_{\delta,m}(\lambda)$.

Let us fix $\sigma_m\in\Sigma^{(1)}_{\delta,m}(\lambda)\bigcap\Sigma^{(2)}_{\delta,m}(\lambda)$ and let L_{σ_m} be the subspace of $W_2^1(\Omega)$ generated by the corresponding eigenvectors of B_{σ_m}. Then L_{σ_m} satisfies the conditions (4.4) due to (4.8), and

$$\dim L_{\sigma_m} \geqslant m - C'_{\delta,\Omega}\,\delta^{-4}\,m^2\,\lambda^{4-d}\,|\Omega_{\lambda^{-1}}|_d\,.$$

Optimizing in m the expression in the right hand side we obtain $\dim L_{\sigma_m} \geqslant C_\Omega\,\lambda^{d-4}|\Omega_{\lambda^{-1}}|_d^{-1}$ for all sufficiently large λ. This inequality and (4.5) imply (4.1). □

5. Appendix. Proof of the estimate (4.10)

We shall deduce (4.10) from the following two lemmas.

Lemma 5.1. *$\int_{|\theta|\leqslant 2\lambda} |\theta|^2\,|\hat\chi(\theta)|^2\,\mathrm{d}\theta \leqslant 9\lambda^2\,|\Omega_{\lambda^{-1}}|_d$ for all $d\geqslant 1$ and all $\lambda > 0$.*

Proof. Let

$$\psi(x) = \begin{cases} 1, & x\in\Omega\setminus\Omega_{\lambda^{-1}}\,,\\ \lambda\,\mathrm{dist}(x,\partial\Omega), & x\in\Omega_{\lambda^{-1}}\,,\\ 0, & x\notin\Omega\,.\end{cases}$$

The triangle inequality implies that

$$|\psi(x)-\psi(y)| \leqslant \lambda\,|x-y| \quad \forall x,y\in\mathbb{R}^d\,.$$

Therefore $\psi \in W^1_\infty(\mathbb{R}^d)$ and $\|\nabla\psi\|_{L_\infty} = \lambda$. Since the function $\chi - \psi$ vanishes outside $\Omega_{\lambda^{-1}}$,

$$\int_{|\theta|\leqslant 2\lambda} |\theta|^2\,|\hat\chi(\theta) - \hat\psi(\theta)|^2\,\mathrm{d}\theta \;\leqslant\; 4\lambda^2\,\|\hat\chi - \hat\psi\|^2_{L_2(\mathbb{R}^d)} \;=\; 4\lambda^2\,\|\chi - \psi\|^2_{L_2(\mathbb{R}^d)} \;\leqslant\; 4\lambda^2\,|\Omega_{\lambda^{-1}}|_d\,.$$

The gradient $\nabla\psi$ vanishes outside $\Omega_{\lambda^{-1}}$, so

$$\int_{|\theta|\leqslant 2\lambda} |\theta|^2\,|\hat\psi(\theta)|^2\,\mathrm{d}\theta \;\leqslant\; \int_{\mathbb{R}^d} |\theta|^2\,|\hat\psi(\theta)|^2\,\mathrm{d}\theta \;=\; \|\nabla\psi\|^2_{L_2(\mathbb{R}^d)} \;\leqslant\; \lambda^2\,|\Omega_{\lambda^{-1}}|_d\,.$$

These inequalities and the elementary estimate $|\hat\chi|^2 \leqslant 3|\hat\psi|^2 + \frac{3}{2}|\hat\chi - \hat\psi|^2$ imply the required result. □

Lemma 5.2. *If $d \geqslant 2$ then*

$$\int_{|\xi|=\lambda}\int_{|\eta|=\lambda} f(\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \;=\; (d-1)\,\omega_{d-1}\lambda^2 \int_{|\theta|\leqslant 2\lambda} f(\theta)\,|\theta|^{-1}(\lambda^2 - |\theta|^2/4)^{(d-3)/2}\,\mathrm{d}\theta \tag{5.1}$$

for all $\lambda > 0$, $f \in C(\mathbb{R}^d)$.

Proof. Let $r \in (0,\lambda]$. First, we consider the integral of $f(\xi-\eta)$ over the product of two balls $\{\xi : |\xi| < \lambda\}$ and $\{\eta : |\eta| < r\}$. Changing variables $\xi = \eta + \theta$, we see that

$$\int_{|\xi|<\lambda}\int_{|\eta|<r} f(\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \;=\; \int_{|\theta|<\lambda+r} f(\theta)\,F(\theta)\,\mathrm{d}\theta\,, \tag{5.2}$$

where $F(\theta)$ is the volume of intersection of the d-dimensional balls $\{\eta \in \mathbb{R}^d : |\eta| < r\}$ and $\{\eta \in \mathbb{R}^d : |\theta + \eta| < \lambda\}$.

Obviously, $F(\theta) = \omega_d\, r^d$ if $|\theta| \leqslant \lambda - r$. By direct calculation,

$$F(\theta) \;=\; (d-1)\,\omega_{d-1}\left(\int_0^\rho \left(\sqrt{\lambda^2 - s^2} + \sqrt{r^2 - s^2} - |\theta|\right) s^{d-2}\,\mathrm{d}s + 2\int_\rho^r \sqrt{r^2 - s^2}\; s^{d-2}\,\mathrm{d}s\right)$$

whenever $\lambda - r < |\theta| \leqslant \sqrt{\lambda^2 - r^2}$, and

$$F(\theta) \;=\; (d-1)\,\omega_{d-1}\int_0^\rho \left(\sqrt{\lambda^2 - s^2} + \sqrt{r^2 - s^2} - |\theta|\right) s^{d-2}\mathrm{d}s$$

whenever $\sqrt{\lambda^2 - r^2} < |\theta| \leqslant \lambda + r$, where

$$\rho \;=\; \rho(\lambda, r, \theta) \;:=\; (2|\theta|)^{-1}\sqrt{2\lambda^2 r^2 + 2\lambda^2|\theta|^2 + 2r^2|\theta|^2 - \lambda^4 - r^4 - |\theta|^4}. \tag{5.3}$$

Differentiating (5.2) with respect to λ and taking into account the above formulae for F, we obtain

$$\int_{|\xi|=\lambda}\int_{|\eta|<r} f(\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \;=\; (d-1)\,\omega_{d-1}\,\lambda \int_{\lambda-r<|\theta|<\lambda+r} f(\theta)\left(\int_0^\rho \frac{s^{d-2}\,\mathrm{d}s}{\sqrt{\lambda^2 - s^2}}\right)\mathrm{d}\theta\,. \tag{5.4}$$

Now, differentiating (5.4) with respect to r, we see that

$$\int_{|\xi|=\lambda}\int_{|\eta|=r} f(\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \;=\; (d-1)\,\omega_{d-1}\,\lambda\, r \int_{\lambda-r<|\theta|<\lambda+r} f(\theta)\,|\theta|^{-1}\rho^{d-3}\,\mathrm{d}\theta\,. \tag{5.5}$$

Finally, substituting $r = \lambda$ in (5.3) and (5.5), we arrive at (5.1). □

If $d \geqslant 3$ then the right hand side of (5.1) is estimated by

$$(d-1)\,\omega_{d-1}\lambda^{d-1}\int_{|\theta|\leqslant 2\lambda} f(\theta)|\theta|^{-1}\,\mathrm{d}\theta\,.$$

Therefore, Lemma 5.1 and Lemma 5.2 with $f(\theta)=|\theta|^4\,\hat\chi(\theta)$ imply that

$$\int_{|\eta|=\lambda}\int_{|\xi|=\lambda}|\xi-\eta|^4\,|\hat\chi(\xi-\eta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta \;\leqslant\; 18(d-1)\,\omega_{d-1}\,\lambda^{d+2}\,|\Omega_{\lambda^{-1}}|_d \tag{5.6}$$

for all $d \geqslant 3$ and all $\lambda>0$.

In order to prove (4.10), let us note that

$$\int_{|\xi|=\lambda}\mathrm{d}\xi\int_{\eta\in S_\delta(\xi)} g(\xi)\,\mathrm{d}\eta \;=\; |S_\delta(\lambda)|_{d-2}\int_{|\xi|=\lambda} g(\xi)\,\mathrm{d}\xi \;=\; \int_{|\xi|=\lambda}\mathrm{d}\xi\int_{\eta\in S_\delta(\xi)} g(\eta)\,\mathrm{d}\eta$$

for all continuous functions g. It follows that for every continuous function f we have

$$\int_{|\xi^{(1)}|=\lambda}\mathrm{d}\xi^{(1)}\int_{\xi^{(2)}\in S_\delta(\xi^{(1)})}\mathrm{d}\xi^{(2)}\int_{|\eta^{(1)}|=\lambda}\mathrm{d}\eta^{(1)}\int_{\eta^{(2)}\in S_\delta(\eta^{(1)})}\mathrm{d}\eta^{(2)}\sum_{p,q=1,2} f(\xi^{(p)}-\eta^{(q)})$$
$$=\; 4\,|S_\delta(\lambda)|_{d-2}^2\int_{|\xi|=\lambda}\int_{|\eta|=\lambda} f(\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

and, consequently,

$$\int_{\Sigma_{\delta,m}(\lambda)}\sum_{j\neq k}\sum_{p,q=1,2} f(\xi_j^{(p)}-\xi_k^{(q)})\,\mathrm{d}\xi_1^{(1)}\,\ldots\,\mathrm{d}\xi_m^{(2)}$$
$$=\; m(m-1)\int_{\Sigma_{\delta,m}(\lambda)}\sum_{p,q=1,2} f(\xi_1^{(p)}-\xi_2^{(q)})\,\mathrm{d}\xi_1^{(1)}\,\ldots\,\mathrm{d}\xi_m^{(2)}$$
$$=\; 4m(m-1)\,((d-1)\,\omega_{d-1}\lambda^{d-1})^{m-2}\,|S_\delta(\lambda)|_{d-2}^m\int_{|\xi|=\lambda}\int_{|\eta|=\lambda} f(\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$=\; 4m(m-1)((d-1)\,\omega_{d-1}\lambda^{d-1})^{-2}\,|\Sigma_{\delta,m}(\lambda)|_{(2d-3)m}\int_{|\xi|=\lambda}\int_{|\eta|=\lambda} f(\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,.$$

Substituting $f(\theta)=|\theta|^4\,|\hat\chi(\theta)|^2$ and applying (5.6), we obtain (4.10).

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