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On the Comparison of the Dirichlet and Neumann Counting Functions

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To Mikhail Shlёmovich Birman on his 80th birthday

Introduction

Let $N_{\mathrm{N}}(\lambda)$ and $N_{\mathrm{D}}(\lambda)$ be the counting functions of the Dirichlet and Neumann Laplacian on a domain $\Omega \subset \mathbb{R}^n$. If λ is not a Dirichlet or Neumann eigenvalue, then

$$N_{\mathrm{N}}(\lambda) = N_{\mathrm{D}}(\lambda) + g^-(\lambda)\,, \qquad (*)$$

where $g^-(\lambda)$ denotes the number of negative eigenvalues of the Dirichlet-to-Neumann map at $\lambda \in \mathbb{R}$. The equality (*) was proved in [**Fr1**] for domains with sufficiently smooth boundaries. L. Friedlander also noticed that (*) immediately implies Payne's conjecture for the Laplacian on a bounded domain, according to which the $(k+1)$th Neumann eigenvalue does not exceed the kth Dirichlet eigenvalue. Later R. Mazzeo remarked that (*) remains valid for domains with smooth boundaries in any Riemannian symmetric space of noncompact type and gave a geometric explanation of Friedlander's result [**M**].

For irregular boundaries, the Dirichlet-to-Neumann map may not be well-defined and then (*) does not make sense. In 2004, N. Filonov suggested another proof of Payne's conjecture for the Laplacian [**Fi**]. This proof does not use (*) and works for nonsmooth boundaries. The author assumed that the resolvent of the Neumann Laplacian on Ω is compact, but this condition can be removed (see Remark 1.9).

The aim of this note is to show that (*) holds for abstract operators in a Hilbert space H, provided that the Dirichlet-to-Neumann map is understood in a proper sense. Traditionally, one assumes that the Dirichlet-to-Neumann map is a family of operators acting in the same space and depending on the spectral parameter λ (see Subsection 1.3). In our understanding, it is a family of operators $\mathcal{B}_\lambda$ generated by the restrictions of the same sesquilinear form to different subspaces $G_\lambda \subset H^1$. The

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identity (*) is proved with the use of special isomorphisms between the subspaces G_λ with different values of λ.

This approach is close in spirit to Birman's paper [**B1**] on selfadjoint extensions of symmetric operators. In particular, it removes technical problems related to nonsmooth boundaries and allows one to extend Payne's conjecture to all operators generated by differential quadratic forms with constant coefficients on an arbitrary domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ (see Corollary 1.13). Another advantage of our scheme is that, unlike the classical Dirichlet-to-Neumann map, the operators $\mathcal{B}_\lambda$ do not blow up as λ passes through isolated eigenvalues. This enables one to perform a more detailed analysis of the relation between their properties and the spectral characteristics of the Dirichlet and Neumann problems.

The paper is constructed as follows. In Section 1 we introduce some necessary notation and state the main results. Note that the notation is deliberately chosen as if A is a second-order elliptic differential operator acting in the Sobolev spaces on a domain, subject to the Dirichlet or Neumann boundary condition (even though H does not have to be a function space and the ellipticity is irrelevant). In Section 2 we prove some simple auxiliary lemmas on abstract selfadjoint operators. Section 3 is devoted to the proof of the main statements. Finally, Section 4 contains some remarks and by-product results, which are not needed in our proofs but may be of interest in themselves.

1. Basic notation and main results

1.1. Notation. We shall always be assuming that $\lambda, \mu \in \mathbb{R}$ and $z \in \mathbb{C}$.

Let H be an infinite-dimensional separable complex Hilbert space. As usual, $(\cdot,\cdot)$ and $\|\cdot\|$ are the inner product and norm in H, and $\dotplus$ denotes a direct sum in H. Let

- H^1 be a dense subspace of H;
- $\mathbf{a}[\cdot]$ be a closed positive quadratic form on H^1 and $\mathbf{a}[\cdot,\cdot]$ be the corresponding sesquilinear form;
- A_{N} be the selfadjoint operator in H generated by the form $\mathbf{a}[\cdot]$.

We shall consider H^1 as a Hilbert space provided with the inner product $\mathbf{a}[\cdot,\cdot]$. Let

- H^1_0 be a closed subspace of H^1 which is dense in H;
- A_{D} be the selfadjoint operator in H generated by the restriction of the form $\mathbf{a}[\cdot]$ to H^1_0.

Further on we shall write B instead of N or D in the case where the corresponding statement holds or the definition refers to both operators A_{N} and A_{D}. In particular, we shall be using the following notation.

- $\sigma(A_{\mathrm{B}})$ and $\sigma_{\mathrm{ess}}(A_{\mathrm{B}})$ denote the spectrum and the essential spectrum of the operator A_{B}.
- $\lambda_{B,\infty} := \inf \sigma_{\mathrm{ess}}(A_{\mathrm{B}})$.
- $\lambda_{B,1} \leqslant \lambda_{B,2} \leqslant \lambda_{B,3} \ldots$ are the eigenvalues of the operator A_{B} lying in the interval $(-\infty, \lambda_{B,\infty})$ and counted with their multiplicities.
- χ_Λ denotes the characteristic function of the Borel set $\Lambda \subset \mathbb{R}$, so that
- $\chi_\Lambda(A_{\mathrm{B}})$ is the spectral projection of A_{B} corresponding to Λ.

- $E_{\rm B}(z)$ is the orthogonal projection onto $\ker(A_{\rm B} - zI)$ and $E'_{\rm B}(z) := I - E_{\rm B}(z)$.
- $N_{\rm B}(\lambda) := \dim \chi_{(-\infty,\lambda)}(A_{\rm B})\, H$ is the left continuous counting function of the operator $A_{\rm B}$.

The Rayleigh–Ritz variational formula implies that $N_{\rm D}(\lambda) \leqslant N_{\rm N}(\lambda)$ or, in other words, $0 < \lambda_{{\rm N},j} \leqslant \lambda_{{\rm D},j}$ for all $j = 1, 2, \dots, \infty$. We have $N_{\rm B}(\lambda) = \#\{j : \lambda_{{\rm B},j} < \lambda\}$ whenever $\lambda \leqslant \lambda_{{\rm B},\infty}$ and $N_{\rm B}(\lambda) = \infty$ otherwise.

Let

- H^1_A be the set of vectors $u \in H^1$ such that the functionals $v \to \mathbf{a}[u,v]$ on H^1_0 are H-continuous;
- A be the operator acting from H^1_A to H such that $(Au, v) = \mathbf{a}[u,v]$ for all $v \in H^1_0$;
- $G_z := \{u \in H^1_A \,:\, Au = zu\}$, where $z \in \mathbb{C}$;
- $\mathbf{b}[u,v] := \mathbf{a}[u,v] - (Au, v)$ and $\mathbf{b}[u] := \mathbf{b}[u,u]$, where $u \in H^1_A$ and $v \in H^1$.

Since the operator A is H^1-closed, G_z are closed subspaces of H^1. Denote

- $\mathcal{B}_\lambda := (I - \lambda \Pi'_\lambda A_{\rm N}^{-1})\big|_{G_\lambda}$, where
- Π'_λ is the H^1-orthogonal projection onto G_λ

(an explicit formula for Π'_λ is given in Subsection 2.3). We shall consider $\mathcal{B}_\lambda$ as an operator in G_λ. Obviously,

$$\mathbf{a}[\mathcal{B}_\lambda u, v] \;=\; \mathbf{a}[u,v] - \lambda\,(u,v) \;=\; \mathbf{b}[u,v]\,, \qquad \forall u, v \in G_\lambda\,. \tag{1.1}$$

Therefore $\mathcal{B}_\lambda$ is a bounded selfadjoint operator in the Hilbert space G_λ provided with the inner product $\mathbf{a}[\cdot,\cdot]$.

Let

- $\sigma(\mathcal{B}_\lambda)$ and $\sigma_{\rm ess}(\mathcal{B}_\lambda)$ be the spectrum and essential spectrum of $\mathcal{B}_\lambda$,
- $G^0_\lambda := \ker \mathcal{B}_\lambda$, $G^-_\lambda := \chi_{(-\infty,0)}(\mathcal{B}_\lambda) G_\lambda$ and $G^+_\lambda := \chi_{(0,+\infty)}(\mathcal{B}_\lambda) G_\lambda$,

where $\chi_{(-\infty,0)}(\mathcal{B}_\lambda)$ and $\chi_{(0,+\infty)}(\mathcal{B}_\lambda)$ are the corresponding spectral projections of the operator $\mathcal{B}_\lambda$.

Finally, let

- $\mathcal{H}_0$ be the subspace of H spanned by all common eigenvectors of the operators $A_{\rm N}$ and $A_{\rm D}$;
- $\mathcal{H}$ be the H-orthogonal complement of $\mathcal{H}_0$;
- $n_{\rm B}(\lambda) := \dim E_{\rm B}(\lambda)\mathcal{H}$ and $n_{\rm N,D}(\lambda) := \dim E_{\rm B}(\lambda)\mathcal{H}_0$, so that $n_{\rm B}(\lambda) = \dim E_{\rm B}(\lambda) H - n_{\rm N,D}(\lambda)$.

Clearly, $\mathcal{H}$ and $\mathcal{H}_0$ are invariant subspaces of the operators $A_{\rm N}$ and $A_{\rm D}$, whose intersections with H^1 are H^1-orthogonal. Similarly, $G_\lambda \cap \mathcal{H}$ and $G_\lambda \cap \mathcal{H}_0$ are invariant subspaces of $\mathcal{B}_\lambda$. One can easily see that $G_\lambda \cap \mathcal{H}_0 = G^0_\lambda \cap \mathcal{H}_0 = E_{\rm B}(\lambda)\mathcal{H}_0$ and $\mathcal{B}_\lambda|_{G_\lambda \cap \mathcal{H}_0} = 0$. In particular, $G_\lambda \cap \mathcal{H}_0 = \{0\}$ whenever λ is not an eigenvalue corresponding to a common eigenvector of $A_{\rm N}$ and $A_{\rm D}$.

1.2. Main results. The following lemma implies that the restriction $\mathcal{B}_\lambda|_{\mathcal{H}}$ analytically depends on λ outside the intersection of the essential spectra $\sigma_{\rm ess}(A_{\rm N})$ and $\sigma_{\rm ess}(A_{\rm D})$.

LEMMA 1.1. *The H^1-orthogonal projection onto $G_\lambda \cap \mathcal{H}$ is an analytic operator-valued function of λ on the set $\mathbb{R} \setminus (\sigma_{\rm ess}(A_{\rm N}) \cap \sigma_{\rm ess}(A_{\rm D}))$.*

One can easily show that

$$E_{\mathrm{N}}(\lambda)H + E_{\mathrm{D}}(\lambda)H \ \subset\ G_\lambda^0\,, \qquad \forall\lambda\in\mathbb{R} \tag{1.2}$$

(see Subsection 3.3). The next lemma is less obvious.

Lemma 1.2. *If* $\lambda\notin\sigma_{\mathrm{ess}}(A_{\mathrm{N}})\cap\sigma_{\mathrm{ess}}(A_{\mathrm{D}})$ *, then* $G_\lambda^0 = E_{\mathrm{D}}(\lambda)H + E_{\mathrm{N}}(\lambda)H$ *. If* $\lambda\notin\sigma_{\mathrm{ess}}(A_{\mathrm{N}})\cup\sigma_{\mathrm{ess}}(A_{\mathrm{D}})$ *, then the point* 0 *does not belong to the essential spectrum of the operator* $\mathcal{B}_\lambda$ *.*

Lemmas 1.1 and 1.2 imply

Theorem 1.3. *Let* $\lambda\notin\sigma_{\mathrm{ess}}(A_{\mathrm{N}})\cup\sigma_{\mathrm{ess}}(A_{\mathrm{D}})$ *. Then for each sufficiently small* $\varepsilon>0$ *there exists* $\delta>0$ *such that the intersection* $(-\varepsilon,\varepsilon)\cap\sigma(\mathcal{B}_\mu)$ *consists of*

(1) $n_{\mathrm{N}}(\mu)+n_{\mathrm{D}}(\mu)+n_{\mathrm{N,D}}(\mu)$ *zero eigenvalues if* $\mu=\lambda$ *,*
(2) $n_{\mathrm{D}}(\lambda)$ *negative and* $n_{\mathrm{N}}(\lambda)$ *positive eigenvalues if* $\mu\in(\lambda-\delta,\lambda)$ *,*
(3) $n_{\mathrm{N}}(\lambda)$ *negative and* $n_{\mathrm{D}}(\lambda)$ *positive eigenvalues if* $\mu\in(\lambda,\lambda+\delta)$

(as usual, the eigenvalues are counted according to their multiplicities).

Remark 1.4. By Lemma 1.2, if $\lambda\notin\sigma_{\mathrm{ess}}(A_{\mathrm{N}})\cup\sigma_{\mathrm{ess}}(A_{\mathrm{D}})$, then we have $[-\varepsilon,\varepsilon]\cap\sigma_{\mathrm{ess}}(\mathcal{B}_\mu)=\varnothing$ for all sufficiently small $\varepsilon,\delta>0$ and all $\mu\in[\lambda-\delta,\lambda+\delta]$. By Lemma 1.1, the eigenvalues $\nu_j(\mu)$ of the restrictions $\mathcal{B}_\mu|_{G_\mu\cap\mathcal{H}}$ lying in $(-\varepsilon,\varepsilon)$ are continuous function of $\mu\in(\lambda-\delta,\lambda+\delta)$. Therefore, if ε and δ are small enough, then $\nu_j(\mu)\in(-\varepsilon,\varepsilon)$ for some $\mu\in(\lambda-\delta,\lambda+\delta)$ if and only if $\nu_j(\lambda)=0$. Theorem 1.3 states that $n_{\mathrm{D}}(\mu)$ eigenvalues $\nu_j(\mu)$ change their sign from minus to plus and $n_{\mathrm{N}}(\mu)$ eigenvalues $\nu_j(\mu)$ change their sign from plus to minus as μ passes through the eigenvalue λ . At the point λ all these eigenvalues are equal to zero and, in addition, there are $n_{\mathrm{N,D}}(\lambda)$ zero eigenvalues of the restriction $\mathcal{B}_\lambda|_{G_\lambda\cap\mathcal{H}_0}$.

Remark 1.5. A similar result was obtained in [**Fr1**] and [**M**] for differential operators on domains with smooth boundaries under the additional assumption that their spectra are discrete. Theorem 1.3 holds in the abstract setting and remains valid for λ lying in the gaps of the essential spectra.

Corollary 1.6. *Let* $a<b$ *. If* $[a,b]\cap\sigma_{\mathrm{ess}}(A_{\mathrm{N}})=[a,b]\cap\sigma_{\mathrm{ess}}(A_{\mathrm{D}})=\varnothing$ *, then*

$$\dim G_b^- \ =\ \dim G_a^- \,+\, \dim\chi_{[a,b)}(A_{\mathrm{N}})\mathcal{H} \,-\, \dim\chi_{(a,b]}(A_{\mathrm{D}})\mathcal{H}\,. \tag{1.3}$$

If $a<\inf\sigma(A_{\mathrm{N}})$, then $G_a^-=\{0\}$ and, according to the next theorem, the equality (1.3) remains valid for $b\in[\lambda_{\mathrm{N},\infty},\lambda_{\mathrm{D},\infty})$.

Theorem 1.7. $N_{\mathrm{N}}(\lambda)=N_{\mathrm{D}}(\lambda)+n_{\mathrm{D}}(\lambda)+\dim G_\lambda^-$ *for all* $\lambda<\lambda_{\mathrm{D},\infty}$ *.*

Remark 1.8. By Theorem 1.7, $N_{\mathrm{N}}(\lambda)=\dim G_\lambda^-$ for all λ lying below $\sigma(A_{\mathrm{D}})$. In the case where A_{N} and A_{D} are selfadjoint extensions of the same symmetric operator defined on $\mathcal{D}(A_{\mathrm{N}})\cap\mathcal{D}(A_{\mathrm{D}})$, the above identity was obtained by M.S. Birman [**B1**] (see also [**B2**]). Theorem 1.7 extends Birman's result to all $\lambda<\lambda_{\mathrm{D},\infty}$ in a slightly more general setting (see Subsection 4.1).

Remark 1.9. N. Filonov noticed in [**Fi**] that for the Laplacian on an arbitrary domain $\Omega\subset\mathbb{R}^n$ we have

$$\mathbf{a}[u] \ \leqslant\ \lambda\,\|u\|^2\,, \qquad \forall u\in\chi_{[0,\lambda]}(A_{\mathrm{D}})H+E_{\mathrm{N}}(\lambda)H+G_\lambda^0+G_\lambda^-\,. \tag{1.4}$$

Similar arguments show that (1.4) holds for any pair of abstract operators A_{D} and A_{N} (see Subsection 3.2). The estimate (1.4) immediately implies that

$$N_{\mathrm{N}}(\lambda) \ \geqslant\ N_{\mathrm{D}}(\lambda) \,+\, n_{\mathrm{D}}(\lambda) \,+\, \dim G_\lambda^-\,, \qquad \forall\lambda\in\mathbb{R}\,. \tag{1.5}$$

The inequality (1.5) is sufficient to prove Payne's conjecture for the Laplacian on a bounded domain (see the proof of Corollary 1.13).

REMARK 1.10. The equality $N_{\mathrm{N}}(\lambda) = N_{\mathrm{D}}(\lambda) + n_{\mathrm{D}}(\lambda) + \dim G^-_\lambda$ remains valid for all $\lambda > \lambda_{\mathrm{D},\infty}$ because $N_{\mathrm{N}}(\lambda) = N_{\mathrm{D}}(\lambda) = \infty$. However, as was pointed out by N. Filonov, it may not be true for $\lambda = \lambda_{\mathrm{D},\infty}$.

REMARK 1.11. Let $\lambda = \lambda_{\mathrm{D},k} < \lambda_{\mathrm{N},\infty}$. Theorem 1.7 implies that the number of eigenvalues $\lambda_{\mathrm{N},j}$ lying below $\lambda_{\mathrm{D},k}$ is equal to $k - 1 + n_{\mathrm{D}}(\lambda_{\mathrm{D},k}) + \dim G^-_{\lambda_{\mathrm{D},k}}$. Therefore

(1) $\lambda_{\mathrm{N},k+q_k+p_k-1} < \lambda_{\mathrm{D},k}$, where $p_k := \dim G^-_{\lambda_{\mathrm{D},k}}$ and $q_k := n_{\mathrm{D}}(\lambda_{\mathrm{D},k})$.

If $n_{\mathrm{D}}(\lambda_{\mathrm{D},k}) = 0$, then $\lambda_{\mathrm{N},k+q_k+p_k} = \lambda_{\mathrm{N},k+p_k} = \lambda_{\mathrm{D},k}$; if $n_{\mathrm{D}}(\lambda_{\mathrm{D},k}) \neq 0$, then $q_k \geqslant 1$. Thus we always have

(2) $\lambda_{\mathrm{N},k+p_k} \leqslant \lambda_{\mathrm{D},k}$.

Note that the estimates (1) and (2) are actually consequences of (1.5). These estimates and Lemma 1.2 imply that

(3) $\lambda_{\mathrm{N},k+1} \leqslant \lambda_{\mathrm{D},k}$ whenever there exists a vector $u \in G_{\lambda_{\mathrm{D},k}}$, such that $\mathbf{b}[u] \leqslant 0$ and $u \notin \mathcal{D}(A_{\mathrm{D}})$;
(4) $\lambda_{\mathrm{N},k+1} < \lambda_{\mathrm{D},k}$ whenever $n_{\mathrm{D}}(\lambda_{\mathrm{D},k}) \geqslant 1$ and there exist two vectors $u_1, u_2 \in G_{\lambda_{\mathrm{D},k}}$, such that $\mathbf{b}[u_1] \leqslant 0$, $\mathbf{b}[u_2] \leqslant 0$ and the linear subspace spanned by u_1 and u_2 does not contain Neumann eigenvectors.

Indeed, if $G^-_{\lambda_{\mathrm{D},k}} \geqslant 1$, then (3) and (4) follow from (2) and (1) respectively. If $G^-_{\lambda_{\mathrm{D},k}} = \{0\}$, then $u \in G^0_{\lambda_{\mathrm{D},k}}$ and $u_1, u_2 \in G^0_{\lambda_{\mathrm{D},k}}$. The inclusion $u \in G^0_{\lambda_{\mathrm{D},k}}$ implies that $\lambda_{\mathrm{D},k}$ is also a Neumann eigenvalue and, consequently, $\lambda_{\mathrm{N},k+1} = \lambda_{\mathrm{D},k}$. The inclusions $u_1, u_2 \in G^0_{\lambda_{\mathrm{D},k}}$ imply that $n_{\mathrm{D}}(\lambda_{\mathrm{D},k}) \geqslant 2$ (otherwise a linear combination of u_1 and u_2 would belong to $E_{\mathrm{N}}(\lambda_{\mathrm{D},k})H$).

Lemma 1.2 and Theorem 1.7 also imply

COROLLARY 1.12. *If $\lambda < \lambda_{\mathrm{D},\infty}$ and $\lambda \notin \sigma(A_{\mathrm{N}}) \cup \sigma(A_{\mathrm{D}})$, then the number of negative eigenvalues of the selfadjoint operator $R'(\lambda) := (A_{\mathrm{N}} - \lambda I)^{-1} - (A_{\mathrm{D}} - \lambda I)^{-1}$ in H coincides with $N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda)$.*

Obviously, the number of negative eigenvalues of the operator $(A_{\mathrm{B}} - \lambda I)^{-1}$ jumps by $n_{\mathrm{B}}(\lambda_0) + n_{\mathrm{N},\mathrm{D}}(\lambda_0)$ as λ passes through an eigenvalue λ_0. Corollary 1.12 shows that the corresponding jump for $R'(\lambda)$ is equal to $n_{\mathrm{N}}(\lambda_0) - n_{\mathrm{D}}(\lambda_0)$, as if $R'(\lambda)$ were the orthogonal sum of the operators $(A_{\mathrm{N}} - \lambda I)^{-1}$ and $-(A_{\mathrm{D}} - \lambda I)^{-1}$.

1.3. The Dirichlet-to-Neumann map. In the theory of boundary value problems, it is often possible to construct a linear isomorphism $W : G_\lambda \to \mathfrak{H}$, where $\mathfrak{H}$ is a Hilbert space of functions defined on the boundary. Then one can consider the operator $W\mathcal{B}_\lambda W^{-1} : \mathfrak{H} \to \mathfrak{H}$ instead of $\mathcal{B}_\lambda$. Clearly, these two operators have the same eigenvalues. If H^1, H^1_0 are the Sobolev spaces and Wv is the restriction of v to the boundary, then $W\mathcal{B}_\lambda W^{-1}$ is usually called the Dirichlet-to-Neumann map. This scheme works under certain smoothness conditions on the boundary and the coefficients, whereas our approach does not rely on the existence of an auxiliary operator W and does not require any additional assumptions.

1.4. Applications to boundary value problems. Let Ω be an arbitrary open subset of $\mathbb{R}^n$ with $n \geqslant 2$. Consider a differential operator L acting from the space of m-vector functions $C^\infty(\Omega, \mathbb{C}^m)$ into the space of l-vector functions $C^\infty(\Omega, \mathbb{C}^l)$ and denote by L^* its formal adjoint. Let us assume that the form $\int_\Omega |Lu(x)|^2\, \mathrm{d}x$ with domain $C^\infty(\Omega, \mathbb{C}^m) \cap L_2(\Omega, \mathbb{C}^m)$ is strictly positive and closable in $H = L_2(\Omega, \mathbb{C}^m)$, and denote its closure by $\mathbf{a}[u]$. If $H^1 := \mathcal{D}(\mathbf{a})$ and H_0^1 is the H^1-closure of $C_0^\infty(\Omega)$, then $A = L^*L$ and A_{B} is the differential operator A with the corresponding boundary condition.

Corollary 1.13. *Let L be an operator with constant coefficients. Then we have $\lambda_{\mathrm{N},k+1} \leqslant \lambda_{\mathrm{D},k}$ for all eigenvalues $\lambda_{\mathrm{D},k} \in (0, \lambda_{\mathrm{N},\infty})$. If at least one Dirichlet eigenfunction corresponding to $\lambda_{\mathrm{D},k}$ does not satisfy the Neumann boundary condition, then $\lambda_{\mathrm{N},k+1} < \lambda_{\mathrm{D},k}$.*

Remark 1.14. Our proof of Corollary 1.13 uses the exponential functions $u_\xi(x) = e^{ix\cdot\xi}$ and is very similar to the proof of the Payne conjecture given in [**Fr1**]. The main difference is that L. Friendlander considered the Dirichlet-to-Neumann map and therefore had to assume that the boundary is smooth enough.

Remark 1.15. If A is the Laplacian on a convex n-dimensional domain with sufficiently smooth boundary, then $\lambda_{\mathrm{N},k+n} < \lambda_{\mathrm{D},k}$. This estimate was obtained in [**LW**]. Later L. Friedlander found another proof, based on the fact that $G_\lambda^0 \cup G_\lambda^-$ contains all first-order derivatives $D_j u$ of the Dirichlet eigenfunctions $u \in E_{\mathrm{D}}(\lambda)H$ (the derivatives obviously belong to G_λ, and the estimate $\mathbf{b}[D_j u] \leqslant 0$ is a consequence of the convexity). The inclusion $D_j u \in G_\lambda^0 \cup G_\lambda^-$ also implies that $N_{\mathrm{N}}(\lambda) \geqslant N_{\mathrm{D}}(\lambda) + 2n_{\mathrm{D}}(\lambda)$ (see [**Fr2**] for details).

2. Further notation and auxiliary results

2.1. The inverse A_{N}^{-1} is a bounded selfadjoint operator in H^1 because $(u,v) = \mathbf{a}[A_{\mathrm{N}}^{-1}u, v]$ for all $u, v \in H^1$. Since $\mathbf{a}[u,v] = \lambda\,(u,v)$ for all $v \in E_{\mathrm{N}}(\lambda)H$ and $v \in H^1$, its spectral projections $E_{\mathrm{N}}(\lambda)$ are H^1-orthogonal.

Let

- Π_0 be the orthogonal projection in H^1 onto H_0^1.

From the definition of G_z it clear that $\Pi_0' = I - \Pi_0$ (this well-known result can be found, for example, in [**K**] or [**BS**, Chapter 10, Section 3]). Since $\mathbf{a}[A_{\mathrm{D}}^{-1}u, v] = (u,v) = \mathbf{a}[A_{\mathrm{N}}^{-1}u, v]$ for all $u \in H^1$ and $v \in H_0^1$, we have $A_{\mathrm{D}}^{-1} = \Pi_0 A_{\mathrm{N}}^{-1}$ and $\mathcal{D}(A_{\mathrm{D}}) = \Pi_0 \mathcal{D}(A_{\mathrm{N}})$. The following simple lemma is also well known in the theory of selfadjoint extensions.

Lemma 2.1. *We have $H_A^1 = G_0 \dotplus \mathcal{D}(A_{\mathrm{B}})$. If $w_0 \in G_0$ and $w_{\mathrm{B}} \in \mathcal{D}(A_{\mathrm{B}})$, then $A(w_0 + w_{\mathrm{B}}) = A_{\mathrm{B}} w_{\mathrm{B}}$.*

Proof. Obviously, $G_0 \dotplus \mathcal{D}(A_{\mathrm{B}}) \subset H_A^1$. On the other hand, if $v \in H_A^1$, then there exists $\tilde v \in H$ such that $(u, \tilde v) = \mathbf{a}[u,v]$ for all $u \in H_0^1$. Since $(u,v) = \mathbf{a}[u, A_{\mathrm{N}}^{-1}v]$, this implies that $\Pi_0 v = \Pi_0 A_{\mathrm{N}}^{-1}\tilde v = A_{\mathrm{D}}^{-1}\tilde v$. Therefore $v = \Pi_0' v + A_{\mathrm{D}}^{-1}\tilde v$ and $v = \Pi_0'\left(v - A_{\mathrm{N}}^{-1}\tilde v\right) + A_{\mathrm{N}}^{-1}\tilde v$. These equalities imply the first statement.

If $w_0 \in G_0$ and $w_{\mathrm{B}} \in \mathcal{D}(A_{\mathrm{B}})$, then $\mathbf{a}[w_0 + w_{\mathrm{B}}, v] = \mathbf{a}[w_{\mathrm{B}}, v] = (A_{\mathrm{B}} w_{\mathrm{B}}, v)$, for all $v \in H_0^1$. This proves the second statement. □

2.2. By Lemma 2.1, H^1_A is dense in H^1. If $u,v\in H^1_A$, then

$$\mathbf{a}[A_{\rm D}^{-1}Au,v]=\mathbf{a}[A_{\rm D}^{-1}Au,\Pi_0v]=(AA_{\rm D}^{-1}Au,\Pi_0v)=(Au,\Pi_0v)=\mathbf{a}[\Pi_0u,v]\,.$$

This implies that $A_{\rm D}^{-1}A=\Pi_0|_{H^1_A}$ and, consequently,

$$G_z\;:=\;\ker(A-zI)\;=\;\ker A_{\rm D}^{-1}(A-zI)\;=\;\ker\Pi_0(I-zA_{\rm B}^{-1})\,,\qquad\forall z\in\mathbb{C}\,. \tag{2.1}$$

By (2.1), we have $(I-zA_{\rm B}^{-1})\,G_z\subset G_0\cap(I-zA_{\rm B}^{-1})H^1$. On the other hand, if $(I-zA_{\rm B}^{-1})u\in G_0$, then $u\in G_z$ because $(A-zI)u=A(I-zA_{\rm B}^{-1})u=0$. Therefore

$$(I-zA_{\rm B}^{-1})\,G_z\;=\;G_0\cap(I-zA_{\rm B}^{-1})H^1\,,\qquad\forall z\in\mathbb{C}\,. \tag{2.2}$$

Let

- $R_{\rm B}(z):=(A_{\rm B}-zI)^{-1}$ be the resolvent of $A_{\rm B}$.

For each $z\notin\sigma_{\rm ess}(A_{\rm B})$, the operator $R_{\rm B}(z)E'_{\rm B}(z)$ is bounded from H to H^1, $\ker\left(R_{\rm B}(z)E'_{\rm B}(z)\right)=\ker E'_{\rm B}(z)=E_{\rm B}(z)H$, $R_{\rm B}(z)E'_{\rm B}(z)H\subset E'_{\rm B}(z)\mathcal{D}(A_{\rm B})\subset H^1_A$ and $(A-zI)R_{\rm B}(z)E'_{\rm B}(z)=E'_{\rm B}(z)$. We also have

$$(I-zA_{\rm B}^{-1})^{-1}\big|_{E'_{\rm B}(z)H}\;=\;(I+zR_{\rm B}(z))\,E'_{\rm B}(z)\,,\qquad\forall z\notin\sigma_{\rm ess}(A_{\rm B})\,, \tag{2.3}$$

where the operators in the right and left hand sides map $E'_{\rm B}(\lambda)H^1$ onto $E'_{\rm B}(\lambda)H^1$ and are H^1-bounded. This implies that $(I-zA_{\rm B}^{-1})H^1=E'_{\rm B}(z)H^1$ and, by (2.2),

$$(I-zA_{\rm B}^{-1})\,G_z\;=\;G_0\cap E'_{\rm B}(z)H^1\,,\qquad\forall z\notin\sigma_{\rm ess}(A_{\rm B})\,. \tag{2.4}$$

2.3. Denote $T_z:=(I-zA_{\rm N}^{-1})\big|_{H^1_0\cap\mathcal{H}}$ and $T_z^\star:=\Pi_0(I-zA_{\rm N}^{-1})\big|_{\mathcal{H}}$. Let Σ be the set of $z\in\mathbb{C}$ such that the spectrum of the operator $T_z^\star T_z:H^1_0\cap\mathcal{H}\to H^1_0\cap\mathcal{H}$ contains the point 0, and let

- $\Pi(z):=T_z\,(T_z^\star T_z)^{-1}\,T_z^\star$ and $\Pi'(z):=I-\Pi(z)$, where $z\in\mathbb{C}\setminus\Sigma$.

By (2.1), we have $G_z\cap\mathcal{H}=\ker T_z^\star$. Since $T_z^\star\Pi'(z)=0$ and $\Pi'(z)u=u$ for all $u\in\ker T_z^\star$, this implies that $\Pi'(z)$ is a projection onto $G_z\cap\mathcal{H}$ in $H^1\cap\mathcal{H}$. Its H^1-adjoint coincides with $\Pi'(\bar z)$; in particular, $\Pi'(\lambda)$ is the H^1-orthogonal projection onto $G_z\cap\mathcal{H}$ in $\mathcal{H}$. Thus we obtain

$$\Pi'_\lambda\;=\;\Pi'(\lambda)\oplus E_{\rm B}(\lambda)|_{H^1\cap\mathcal{H}_0}\,,\qquad\forall\lambda\in\mathbb{R}\setminus\Sigma\,. \tag{2.5}$$

2.4. If $z\in\mathbb{C}\setminus\sigma_{\rm ess}(A_{\rm B})$, let

- $P_{\rm B}(z):=R_{\rm B}(z)E'_{\rm B}(z)\,(A-zI)$ and $P'_{\rm B}(z):=I-P_{\rm B}(z)$;
- $G_{z,B}:=\{v\in H^1_A:(A-zI)v\in E_{\rm B}(z)H\}$.

The operators $P_{\rm B}(z)$ and $P'_{\rm B}(z)$ are projections in H^1_A because

$$\begin{aligned}P^2_{\rm B}(z)&=R_{\rm B}(z)E'_{\rm B}(z)\,(A_{\rm B}-zI)R_{\rm B}(z)E'_{\rm B}(z)\,(A-zI)\\&=E'_{\rm B}(z)R_{\rm B}(z)E'_{\rm B}(z)\,(A-zI)=P_{\rm B}(z)\,.\end{aligned}$$

One can easily show that $P_{\rm B}(z)H^1_A=E'_{\rm B}(z)\mathcal{D}(A_{\rm B})$ and $P'_{\rm B}(z)H^1_A=G_{z,B}$. The subspace $G_{z,B}$ is the inverse image of $E_{\rm B}(z)H$ by the map $A-zI$, whereas G_z is the kernel of $A-zI$. Therefore $G_z\subset G_{z,B}$ and the dimension of the quotient space $G_{z,B}/G_z$ does not exceed $n_{\rm B}(z)+n_{\rm N,D}(z)$. This implies that the subspaces $G_{z,B}$ are H^1-closed for all $z\in\mathbb{C}\setminus\sigma_{\rm ess}(A_{\rm B})$.

If $z\notin\sigma(A_{\rm B})$, then $P_{\rm B}(z)H^1_A=\mathcal{D}(A_{\rm B})$ and $P'_{\rm B}(z)H^1_A=G_z=G_{z,B}$. In particular, $P_{\rm D}(0)|_{H^1_A}=\Pi_0|_{H^1_A}$, $P'_{\rm D}(0)|_{H^1_A}=\Pi'_0|_{H^1_A}$ and $H^1_A=P'_{\rm B}(0)H^1_A\dotplus P_{\rm B}(0)H^1_A$

is the decomposition discussed in Lemma 2.1. By direct calculation, if $u,v\in H^1_A$ and $\lambda,\mu\in\mathbb{R}$, then

$$\text{(2.6)}\quad \mathbf{b}[P'_{\mathrm{N}}(\lambda)u,P'_{\mathrm{N}}(\mu)v] = \mathbf{b}[u,v] - \big(u,E'_{\mathrm{N}}(\mu)(A-\mu I)v\big) + \big((A-\mu I)u,R_{\mathrm{N}}(\mu)E'_{\mathrm{B}}(z)(A-\mu I)v\big)\,,$$

$$\text{(2.7)}\quad \mathbf{b}[P'_{\mathrm{D}}(\lambda)u,P'_{\mathrm{D}}(\mu)v] = \mathbf{b}[u,v] + \big(E'_{\mathrm{D}}(\lambda)(A-\lambda I)u,v\big) - \big(R_{\mathrm{D}}(\lambda)E'_{\mathrm{B}}(z)(A-\lambda I)u,(A-\lambda I)v\big).$$

3. Proofs of main results

3.1. Proof of Lemma 1.1. Since the operator-valued function $T_z^\star T_z$ is analytic, the inverse operator $(T_z^\star T_z)^{-1}$ exists and analytically depends on z in a sufficiently small neighbourhood of each point $z_0\notin\Sigma$. Thus it is sufficient to show that $\Sigma\subset(\sigma_{\mathrm{ess}}(A_{\mathrm{N}})\cap\sigma_{\mathrm{ess}}(A_{\mathrm{D}}))$.

Let us fix $\lambda\in\mathbb{R}$, and let $u\in\mathcal{H}$. Since $T_\lambda^\star$ is the H^1-adjoint to T_λ, we have $T_\lambda^\star T_\lambda u=0$ if and only if $T_\lambda u=0$. The latter means that $u\in\mathcal{D}(A_{\mathrm{N}})\cap\mathcal{D}(A_{\mathrm{D}})$ and $A_{\mathrm{N}}u=A_{\mathrm{D}}u=\lambda u$. Since $u\in\mathcal{H}$, it is only possible if $u=0$. This implies that $\ker(T_\lambda^\star T_\lambda)=\{0\}$.

Assume that the essential spectrum of the operator $T_\lambda^\star T_\lambda$ contains 0. Then, for any given finite-dimensional subspace $\mathcal{L}$, there exists a sequence of H^1-orthogonal vectors $u_n\in H^1_0\cap\mathcal{H}$ such that u_n are H-orthogonal to $\mathcal{L}$, $\mathbf{a}[u_n]=1$ and $\mathbf{a}[T_\lambda^\star T_\lambda u_n,u_n]=\mathbf{a}[T_\lambda u_n]\to0$ as $n\to\infty$. Clearly, $T_\lambda u_n=u_n-\lambda\,A_{\mathrm{N}}^{-1}u_n\to0$ and $\Pi_0T_\lambda u_n=u_n-\lambda\,A_{\mathrm{D}}^{-1}u_n\to0$ in H^1. If $\lambda\notin\sigma_{\mathrm{ess}}(A_{\mathrm{B}})$, then, by (2.3), we have $\mathbf{a}\left[u-\lambda\,A_{\mathrm{B}}^{-1}u\right]\geqslant C\,\mathbf{a}[u]$ with some positive constant C for all vectors $u\in H^1$ which are H-orthogonal to $\mathcal{L}=E_{\mathrm{B}}(\lambda)H$. Therefore, by the above, $\lambda\in\sigma_{\mathrm{ess}}(A_{\mathrm{N}})\cap\sigma_{\mathrm{ess}}(A_{\mathrm{D}})$.

3.2. Proof of the estimate (1.4)**.** We have

$$\text{(3.1)}\qquad \mathbf{a}[u,v] \;=\; \lambda\,(u,v)\,,\quad \forall u\in E_{\mathrm{N}}(\lambda)H\,,\ \forall v\in H^1\,,$$

$$\text{(3.2)}\qquad \mathbf{a}[u,v] \;=\; \lambda\,(u,v)\,,\quad \forall u\in \mathcal{D}(A_{\mathrm{D}})\,,\ \forall v\in G_\lambda\,.$$

Let $u=u_1+u_2+u_3$, where $u_1\in\chi_{[0,\lambda]}(A_{\mathrm{D}})H$, $u_2\in E_{\mathrm{N}}(\lambda)$ and $u_3\in G^0_\lambda+G^-_\lambda$. Then (3.1) and (3.2) imply

$$\begin{aligned}\mathbf{a}[u]-\lambda\,\|u\|^2 &= \mathbf{a}[u_1]-\lambda\,\|u_1\|^2+\mathbf{a}[u_3]-\lambda\,\|u_3\|^2\\ &= ((A_{\mathrm{D}}-\lambda I)u_1,u_1)+\mathbf{a}[\mathcal{B}_\lambda u_3,u_3]\;\leqslant\;0\,.\end{aligned}$$

3.3. Proof of Lemma 1.2. The inclusion (1.2) immediately follows from (3.1) and (3.2).

Assume that $\lambda\notin\sigma_{\mathrm{ess}}(A_{\mathrm{N}})\cap\sigma_{\mathrm{ess}}(A_{\mathrm{D}})$ and $u\in G^0_\lambda$. Then $(I-\lambda A_{\mathrm{N}}^{-1})u$ is H^1-orthogonal to G_λ. In view of (2.5), this means that $(I-\lambda A_{\mathrm{N}}^{-1})u=\Pi(\lambda)v$ for some $v\in H^1\cap\mathcal{H}$. Therefore $(I-\lambda A_{\mathrm{N}}^{-1})u=(I-\lambda A_{\mathrm{N}}^{-1})w$ for some $w\in H^1_0$, which is equivalent to the inclusion $u\in G_\lambda\cap\big(E_{\mathrm{N}}(\lambda)H+H^1_0\big)=E_{\mathrm{N}}(\lambda)H+E_{\mathrm{D}}(\lambda)H$.

Assume now that $\lambda\notin\sigma_{\mathrm{ess}}(A_{\mathrm{N}})\cup\sigma_{\mathrm{ess}}(A_{\mathrm{D}})$. If the point 0 belongs to the essential spectrum of the operator $\mathcal{B}_\lambda$, then there exists a sequence of vectors $u_n\in G_\lambda\ominus G^0_\lambda$ such that $\mathbf{a}[u_n]=1$ and $\mathbf{a}[\mathcal{B}_\lambda u_n]\to0$ as $n\to\infty$. Moreover, since $\dim E_{\mathrm{N}}(\lambda)H<\infty$, we can choose the sequence $\{u_n\}$ in such a way that

$$u_n-(T_\lambda^\star T_\lambda)^{-1}\,\Pi_0(I-\lambda A_{\mathrm{N}}^{-1})^2u_n\;\in\;E'_{\mathrm{N}}(\lambda)\,,\qquad\forall n=1,2,\dots,$$

where T_λ and $T_\lambda^\star$ are the operators defined in Subsection 2.3. Then, by (2.5),

$$\begin{aligned}\mathcal{B}_\lambda u_n &= (I-\lambda A_{\mathrm{N}}^{-1})u_n - \Pi_\lambda(I-\lambda A_{\mathrm{N}}^{-1})u_n\\ &= \left(I-\lambda A_{\mathrm{N}}^{-1}\right)\left(u_n-(T_\lambda^\star T_\lambda)^{-1}\,\Pi_0(I-\lambda A_{\mathrm{N}}^{-1})^2u_n\right) \;\to\; 0\end{aligned}$$

in H^1, and (2.3) implies that

$$\begin{aligned}&\mathbf{a}\left[u_n-(T_\lambda^\star T_\lambda)^{-1}\,\Pi_0(I-\lambda A_{\mathrm{N}}^{-1})^2u_n\right]\\ &\qquad= \;\mathbf{a}\,[\Pi_0'u_n]+\mathbf{a}\left[\Pi_0u_n-(T_\lambda^\star T_\lambda)^{-1}\,\Pi_0(I-\lambda A_{\mathrm{N}}^{-1})^2u_n\right] \;\to\; 0\,.\end{aligned}$$

Therefore $\mathbf{a}[\Pi_0'u_n] = \mathbf{a}[(I-A_{\mathrm{D}}^{-1}A)u_n] = \mathbf{a}[(I-\lambda A_{\mathrm{D}}^{-1})u_n] \to 0$ as $n\to\infty$. However, this is not possible because $\lambda\notin\sigma_{\mathrm{ess}}(A_{\mathrm{D}})$ and the u_n are orthogonal to $E_{\mathrm{D}}(\lambda)H\subset G_\lambda^0$. The obtained contradiction proves the second statement of the lemma.

3.4. Proof of Theorem 1.3. If $\lambda\notin\sigma(A_{\mathrm{N}})\cup\sigma(A_{\mathrm{D}})$, then the theorem is obvious because, in view of Lemmas 1.2 and 1.1, we have $n_{\mathrm{N}}(\lambda)=n_{\mathrm{D}}(\lambda)=n_{\mathrm{N,D}}(\lambda)=0$ and $(-\varepsilon,\varepsilon)\cap\sigma(\mathcal{B}_\mu)=\varnothing$ for all sufficiently small $\varepsilon,\delta>0$ and all $\mu\in(\lambda-\varepsilon,\lambda+\varepsilon)$.

Suppose that λ is an isolated eigenvalue. The first statement of the theorem is an immediate consequence of Lemma 1.2, so we only need to prove (2) and (3). Let us choose ε and δ as explained in Remark 1.4 and assume, in addition, that δ is so small that $\lambda-\delta>0$ and the interval $[\lambda-\delta,\lambda+\delta]$ does not contain any points from $\sigma(A_{\mathrm{N}})\cup\sigma(A_{\mathrm{D}})$ with the exception of λ.

Let $\mathcal{L}_\mu$ be the subspace of $G_\mu\cap\mathcal{H}$ spanned by the eigenfunction corresponding to the eigenvalues $\nu_j(\mu)$ (see Remark 1.4). By Lemma 1.2, we have $\mathcal{L}_\lambda=E_{\mathrm{N}}(\lambda)\mathcal{H}\dot{+}E_{\mathrm{D}}(\lambda)\mathcal{H}\subset G_\lambda^0$. Therefore $E_{\mathrm{B}}'(\lambda)\mathcal{L}_\lambda\subset\mathcal{L}_\lambda$, $\dim E_{\mathrm{N}}'(\lambda)\mathcal{L}_\lambda=n_{\mathrm{D}}(\lambda)$ and $\dim E_{\mathrm{D}}'(\lambda)\mathcal{L}_\lambda=n_{\mathrm{N}}(\lambda)$.

We are going to show that

$$\begin{aligned}&\left|\,\mathbf{a}\left[\mathcal{B}_\mu P_{\mathrm{N}}'(\mu)u,\chi_{(-\varepsilon,\varepsilon)}(\mathcal{B}_\mu)P_{\mathrm{N}}'(\mu)u\right]-(\mu-\lambda)\,\|u\|^2\,\right|\\ &\qquad\leqslant\; C\,(\lambda-\mu)^2\,\|u\|^2\,,\qquad \forall u\in E_{\mathrm{N}}'(\lambda)\mathcal{L}_\lambda\,,\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\left|\,\mathbf{a}\left[\mathcal{B}_\mu P_{\mathrm{D}}'(\mu)u,\chi_{(-\varepsilon,\varepsilon)}(\mathcal{B}_\mu)P_{\mathrm{D}}'(\mu)u\right]-(\lambda-\mu)\,\|u\|^2\,\right|\\ &\qquad\leqslant\; C\,(\lambda-\mu)^2\,\|u\|^2\,,\qquad \forall u\in E_{\mathrm{D}}'(\lambda)\mathcal{L}_\lambda\,,\end{aligned}\tag{3.4}$$

where C is a constant independent of u and $\mu\in(\lambda-\varepsilon,\lambda+\varepsilon)$. From (3.3) and (3.4) it follows that

$$(\mu-\lambda)\,\mathbf{a}[\mathcal{B}_\mu w,w]\;\geqslant\;0\,,\qquad \forall w\in\chi_{(-\varepsilon,\varepsilon)}(\mathcal{B}_\mu)P_{\mathrm{N}}'(\mu)E_{\mathrm{N}}'(\lambda)\mathcal{L}_\lambda\,,\tag{3.5}$$

$$(\lambda-\mu)\,\mathbf{a}[\mathcal{B}_\mu w,w]\;\geqslant\;0\,,\qquad \forall w\in\chi_{(-\varepsilon,\varepsilon)}(\mathcal{B}_\mu)P_{\mathrm{D}}'(\mu)E_{\mathrm{D}}'(\lambda)\mathcal{L}_\lambda\,,\tag{3.6}$$

whenever $|\lambda-\mu|$ is small enough. If $\mu=\lambda$, then $\chi_{(-\varepsilon,\varepsilon)}(\mathcal{B}_\mu)P_{\mathrm{B}}'(\mu)E_{\mathrm{B}}'(\lambda)u = E_{\mathrm{B}}'(\lambda)u$ for all $u\in\mathcal{L}_\lambda$. By continuity, we have $\dim\chi_{(-\varepsilon,\varepsilon)}(\mathcal{B}_\mu)P_{\mathrm{B}}'(\mu)E_{\mathrm{B}}'(\lambda)\mathcal{L}_\lambda = \dim E_{\mathrm{B}}'(\lambda)\mathcal{L}_\lambda = n_{\mathrm{N}}(\lambda)+n_{\mathrm{D}}(\lambda)-n_{\mathrm{B}}(\lambda)$ for all μ sufficiently close to λ. Therefore the estimates (3.5) and (3.6) imply the theorem (with another positive δ).

To prove (3.3) and (3.4), note that $\mathbf{b}[u]=0$ for all $u\in\mathcal{L}_\lambda$ and, in view of (2.6) and (2.7),

$$\begin{aligned}\mathbf{a}\,[\mathcal{B}_\mu P_{\mathrm{N}}'(\mu)u,P_{\mathrm{N}}'(\mu)u] \;&=\; \mathbf{b}[P_{\mathrm{N}}'(\mu)u]\\ &=\;(\mu-\lambda)\,\|u\|^2+(\lambda-\mu)^2\left(u,R_{\mathrm{N}}(\mu)u\right),\qquad\forall u\in\mathcal{L}_\lambda\,,\end{aligned}\tag{3.7}$$

$$\begin{aligned}(3.8)\quad \mathbf{a}\,[\mathcal{B}_\mu P'_{\mathrm{D}}(\mu)u, P'_{\mathrm{D}}(\mu)u] \;&=\; \mathbf{b}[P'_{\mathrm{D}}(\mu)u]\\ &=\; (\lambda-\mu)\,\|u\|^2-(\lambda-\mu)^2\,\big(u,R_{\mathrm{D}}(\mu)u\big)\,,\qquad \forall u\in\mathcal{L}_\lambda\,.\end{aligned}$$

Therefore, for all $\mu\in(\lambda-\delta,\lambda+\delta)$ we have

$$\begin{aligned}(3.9)\quad &\big|\,\mathbf{a}\,[\mathcal{B}_\mu P'_{\mathrm{N}}(\mu)u, P'_{\mathrm{N}}(\mu)u]-(\mu-\lambda)\,\|u\|^2\,\big|\\ &\qquad\leqslant\; C_1^{-1}\,(\lambda-\mu)^2\,\|u\|^2\,,\qquad \forall u\in E'_{\mathrm{N}}(\lambda)\mathcal{L}_\lambda\,,\end{aligned}$$

$$\begin{aligned}(3.10)\quad &\big|\,\mathbf{a}\,[\mathcal{B}_\mu P'_{\mathrm{D}}(\mu)u, P'_{\mathrm{D}}(\mu)u]-(\lambda-\mu)\,\|u\|^2\,\big|\\ &\qquad\leqslant\; C_1^{-1}\,(\lambda-\mu)^2\,\|u\|^2\,,\qquad \forall u\in E'_{\mathrm{D}}(\lambda)\mathcal{L}_\lambda\,,\end{aligned}$$

where C_1 is the distance from $[\lambda-\delta,\lambda+\delta]$ to $(\sigma(A_{\mathrm{N}}\cup\sigma(A_{\mathrm{D}}))\setminus\{\lambda\}$.

Let S_{B} be the projections onto $E_{\mathrm{B}}(\lambda)\mathcal{H}$ in $\mathcal{L}_\lambda$ such that $S_{\mathrm{N}}E_{\mathrm{D}}(\lambda)=0$ and $S_{\mathrm{D}}E_{\mathrm{N}}(\lambda)=0$. Then $u=S_{\mathrm{N}}u+S_{\mathrm{D}}u$ for all $u\in\mathcal{L}_\lambda$. Since $\dim\mathcal{L}_\lambda<\infty$ and $E_{\mathrm{N}}(\lambda)\mathcal{H}\cap E_{\mathrm{D}}(\lambda)\mathcal{H}=\{0\}$, the projections S_{B} are well defined and bounded as operators from H to H^1.

If $u\in\mathcal{L}_\lambda$, then $P'_{\mathrm{B}}(\mu)u=P'_{\mathrm{B}}(\mu)(u-S_{\mathrm{B}}u)=(I-(\lambda-\mu)R_{\mathrm{B}}(\mu))(u-S_{\mathrm{B}}u)$ for all $\mu\neq\lambda$ and, by (1.1),

$$\begin{aligned}\mathbf{a}\,[\mathcal{B}_\mu P'_{\mathrm{N}}(\mu)u,v] \;&=\; \mathbf{a}\,[P'_{\mathrm{N}}(\mu)u,v]-\mu\,(P'_{\mathrm{N}}(\mu)u,v)=(\mu-\lambda)\,(S_{\mathrm{D}}u,v)\,,\qquad \forall v\in G_\mu\,,\\ \mathbf{a}\,[\mathcal{B}_\mu P'_{\mathrm{D}}(\mu)u,v] \;&=\; \mathbf{a}\,[P'_{\mathrm{D}}(\mu)u,v]-\mu\,(P'_{\mathrm{D}}(\mu)u,v)=(\lambda-\mu)\,(S_{\mathrm{N}}u,v)\,,\qquad \forall v\in G_\mu\,,\end{aligned}$$

for all $\mu\in(\lambda-\delta,\lambda+\delta)$. Since $(S_{\mathrm{N}}u,v)=\lambda^{-1}\,\mathbf{a}[S_{\mathrm{N}}u,v]$ and $(S_{\mathrm{D}}u,v)=\mu^{-1}\,\mathbf{a}[S_{\mathrm{D}}u,v]$ whenever $v\in G_\mu$, the above identities imply that

$$(3.11)\qquad \mathbf{a}\,[\mathcal{B}_\mu P'_{\mathrm{N}}(\mu)u,v] \;=\; \mu^{-1}(\mu-\lambda)\,\mathbf{a}\,[S_{\mathrm{D}}u,v]\,,\qquad \forall u\in\mathcal{L}_\lambda\,,\ \forall v\in G_\mu\,,$$

$$(3.12)\qquad \mathbf{a}\,[\mathcal{B}_\mu P'_{\mathrm{D}}(\mu)u,v] \;=\; \lambda^{-1}(\lambda-\mu)\,\mathbf{a}\,[S_{\mathrm{N}}u,v]\,,\qquad \forall u\in\mathcal{L}_\lambda\,,\ \forall v\in G_\mu\,.$$

In view of Lemma 1.1, we have that $\big(I-\chi_{(-\varepsilon,\varepsilon)}(\mathcal{B}_\mu)\big)\,P'_{\mathrm{B}}(\mu)E'_{\mathrm{B}}(\lambda)\big|_{\mathcal{L}_\lambda}$ is an analytic operator-valued function of $\mu\in(\lambda-\delta,\lambda+\delta)$. Since this operator-valued function vanishes at $\mu=\lambda$, we have

$$\begin{aligned}(3.13)\quad &\mathbf{a}\,\big[\big(I-\chi_{(-\varepsilon,\varepsilon)}(\mathcal{B}_\mu)\big)\,P'_{\mathrm{B}}(\mu)E'_{\mathrm{B}}(\lambda)u\big]\\ &\qquad\leqslant\; C_2\,(\lambda-\mu)^2\,\mathbf{a}[u] \;=\; C_2\,(\lambda-\mu)^2\,\lambda\,\|u\|^2\,,\qquad \forall u\in\mathcal{L}_\lambda\,,\end{aligned}$$

with some positive constant C_2 independent of μ and u. Substituting $v=\big(I-\chi_{(-\varepsilon,\varepsilon)}(\mathcal{B}_\mu)\big)\,P'_{\mathrm{B}}(\mu)u$ into (3.11), (3.12) and applying (3.13), we obtain

$$\begin{aligned}(3.14)\quad &\mathbf{a}\,\big[\mathcal{B}_\mu P'_{\mathrm{B}}(\mu)u,\big(I-\chi_{(-\varepsilon,\varepsilon)}(\mathcal{B}_\mu)\big)\,P'_{\mathrm{B}}(\mu)u\big]\\ &\qquad\leqslant\; C_3\,(\lambda-\mu)^2\,\|u\|^2\,,\qquad \forall u\in E'_{\mathrm{B}}(\lambda)\mathcal{L}_\lambda\,,\end{aligned}$$

with some constant C_3 independent of μ and u. Now (3.3) and (3.4) follow from (3.9), (3.10) and (3.14).

3.5. Proof of Corollary 1.6. Lemma 1.1 and Theorem 1.3 imply that $\dim G_\lambda^-$ is constant on every connected component of the set $\mathbb{R}\setminus(\sigma(A_{\mathrm{N}})\cup\sigma(A_{\mathrm{D}}))$. If $\lambda\notin\sigma_{\mathrm{ess}}(A_{\mathrm{N}})\cup\sigma_{\mathrm{ess}}(A_{\mathrm{D}})$ and $\lambda\in\Lambda$ is an eigenvalue, then, by Theorem 1.3,

$$(3.15)\qquad \dim G_\mu^- \;=\; \begin{cases}\dim G_\lambda^-+n_{\mathrm{D}}(\lambda)\,, & \forall\mu\in(\lambda-\delta,\lambda),\\ \dim G_\lambda^-+n_{\mathrm{N}}(\lambda)\,, & \forall\mu\in(\lambda,\lambda+\delta)\,,\end{cases}$$

provided that $\delta > 0$ is small enough. In other words, the value of $\dim G_\mu^-$ jumps by $n_{\mathrm{N}}(\lambda) - n_{\mathrm{D}}(\lambda)$ as μ passes through the eigenvalue λ. Summing up these jumps over all the eigenvalues lying between a and b and taking into account that $\dim G_\lambda^- = \dim G_{\lambda-0}^- - n_{\mathrm{D}}(\lambda)$, we obtain (1.3).

3.6. Proof of Theorem 1.7. Let $\mathcal{L}$ be the subspace of $\chi_{(-\infty,\lambda)}(A_{\mathrm{N}})H$ spanned by all the vectors $v \in \chi_{(-\infty,\lambda)}(A_{\mathrm{N}})H$ such that

$$\chi_{(-\infty,\lambda]}(A_{\mathrm{D}})\,(A - \lambda I)v \;=\; 0\,. \tag{3.16}$$

The inclusion $\mathcal{L} \subset \mathcal{D}(A_{\mathrm{N}})$ implies that $\dim \mathcal{L} \geqslant N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) - n_{\mathrm{D}}(\lambda)$ and $\mathbf{b}[v] = 0$ for all $v \in \mathcal{L}$. From the latter identity, (3.16) and (2.7) it follows that $\mathbf{b}[P'_{\mathrm{D}}(\lambda)v] < \mathbf{b}[v]$ for all nonzero $v \in \mathcal{L}$. Since $((A-\lambda I)v, v)) < 0$ for all nonzero $v \in \chi_{(-\infty,\lambda)}(A_{\mathrm{N}})H$ and $((A-\lambda I)v, v) \geqslant 0$ for all $v \in \mathcal{D}(A_{\mathrm{D}})$ satisfying (3.16), we have $\mathcal{L} \cap \mathcal{D}(A_{\mathrm{D}}) = \{0\}$. Therefore $\ker P'_{\mathrm{D}}(\lambda)|_{\mathcal{L}} \subset L \cap \mathcal{D}(A_{\mathrm{D}}) = \{0\}$ and, consequently, $\dim P'_{\mathrm{D}}(\lambda)\mathcal{L} \geqslant N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) - n_{\mathrm{D}}(\lambda)$. Thus we have $\dim G_\lambda^- \geqslant \dim P'_{\mathrm{D}}(\lambda)\mathcal{L} \geqslant N_{\mathrm{N}}(\lambda) - N_{\mathrm{D}}(\lambda) - n_{\mathrm{D}}(\lambda)$. Now the theorem follows from (1.5).

3.7. Proof of Corollary 1.12. We have $R'(\lambda)H \subset G_\lambda$ and, by (1.1),

$$\begin{aligned} \mathbf{a}[\mathcal{B}_\lambda R'(\lambda)u, v] &= \mathbf{a}[R'(\lambda)u, v] - \lambda(R'(\lambda)u, v) \\ &= (u, v) = \mathbf{a}[A_{\mathrm{N}}^{-1}u, v]\,, \qquad \forall u \in H\,, \quad \forall v \in G_\lambda\,. \end{aligned} \tag{3.17}$$

The above identity implies that $\mathcal{B}_\lambda R'(\lambda)u = \Pi'_\lambda A_{\mathrm{N}}^{-1}u$ for all $u \in H$. In view of Lemma 1.2, the operator $\mathcal{B}_\lambda$ is invertible and, consequently, $R'(\lambda) = \mathcal{B}_\lambda^{-1}\Pi'_\lambda A_{\mathrm{N}}^{-1}$. Since $\mathbf{a}[A_{\mathrm{N}}^{-1}u, v] = (u, v)$, the subspace $\Pi'_\lambda A_{\mathrm{N}}^{-1}H$ is H^1-dense in G_λ. Therefore $R'(\lambda)H$ is an H^1-dense subspace of G_λ. Finally, by (3.17),

$$\mathbf{a}[\mathcal{B}_\lambda R'(\lambda)u, R'(\lambda)v] \;=\; (u, R'(\lambda)v)\,, \qquad \forall u, v \in H\,.$$

Thus we have $\mathbf{a}[\mathcal{B}_\lambda u, u] < 0$ on a k-dimensional subspace of G_λ if and only if $(R'(\lambda)u, u) < 0$ on a k-dimensional subspace of H. Now the corollary follows from Theorem 1.7.

3.8. Proof of Corollary 1.13. Let $a(\xi)$ be the full symbol of L^*L and $\lambda_1(\xi), \ldots, \lambda_m(\xi)$ be the eigenvalues of $a(\xi)$. Then $\lambda_{\mathrm{D},k} > \lambda_* := \min_j \inf_\xi \lambda_j(\xi)$ for all k because $\mathbf{a}[u] \geqslant \lambda_* \|u\|^2$ on $C_0^\infty(\Omega)$. On the other hand, since $\lambda_j(\xi)$ are continuous functions of ξ, the equation $\det(a(\xi) - \lambda I) = 0$ has infinitely many ξ-solutions for each fixed $\lambda > \lambda_*$. Therefore G_λ contains an infinite-dimensional set formed by functions of the form $u_\xi = e^{ix\cdot\xi}\,\vec{c}$, where $\vec{c} \in \ker(a(\xi) - \lambda I)$. For each of these functions we have $\mathbf{a}[u_\xi] = \lambda\,\|u_\xi\|^2$. This implies that either $\dim G_\lambda^- \geqslant 1$ or $\dim G_\lambda^0 = \infty$. By Lemma 1.2, the latter is possible only if $\lambda \geqslant \lambda_{\mathrm{N},\infty}$. Therefore, by Remark 1.11(2), we have $\lambda_{\mathrm{N},k+1} \leqslant \lambda_{\mathrm{D},k}$ for all eigenvalues lying below $\lambda_{\mathrm{N},\infty}$. If at least one Dirichlet eigenfunction corresponding to $\lambda_{\mathrm{D},k}$ does not satisfy the Neumann boundary condition, then $n_{\mathrm{D}}(\lambda_{\mathrm{D},k}) \geqslant 1$ and, by Remark 1.11(1), $\lambda_{\mathrm{N},k+1} < \lambda_{\mathrm{D},k}$.

4. Remarks

4.1. A_{D} and A_{N} as selfadjoint extensions. Denote $H_0^2 := H_0^1 \cap \mathcal{D}(A_{\mathrm{N}})$. Since $\mathcal{D}(A_{\mathrm{D}}) = \Pi_0\mathcal{D}(A_{\mathrm{N}})$ (see Subsection 2.1), we have $H_0^2 = \mathcal{D}(A_{\mathrm{D}}) \cap \mathcal{D}(A_{\mathrm{N}})$.

The H-adjoint A^* coincides with the restriction of A to H_0^2. Indeed, if $(u, Av) = (\tilde{u}, v)$ for some $u, \tilde{u} \in H$ and all $v \in H_A^1$, then, taking $v \in \mathcal{D}(A_{\mathrm{N}})$ or $v \in \mathcal{D}(A_{\mathrm{D}})$, we obtain $u = A_{\mathrm{N}}^{-1}\tilde{u} = A_{\mathrm{D}}^{-1}\tilde{u}$. Therefore $\mathcal{D}(A^*) \subset H_0^2$. On the other hand, if

$u \in H_0^2$, then $(u, Av) = \mathbf{a}[u,v]$ because $u \in H_0^1$ and $\mathbf{a}[u,v] = (A_{\mathrm{N}}u, v)$. Thus $\mathcal{D}(A^*) = H_0^2$ and $A^* = A|_{H_0^2}$.

If H_0^2 is not dense in H, then the second adjoint A^{**} does not exist and the operator A is not closable in H (see, for example, [**BS**, Section 3.3]).

If H_0^2 is dense in H, then A_{D} and A_{N} are selfadjoint extensions of A^*, and A^{**} is the closure of A. Note that $\mathcal{D}(A) = H_A^1$ may be strictly smaller than $H^1 \cap \mathcal{D}(A^{**})$. Also, the H^1-closed subspaces G_z may be strictly smaller than $\ker(A^{**} - zI)|_{H^1}$ and may not be closed in H (see the next subsection).

4.2. An example. Let Ω be a bounded domain with smooth boundary, $H = L_2(\Omega)$ and H^s be the Sobolev spaces. If $\mathbf{a}[u] = \|\nabla u\|^2 + \|u\|^2$ and H_0^1 is the H^1-closure of $C_0^\infty(\Omega)$, then $A = -\Delta + I$, $\mathcal{D}(A) = \{u \in H^1 \,:\, Au \in H\}$ and $G_0 = \{u \in H^1 \,:\, Au = 0\}$. The selfadjoint operators A_{D} and A_{N} are obtained by imposing the Dirichlet and Neumann boundary conditions. The H-adjoint A^* coincides with the restriction of A to $H_0^2 := \{u \in H^2 \,:\, u|_{\partial\Omega} = \partial_n u|_{\partial\Omega} = 0\}$, where ∂_n is the normal derivative. The second H-adjoint A^{**} is the extension of A to $\mathcal{D}(A^{**}) = \{u \in H \,:\, Au \in H\}$, and $\mathcal{D}(A) = H^1 \cap \mathcal{D}(A^{**})$.

Let us choose a nonzero function $v_0 \in G_0$, and define $\tilde H_0^1 = H_0^1 \oplus \mathcal{L}_0$, where $\mathcal{L}_0$ is the one-dimensional subspace spanned by v_0 and $\oplus$ denotes the orthogonal sum in H^1. Then the corresponding operator $\tilde A$ is the same differential operator $-\Delta + I$ but $\mathcal{D}(\tilde A) = \{u \in \mathcal{D}(A) \,:\, \langle \partial_n P_{\mathrm{N}}'(0)u, v_0\rangle_{\partial\Omega} = 0\}$, where $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the sesquilinear pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. The Neumann operator remains the same, and the domain of the new “Dirichlet” operator is $\mathcal{D}(\tilde A_{\mathrm{D}}) = \mathcal{D}(\tilde A) \cap \tilde H_0^1$. Finally, $\mathcal{D}(\tilde A^*) = \mathcal{D}(A_{\mathrm{N}}) \cap \tilde H_0^1 = \{u \in H^2 \cap \tilde H_0^1 \,:\, \partial_n u|_{\partial\Omega} = 0\}$. Note that $H_0^2 \subset \mathcal{D}(\tilde A^*)$.

Let $v_0 \notin H^2$. Then $v_0|_{\partial\Omega} \notin H^{3/2}(\partial\Omega)$ and, consequently, $u|_{\partial\Omega} \notin H^{3/2}(\partial\Omega)$ for all $u \in \tilde H_0^1 \setminus H_0^1$. This implies that $H^2 \cap \tilde H_0^1 = H^2 \cap H_0^1$ and $\mathcal{D}(\tilde A^*) = \mathcal{D}(A^*) = H_0^2$. Thus we have $\tilde A^{**} = A^{**}$. By the above, in this case, $\mathcal{D}(\tilde A) \neq H^1 \cap \mathcal{D}(\tilde A^{**})$.

The H^1-orthogonal complement $\tilde G_0 := H^1 \ominus \tilde H_0^1 = G_0 \ominus \mathcal{L}_0$ coincides with the kernel of the functional $u \to \mathbf{a}[v_0, u] = \langle \partial_n v_0, u\rangle_{\partial\Omega}$ defined on the space G_0. If $v_0 \notin H^2$, then this functional is not H-continuous and $\tilde G_0$ is not H-closed. Now (2.1) implies that $\tilde G_z := \ker(\tilde A - zI)$ are not H-closed for all $z \in \mathbb{C}$.

4.3. The projections $P_{\mathrm{B}}(\lambda)$. Note that by the spectral theorem, the right hand side of (3.7) is a nondecreasing function of μ and the right hand side of (3.8) is a nonincreasing function of μ. This observation allows one to simplify the proof of Theorem 1.3 in the case where $\dim G_\lambda^- < \infty$ or $\dim G_\lambda^+ < \infty$. The monotonicity is an implicit consequence of the following result.

Lemma 4.1. *Let Λ be an arbitrary real interval, and let $v \in H_A^1$.*

(1) *If $\Lambda \cap \sigma_{\mathrm{ess}}(A_{\mathrm{N}}) = \varnothing$ and $\chi_\Lambda(A_{\mathrm{N}}) P_{\mathrm{N}}'(0)v = 0$, then $P_{\mathrm{N}}'(\lambda)v \in G_\lambda$ for all $\lambda \in \Lambda$ and $\mathbf{b}[P_{\mathrm{N}}'(\lambda)v]$ is a nondecreasing function on Λ.*

(2) *If $\Lambda \cap \sigma_{\mathrm{ess}}(A_{\mathrm{D}}) = \varnothing$ and $\chi_\Lambda(A_{\mathrm{D}}) P_{\mathrm{D}}'(0)v = 0$, then $P_{\mathrm{D}}'(\lambda)v \in G_\lambda$ for all $\lambda \in \Lambda$ and $\mathbf{b}[P_{\mathrm{D}}'(\lambda)v]$ is a nonincreasing function on Λ.*

If, in addition, $v \notin \mathcal{D}(A_{\mathrm{B}})$, then the function $\mathbf{b}[P_{\mathrm{B}}'(\lambda)v]$ is strictly monotone.

Proof. Obviously, $(A - \lambda I)P_{\mathrm{B}}'(\lambda)v = E_{\mathrm{B}}(\lambda)(A - \lambda I)v$. Therefore the condition $\chi_\Lambda(A_{\mathrm{B}})P_{\mathrm{B}}'(0)v = 0$ implies that $(A - \lambda I)P_{\mathrm{B}}'(\lambda)v = -\lambda E_{\mathrm{B}}(\lambda)P_{\mathrm{B}}'(0)v = 0$ and, consequently, $P_{\mathrm{B}}'(\lambda)v \in G_\lambda$ for all $\lambda \in \Lambda$.

If $w_{\rm B}\in\mathcal{D}(A_{\rm B})$, then $P'_{\rm B}(\lambda)w_{\rm B}=E_{\rm B}(\lambda)w_{\rm B}$. Using this identity, one can easily show that

$$\text{(4.1)}\qquad \mathbf{b}[P'_{\rm N}(\lambda)(w+w_{\rm N})] = \mathbf{b}[P'_{\rm N}(\lambda)w]-(E_{\rm N}(\lambda)(A-\lambda I)w,w_{\rm N}),$$

$$\text{(4.2)}\qquad \mathbf{b}[P'_{\rm D}(\lambda)(w+w_{\rm D})] = \mathbf{b}[P'_{\rm D}(\lambda)w]+(w_{\rm D},E_{\rm D}(\lambda)(A-\lambda I)w)$$

for all $w\in H^1_A$, $w_{\rm N}\in\mathcal{D}(A_{\rm N})$ and $w_{\rm D}\in\mathcal{D}(A_{\rm D})$. Since $E_{\rm B}(\lambda)P'_{\rm B}(0)v=0$ and $P'_{\rm B}(\lambda)P'_{\rm B}(0)=P'_{\rm B}(\lambda)$ for all $\lambda\in\Lambda$, substituting $w=P'_{\rm B}(0)v$, $w_{\rm B}=P_{\rm B}(0)v$ in (4.1), (4.2) and applying (2.6), (2.7), we obtain

$$\mathbf{b}[P'_{\rm N}(\lambda)v]=\mathbf{b}[P'_{\rm N}(0)v]+\lambda\|P'_{\rm N}(0)v\|^2+\lambda^2(R_{\rm N}(\lambda)P'_{\rm N}(0)v,P'_{\rm N}(0)v), \qquad \forall\lambda\in\Lambda,$$

$$\mathbf{b}[P'_{\rm D}(\lambda)v]=\mathbf{b}[P'_{\rm D}(0)v]-\lambda\|P'_{\rm D}(0)v\|^2-\lambda^2(R_{\rm D}(\lambda)P'_{\rm D}(0)v,P'_{\rm D}(0)v), \qquad \forall\lambda\in\Lambda.$$

Now the required monotonicity results follow from the spectral theorem. □

Note that $P'_{\rm B}(0)=P'_{\rm B}(0)P'_{\rm B}(\mu)=P'_{\rm B}(\mu)-\mu A_{\rm B}^{-1}P'_{\rm B}(\mu)$ whenever $\mu\notin\sigma(A_{\rm B})$. Therefore we have $\chi_\Lambda(A_{\rm B})P'_{\rm B}(0)v=0$ if and only if $\chi_\Lambda(A_{\rm B})P'_{\rm B}(\mu)v=0$ for all $\mu\notin\sigma(A_{\rm B})$.

4.4. Analytic properties of $\Pi(z)$. If the embedding $H^1_0\hookrightarrow H$ is compact, then the operator-valued functions $\Pi(z)$ and $\Pi'(z)$ introduced in Subsection 2.3 are meromorphic in the whole complex plane. Indeed, since $\mathbf{a}[A_{\rm D}^{-1}u]=(A_{\rm D}^{-1}u,u)$, the compactness of the embedding $H^1_0\hookrightarrow H$ implies that $A_{\rm D}^{-1}$ is compact as an operator from H to H^1. Consequently,

$$T_z^\star T_z-I \;=\; z^2\,\Pi_0A_{\rm N}^{-2}\big|_{H^1_0}-2z\,\Pi_0A_{\rm N}^{-1}\big|_{H^1_0} \;=\; z^2\,A_{\rm D}^{-1}A_{\rm N}^{-1}\big|_{H^1_0}-2z\,A_{\rm D}^{-1}\big|_{H^1_0}$$

are compact operators in H^1_0. Now the required result follows from the analytic Fredholm theorem (see, for example, [**Ya**, Section 1.8]).

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