CM418Z FOURIER ANALYSIS

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1. Series expansions

One of the fundamental methods of solving various problems in applied mathematics, such as solving differential equations or partial differential equations, is to express the required function as a series

$$f(x) \;=\; \sum_k c_k\,\varphi_k(x)$$

where $\varphi_k(x)$ are suitable elementary functions, and then to determine the coefficients either exactly or approximately. This procedure is also of value when writing the programmes for pocket calculators where one has very limited storage space. Thus for a function such as $\sin x$ the calculator may just store ten coefficients $c_1, \dots c_{10}$ of an expansion of $\sin x$ using Chebyshef polynomials. Other special functions then just correspond to different coefficients $c_1, \dots c_{10}$. The best values of $c_1, \dots c_{10}$ may have been determined by a large computer.

Power series.
The first way of writing down such an expansion is to try

$$f(x) \;=\; \sum_k c_k\,x^k\,, \qquad -R < x < R\,.$$

This has two bad features.

(1) Even when possible its rate of convergence can be very slow so it is not useful for numerical purposes.
(2) If there is such an expansion then f must be infinitely differentiable on $(-R, R)$ with

$$f'(x) \;=\; \sum_k c_k\,k\,x^{k-1}\,, \qquad f''(x) \;=\; \sum_k c_k\,k(k-1)\,x^{k-2}\,, \qquad \dots$$

Thus the expansion only exists for rather special functions, which are called the *analytic* functions.

Definition of Fourier series.
Fourier analysis consists of the theory and applications of another type of expansion, the simplest example of which is

$$f(x) \;=\; \sum_{k=-\infty}^{\infty} c_k\, e^{ikx}\,, \qquad x \in \mathbb{R}^1\,, \tag{1.1}$$

where $e^{ikx} = \cos kx + i \sin kx$. It is clear that such an expansion can only work for 2π-*periodic* functions

$$f(x) \;=\; f(x + 2\pi)\,, \qquad \forall x \in \mathbb{R}^1\,.$$

But apart from this condition it turns out that the expansion exists, is useful, and has a rich theory for a very large class of functions f. Periodic functions are often restricted to $(-\pi, \pi]$ or $[0, 2\pi)$, and we shall choose the former, in which case one has no limitations on f (yet).

We shall have to assume that f is well enough behaved for various integrals such as

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx\,, \qquad \int_{-\pi}^{\pi} e^{-ijx}\, f(x)\, dx\,, \qquad \text{etc}$$

to exist and be finite. For the time being we assume that f is *piecewise continuous*, i.e.,

$$-\pi = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = \pi$$

> and f is continuous in each interval (y_i, y_{i+1}) with left and right hand limits at the end points.

The actual values $f(y_i)$ are arbitrary apart from the periodicity condition $f(y_0) = f(y_n)$.

We need a method of calculating the coefficients c_k in the hypothetical expansion (1.1). <u>Assuming</u> that we can integrate the series term by term we get

$$\int_{-\pi}^{\pi} e^{-imx}\, f(x)\, dx \;=\; \sum_{k=-\infty}^{\infty} c_k \int_{-\pi}^{\pi} e^{i(k-m)x}\, dx \;=\; 2\pi c_m\,, \tag{1.2}$$

so

$$c_m \;=\; (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-imx}\, f(x)\, dx\,. \tag{1.3}$$

We shall rigorously prove that c_m <u>must</u> be of this form later. But for the time being we shall just <u>call</u> c_m defined by (1.2) the Fourier coefficients of f and ask whether the series $\sum_{k=-\infty}^{\infty} c_k\, e^{ikx}$ does indeed converge to $f(x)$.

Sine and cosine expansions.
Since

$$\cos kx \;=\; \frac{e^{ikx}+e^{-ikx}}{2}\,, \qquad \sin kx \;=\; \frac{e^{ikx}-e^{-ikx}}{2i}\,, \tag{1.4}$$

we have

$$\begin{aligned}
&c_k\, e^{ikx} + c_{-k}\, e^{-ikx} \\
&= (2\pi)^{-1}\int_{-\pi}^{\pi}\left(e^{ik(x-y)}+e^{-ik(x-y)}\right) f(y)\,dy \;=\; \pi^{-1}\int_{-\pi}^{\pi} f(y)\,\cos k(x-y)\,dy \\
&\quad= \pi^{-1}\int_{-\pi}^{\pi} f(y)\,(\cos kx\cos ky+\sin kx\sin ky)\,dy \;=\; a_k\,\cos kx \;+\; b_k\,\sin kx
\end{aligned}$$

for all $k \geqslant 0$. Here

$$a_k \;=\; \pi^{-1}\int_{-\pi}^{\pi} f(x)\cos kx\,dx\,, \qquad b_k \;=\; \pi^{-1}\int_{-\pi}^{\pi} f(x)\sin kx\,dx\,,$$

in particular, $b_0 = 0$. Clearly, if the function f is real then the coefficients a_k and b_k are real as well.

Now we obtain

$$\sum_{k=-\infty}^{\infty} c_k\, e^{ikx} \;=\; \frac{a_0}{2} \;+\; \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)\,.$$

If f is *even*, that is

$$f(x) \;=\; f(-x)\,, \qquad \forall x\in(-\pi,\pi]\,,$$

then $b_k = 0$ for all k. So we expect

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{k=1}^{\infty} a_k\cos kx \qquad \text{(the cosine series)}$$

If f is *odd*, that is

$$f(x) \;=\; -f(-x)\,, \qquad \forall x\in(-\pi,\pi]\,,$$

then $a_k = 0$ for all k. So we expect

$$f(x) \;=\; \sum_{k=1}^{\infty} b_k\sin kx \qquad \text{(the sine series)}$$

The actual convergence of these series is dependent upon the convergence of the Fourier exponential series (1.1), which is much more convenient for theoretical purposes. Once we have proved the convergence we will be able to turn to a variety of applications.

Remark. Let a be an arbitrary positive number. If f is a 2π-periodic, then the function $g(x) = f(2\pi a^{-1}x)$ is a-periodic, that is

$$g(x) \;=\; g(x+a)\,, \qquad \forall x\in\mathbb{R}^1\,.$$

Therefore changing the variable x we can reformulate all the results for a-periodic functions. In particular, (1.1) implies

$$g(x) \;=\; \sum_{k=-\infty}^{\infty} c_k\, e^{2\pi i k a^{-1} x}\,,$$

and by (1.3)

$$\begin{aligned} c_k \;&=\; (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-ikx}\, f(x)\, dx \;=\; (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-ikx}\, g\left(\frac{ax}{2\pi}\right)\, dx \\ &=\; a^{-1} \int_{-a/2}^{a/2} e^{-2\pi i k a^{-1} x}\, g(x)\, dx\,. \end{aligned}$$

2. Convergence of Fourier series

We first need the following lemma.

Lemma 2.1 (Bessel's inequality). *If c_k are the Fourier coefficients of f and $\int_{-\pi}^{\pi} |f(x)|^2\,dx < \infty$ then*

$$\sum_{k=-\infty}^{\infty} |c_k|^2 \;\leqslant\; (2\pi)^{-1} \int_{-\pi}^{\pi} |f(x)|^2\,dx \tag{2.1}$$

Proof. We calculate

$$|f(x) - \sum_{k=-N}^{N} c_k\, e^{ikx}|^2 \;=\; \left(f(x) - \sum_{k=-N}^{N} c_k\, e^{ikx}\right) \left(\overline{f(x)} - \sum_{k=-N}^{N} \overline{c_k\, e^{ikx}}\right)$$

$$= |f(x)|^2 - \sum_{k=-N}^{N} \overline{c_k}\, f(x)e^{-ikx} - \sum_{k=-N}^{N} c_k\, \overline{f(x)}e^{ikx} + \sum_{k,m=-N}^{N} c_k\, \overline{c_m}\, e^{i(k-m)x},$$

so

$$\int_{-\pi}^{\pi} |f(x) - \sum_{k=-N}^{N} c_k\, e^{ikx}|^2\,dx \;=\; \int_{-\pi}^{\pi} |f(x)|^2\,dx \;-\; \sum_{k=-N}^{N} \overline{c_k} \int_{-\pi}^{\pi} e^{-ikx}\, f(x)\,dx$$

$$- \sum_{k=-N}^{N} c_k \int_{-\pi}^{\pi} e^{ikx}\, \overline{f(x)}\,dx \;+\; \sum_{k,m=-N}^{N} c_k\, \overline{c_m} \int_{-\pi}^{\pi} e^{i(k-m)x}\,dx$$

$$= \int_{-\pi}^{\pi} |f(x)|^2\,dx \;-\; 2\pi \sum_{k=-N}^{N} c_k\, \overline{c_k} \;-\; 2\pi \sum_{k=-N}^{N} c_k\, \overline{c_k} \;+\; 2\pi \sum_{k=-N}^{N} c_k\, \overline{c_k}$$

$$= \int_{-\pi}^{\pi} |f(x)|^2 \;-\; 2\pi \sum_{k=-N}^{N} |c_k|^2\,.$$

This implies that

$$\sum_{k=-N}^{N} |c_k|^2 \;\leqslant\; (2\pi)^{-1} \int_{-\pi}^{\pi} |f(x)|^2\,dx\,, \qquad \forall N = 1, 2, \ldots$$

Since we have the required inequality for all finite N, the same holds for the infinite series. □

Remark. Later we shall actually show that (2.1) is an equality.

Corollary 2.2 (Riemann–Lebesgue lemma). *Let $-\infty < a < b < \infty$ and let $\int_a^b |f(x)|^2\,dx < \infty$. Then*

$$\int_a^b e^{ikx}\, f(x)\,dx \to 0\,, \qquad k \to \pm\infty\,. \tag{2.2}$$

Proof. For sufficiently large N we have $-2N\pi-\pi < a < b < 2N\pi+\pi$. If we extend f by zero to the interval $(-2N\pi-\pi, 2N\pi+\pi)$ then

$$\int_a^b e^{ikx} f(x)\,dx = \int_{-2N\pi-\pi}^{2N\pi+\pi} e^{ikx} f(x)\,dx$$
$$= \sum_{j=-N}^{j=N} \int_{2j\pi-\pi}^{2j\pi+\pi} e^{ikx} f(x)\,dx = \sum_{j=-N}^{j=N} \int_{-\pi}^{\pi} e^{ikx} f(x+2j\pi)\,dx\,.$$

The functions $f(x+2j\pi)$ are square integrable on the interval $(-\pi,\pi)$, so it is sufficient to prove (2.2) assuming that $a=-\pi$ and $b=\pi$.

In this case $\int_{-\pi}^{\pi} e^{ikx} f(x)\,dx$ are the Fourier coefficients c_k of the function f. In view of Bessel's inequality the series $\sum |c_k|^2$ is convergent. This implies that $|c_k|^2 \to 0$ and, consequently, $c_k \to 0$.

Remark. In view of (1.4) we also have

$$\int_a^b f(x)\,\sin kx\,dx \to 0\,, \qquad k\to\pm\infty\,, \tag{2.3}$$
$$\int_a^b f(x)\,\cos kx\,dx \to 0\,, \qquad k\to\pm\infty\,. \tag{2.4}$$

Moreover, if f is square integrable then for any real c the function $e^{icx}f(x)$ is also square integrable. Therefore

$$\int_a^b e^{i(k+c)x} f(x)\,dx \to 0\,, \qquad k\to\pm\infty$$

for all real c. This implies that

$$\int_a^b f(x)\,\sin(k+c)x\,dx \to 0\,, \qquad k\to\pm\infty\,, \tag{2.5}$$
$$\int_a^b f(x)\,\cos(k+c)x\,dx \to 0\,, \qquad k\to\pm\infty\,.$$

Pointwise convergence of the Fourier series.
We want to evaluate as $N\to\infty$ the sum

$$\sum_{k=-N}^{N} c_k\,e^{ikx} = (2\pi)^{-1} \sum_{k=-N}^{N} e^{ikx} \int_{-\pi}^{\pi} e^{-iky} f(y)\,dy$$
$$= (2\pi)^{-1} \sum_{k=-N}^{N} \int_{-\pi-x}^{\pi-x} e^{-iky} f(x+y)\,dy = (2\pi)^{-1} \sum_{k=-N}^{N} \int_{-\pi}^{\pi} e^{-iky} f(x+y)\,dy\,.$$

Since

$$e^{-iNy} + e^{-i(N-1)y} + \cdots + 1 + e^{iy} + \cdots + e^{iNy}$$
$$= e^{-iNy}\Big(1 + e^{iy} + (e^{iy})^2 + \cdots + (e^{iy})^{2N}\Big)$$
$$= e^{-iNy}\,\frac{(e^{iy})^{2N+1}-1}{e^{iy}-1} = \frac{e^{i(N+1/2)y} - e^{-i(N+1/2)y}}{e^{iy/2} - e^{-iy/2}}$$

we obtain

$$\sum_{k=-N}^{N} c_k\, e^{ikx} \;=\; (2\pi)^{-1} \int_{-\pi}^{\pi} f(x+y)\, \frac{\sin(N+1/2)y}{\sin y/2}\, dy$$
$$= \pi^{-1} \int_{0}^{\pi} \frac{f(x+y)+f(x-y)}{2}\, \frac{\sin(N+1/2)y}{\sin y/2}\, dy\,. \tag{2.6}$$

Now the particular case $f \equiv 1$ has $c_0 = 1$ and $c_k = 0$ for all other k. Therefore

$$1 \;=\; (2\pi)^{-1} \int_{-\pi}^{\pi} \frac{\sin(N+1/2)y}{\sin y/2}\, dy \;=\; \pi^{-1} \int_{0}^{\pi} \frac{\sin(N+1/2)y}{\sin y/2}\, dy\,. \tag{2.7}$$

Let

$$h(y) \;=\; \frac{1}{\pi\, \sin y/2} \left(\frac{f(x+y)+f(x-y)}{2} \;-\; \frac{f(x+0)+f(x-0)}{2} \right)$$
$$= \frac{y}{2\pi\, \sin y/2} \left(\frac{f(x+y)-f(x+0)}{y} \;+\; \frac{f(x-y)-f(x-0)}{y} \right).$$

In view of (2.6) and (2.7) we have

$$\sum_{k=-N}^{N} c_k\, e^{ikx} \;-\; \frac{f(x+0)+f(x-0)}{2}$$
$$= \pi^{-1} \int_{0}^{\pi} \frac{f(x+y)+f(x-y)}{2}\, \frac{\sin(N+1/2)y}{\sin y/2}\, dy$$
$$-\; \pi^{-1} \int_{0}^{\pi} \frac{f(x+0)+f(x-0)}{2}\, \frac{\sin(N+1/2)y}{\sin y/2}\, dy$$
$$= \int_{0}^{\pi} h(y)\, \sin(N+1/2)y\, dy\,.$$

We say that f has finite *right* and *left derivatives* at x if the limits

$$f'(x+0) \;=\; \lim_{y\downarrow 0} \frac{f(x+y)-f(x+0)}{y}\,, \qquad f'(x-0) \;=\; \lim_{y\downarrow 0} \frac{f(x-y)-f(x-0)}{y}$$

exist. These hypotheses imply that $h(y)$ is continuous on $(0,\pi]$ apart from the finite number of jump discontinuities caused by those of f. Indeed, the only troublesome point is $y=0$ where $\sin y/2 = 0$. But since

$$\lim_{y\to 0} \frac{y}{\sin y/2} \;=\; 2\,,$$

$\lim_{y\downarrow 0} h(y)$ exists with

$$\lim_{y\downarrow 0} h(y) \;=\; \pi^{-1}\Big(f'(x+0) + f'(x-0) \Big)\,.$$

Therefore the Riemann–Lebesgue lemma is applicable to h, and by (2.5)

$$\sum_{k=-N}^{N} c_k\, e^{ikx} \;-\; \frac{f(x+0)+f(x-0)}{2} \;\to\; 0\,, \qquad N\to\infty\,.$$

We have proved the following result.

Theorem 2.3. *If f is piecewise continuous on $(-\pi,\pi]$ with finite left and right derivatives at the point x then*

$$\lim_{N\to\infty}\sum_{k=-N}^{N} c_k\, e^{ikx} \;=\; \frac{f(x+0)+f(x-0)}{2}\,.$$

Remark. Since f is periodic, the relevant limit of the series at $x=\pm\pi$ is

$$\frac{1}{2}\left(\lim_{x\uparrow\pi} f(x) + \lim_{x\downarrow -\pi} f(x)\right).$$

Corollary 2.4. *If f is differentiable (and hence also continuous) at x then*

$$\lim_{N\to\infty}\sum_{k=-N}^{N} c_k\, e^{ikx} \;=\; f(x)\,.$$

Uniform convergence.

The above criteria for pointwise convergence are applicable to a wide variety of functions f. Uniform convergence of the series is another matter. It is an elementary consequence of a uniform convergence theorem that if the Fourier series of f converges uniformly to f then f must be continuous on $[-\pi,\pi]$ with $f(-\pi)=f(\pi)$. However, this is not quite sufficient.

The following lemma will be important.

Lemma 2.5. *Let f be periodic and continuously differentiable with derivative g. Then f is continuous and the Fourier coefficients c_k of f and d_k of g are related by*

$$d_k \;=\; ik\, c_k\,.$$

Proof. Integrating by parts we obtain

$$d_k \;=\; (2\pi)^{-1}\int_{-\pi}^{\pi} e^{-ikx}\, f'(x)\, dx$$

$$(2\pi)^{-1}\, e^{-ikx}\, f(x)\Big|_{-\pi}^{\pi} \;-\; (2\pi)^{-1}\int_{-\pi}^{\pi} (-ik\, e^{-ikx})\, f(x)\, dx \;=\; ik\, c_k\,.$$

□

Theorem 2.6. *If f is periodic and continuously differentiable then the Fourier series of f converges to f uniformly on $[-\pi,\pi]$.*

Proof. We know that $f(x)=\sum_{k=-\infty}^{\infty} c_k\, e^{ikx}$ for each x, so it is sufficient to prove that the series converges uniformly and for this it is enough to prove that

$$\sum_{k=-\infty}^{\infty} |c_k| \;<\; \infty\,.$$

By Bessel's inequality

$$\sum_{k=-\infty}^{\infty} |c_k|^2 \; < \; \infty\,, \qquad \sum_{k=-\infty}^{\infty} k^2\,|c_k|^2 \; = \; \sum_{k=-\infty}^{\infty} |d_k|^2 \; < \; \infty\,,$$

where d_k are the Fourier coefficients of f'. Therefore

$$\sum_{k=-\infty}^{\infty} (1+k^2)\,|c_k|^2 \; = \; \gamma \; < \; \infty\,.$$

Now for any $N < \infty$ Schwarz's inequality yields

$$\begin{aligned}\sum_{k=-N}^{N} |c_k| \; &= \; \sum_{k=-N}^{N} (1+k^2)^{-1/2}\,(1+k^2)^{1/2}\,|c_k| \\ &\leqslant \; \left(\sum_{k=-N}^{N} (1+k^2)^{-1}\right)^{1/2} \left(\sum_{k=-N}^{N} (1+k^2)|c_k|^2\right)^{1/2} \\ &\leqslant \gamma^{1/2}\left(1+2\sum_{k=1}^{N} 1/k^2\right)^{1/2} \; \leqslant \; \gamma^{1/2}\left(1+2\sum_{k=1}^{\infty} 1/k^2\right)^{1/2},\end{aligned}$$

so the series $\sum_{k=-\infty}^{\infty} |c_k|$ does converge. □

Convergence of sine and cosine series.

If f is even then

$$\sum_{k=-N}^{N} c_k\, e^{ikx} \; = \; \frac{a_0}{2} \; + \; \sum_{k=1}^{N} a_k \cos kx\,.$$

if f is odd then

$$\sum_{k=-N}^{N} c_k\, e^{ikx} \; = \; \sum_{k=1}^{N} b_k \sin kx\,.$$

Therefore under conditions of Theorem 2.3 the sine and cosine series converge at the point x to $\left(f(x+0)+f(x-0)\right)/2$, and under conditions of Theorem 2.6 they converge uniformly to f.

3. Some applications

Periodic solutions of differential equations.
Let g be a periodic continuous function on $\mathbb{R}^1$. We show how to solve the differential equation

$$f(x) \;-\; f''(x) \;=\; g(x)$$

using the theory of Fourier series.

Let

$$b_k \;=\; (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-ikx}\, g(x)\, dx$$

be the Fourier coefficients of g, and let us assume that f has Fourier coefficients c_k. Then f' has Fourier coefficients $ik\, c_k$ and f'' has Fourier coefficients $-k^2\, c_k$. Equating the Fourier coefficients of the two sides yields $(1+k^2)\, c_k = b_k$, so

$$c_k \;=\; (1+k^2)^{-1}\, b_k$$

and we obtain the solution

$$f(x) \;=\; \sum_{k=-\infty}^{\infty} (1+k^2)^{-1}\, b_k\, e^{ikx}\, .$$

Corollary 2.4 implies that it is the only possible solution which is periodic and twice continuously differentiable. However, there are two gaps.

(1) Is there any solution to the problem at all? Well, the given series is an obvious candidate, and we have to check that it can be differentiated term by term, twice, with the desired result.
(2) Are there any non-periodic solutions? The answer is "yes". If f is a solution then for all constants α and β the functions

$$f(x) \;+\; \alpha\, e^{x} \;+\; \beta\, e^{-x}$$

also solve the equation.

Both of these points are generally not bothered with in applications.

Remark. It is clear that the same approach can be applied to much more general differential equations. However, it is more common to solve these problems using the sine–cosine expansion, and just differentiating it formally term by term.

Convolution equations.
A convolution equation is one of the form

$$f(x) \;+\; \int_{-\pi}^{\pi} h(x-y)\, f(y)\, dy \;=\; g(x)\, ,$$

where g and h are given periodic functions, and we wish to find a solution f. We rewrite this in the form

$$f \;+\; h * f \;=\; g$$

where the convolution $h * f$ of f and h is given by

$$h * f\,(x) \;=\; \int_{-\pi}^{\pi} h(x-y)\, f(y)\, dy\, .$$

Lemma 3.1. *If h and f are piecewise continuous and periodic then $h * f$ is continuous and periodic with $h * f = f * h$.*

Proof. The integral is certainly well-defined for each x and is a periodic function of x. Moreover, by a change of variables

$$h * f\,(x) \;=\; \int_{-\pi}^{\pi} h(-y)\,f(y+x)\,dy \;=\; \int_{-\pi}^{\pi} h(y)\,f(-y+x)\,dy \;=\; f * h\,(x)\,,$$

so we are left with proving continuity.

We have

$$\begin{aligned}|\,h * f\,(x_1) - h * f\,(x_2)\,| \;&=\; |\int_{-\pi}^{\pi} (h(x_1-y) - h(x_2-y))\;f(y)\,dy\,| \\ &\leqslant\; c\int_{-\pi}^{\pi} |h(x_1-y) - h(x_2-y)|\;dy \;=\; c\int_{-\pi}^{\pi} |h(y) - h(y-\delta)|\;dy\end{aligned}$$

where $\delta = x_1 - x_2$ and $c = \sup\limits_{-\pi\leqslant y\leqslant\pi} |f(y)|$. The problem is therefore to show that this integral vanishes as $\delta \to 0$. That is obviously true when h is a step function. But then it is true in the general case because any piecewise continuous function can be uniformly approximated by step functions. □

We now compute the Fourier coefficients of the convolution.

Lemma 3.2. *If the k-th Fourier coefficients of f and h are c_k and d_k then the k-th Fourier coefficient of $h * f$ is* $2\pi\, c_k\, d_k$.

Proof. We calculate

$$\begin{aligned}&(2\pi)^{-1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(x-y)\,f(y)\,dy\,e^{-ikx}\,dx \\ &\qquad\qquad = \;(2\pi)^{-1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(x)\,f(y)\,e^{-ik(x+y)}\,dy\,dx \;=\; 2\pi\,c_k\,d_k\,.\end{aligned}$$

□

We now show how to solve the convolution equations. We assume that g and h are given piecewise continuous periodic functions and compute their Fourier coefficients b_k and d_k respectively. We also assume that there is a solution f which is piecewise continuous and periodic with Fourier coefficients c_k. Then, by Lemma 3.2, $f + h * f = g$ implies $c_k + 2\pi\,c_k\,d_k = b_k$, so

$$c_k \;=\; (1 + 2\pi\,d_k)^{-1}\,b_k$$

and we get the solution

$$f(x) \;=\; \sum_{k=-\infty}^{\infty} (1 + 2\pi\,d_k)^{-1}\,b_k\,e^{ikx}\,.$$

There are three problems in justifying this procedure.

(1) If one of the Fourier coefficients $d_k = -(2\pi)^{-1}$ then the denominator vanishes. The convolution equation may indeed have no solutions in this case.

(2) Even if this does not happen one does not know that the equation is soluble. However if $\sum |b_k| < \infty$, as happens if g is periodic and piecewise continuously differentiable, then since $d_k \to 0$ we see that $\sum |c_k| < \infty$. Thus, the series for f does converge uniformly, so f is defined by the series as a continuous periodic function and can then check it is indeed a solution.

(3) There may be non-periodic solutions not given by the above procedure.

Problem (2) is always and problem (3) is nearly always forgotten about, but problem (1) is an important difficulty.

The vibrating string.

We are concerned with solving the wave equation for $\varphi = \varphi(x,t)$

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2}, \qquad 0 \leqslant x \leqslant a,$$

subject to the conditions

$$\varphi(0,t) = \varphi(a,t) = 0 \qquad \text{for all } t,$$

$$\varphi(x,0) = f(x), \qquad \frac{\partial \varphi}{\partial t}(x,0) = g(x).$$

This equation describes the oscillating string; the conditions $\varphi(0,t) = \varphi(a,t) = 0$ mean that its ends are fixed, and f and g determine the initial position of the string.

The method is to suppose that there is a sine expansion for each instant $t \geqslant 0$,

$$\varphi(x,t) = \sum_{k=1}^{\infty} b_k(t) \sin \frac{\pi k x}{a}$$

and then to determine the functions $b(t)$ without any regard for rigour. We formally obtain

$$\sum_{k=1}^{\infty} b_k''(t) \sin \frac{\pi k x}{a} = -\sum_{k=1}^{\infty} b_k(t)\, c^2 \pi^2 k^2 a^{-2} \sin \frac{\pi k x}{a}$$

so $b_k'' = -\,c^2 \pi^2 k^2 a^{-2}\, b_k$. The solutions of this equation have the form

$$b_k(t) = \alpha_k \sin \frac{c\pi k t}{a} + \beta_k \cos \frac{c\pi k t}{a}$$

where α_k and β_k are some constants, so

$$\varphi(x,t) = \sum_{k=1}^{\infty} \left(\alpha_k \sin \frac{c\pi k t}{a} + \beta_k \cos \frac{c\pi k t}{a} \right) \sin \frac{\pi k x}{a}.$$

This implies

$$f(x) = \sum_{k=1}^{\infty} \beta_k \sin \frac{\pi k x}{a},$$

$$g(x) = \sum_{k=1}^{\infty} \frac{c\pi k}{a} \alpha_k \sin \frac{\pi k x}{a},$$

which enables us to determine α_k and β_k.

In practise this method is extremely successful. The prospects of justifying all the steps rigorously are non-existent. The best that is ever done is to obtain the answer and try to prove rigorously that the series converge uniformly, may be differentiated term by term, and that the various conditions are indeed satisfied.

4. The space $L_2(-\pi, \pi)$

Mean square convergence.
While our series on pointwise convergence and uniform convergence are very useful, they apply only to restricted classes of functions and it turns out to be advantageous to use a weaker form of convergence.

Definition 4.1. We say the sequence of functions f_k defined on $[-\pi, \pi]$ is mean square convergent to f if $\int_{-\pi}^{\pi} |f(x) - f_k(x)|^2 \, dx \to 0$ as $k \to \infty$.

Note that if $|f(x) - g(x)| \leqslant \varepsilon$ for all x then $\int_{-\pi}^{\pi} |f(x) - f_k(x)|^2 \, dx \leqslant 2\pi \, \varepsilon^2$. Hence uniform convergence implies mean square convergence.

Remark. Mean convergence does <u>not</u> imply uniform convergence. For example, if

$$f_k(x) \;=\; \begin{cases} 1, & |x| \leqslant k^{-1} \\ 0, & k^{-1} \leqslant |x| \leqslant \pi, \end{cases}$$

then $f_k(x) \to 0$ for all $x \neq 0$ and $\|f_k\| \to 0$ as $k \to \infty$, but $\sup_x |f_k(x)| = 1$ for all k.

In order to study the notion of mean square convergence it is helpful to introduce the notion of an abstract Hilbert space.

Definition of a Hilbert space.
We say that a complex vector space H has an inner product $(\cdot, \cdot)$ if we are given a map $(\cdot, \cdot) : H \times H \to \mathbb{C}^1$ such that

$$(\alpha f + \beta g, h) \;=\; \alpha(f, h) \;+\; \beta(g, h)\,, \qquad \forall\, \alpha, \beta \in \mathbb{C}^1\,, \tag{4.1}$$

$$(f, g) \;=\; \overline{(g, f)}\,, \tag{4.2}$$

$$(f, f) \;\geqslant\; 0\,, \tag{4.3}$$

$$\text{if } (f, f) \;=\; 0 \;\text{ then } f = 0\,. \tag{4.4}$$

We then define the norm of an element $f \in H$ by

$$\|f\| \;=\; \sqrt{(f, f)}\,.$$

Taking in (4.1) $\alpha = \beta = 0$ and $h = 0$, we obtain that $\|f\| = 0$ for $f = 0$. The equalities (4.1) and (4.2) also imply that

$$(f, \alpha g + \beta h) \;=\; \bar{\alpha}(f, g) \;+\; \bar{\beta}(f, h)\,, \qquad \forall\, \alpha, \beta \in \mathbb{C}^1\,. \tag{4.5}$$

Definition 4.2. We say that f and g are orthogonal if $(f, g) = 0$.

Obviously, if f and g are orthogonal then

$$\|f + g\|^2 \;=\; \|f\|^2 \;+\; \|g\|^2\,. \tag{4.6}$$

Definition 4.3. We say that $e_1, e_2, \ldots$ is an orthonormal set in an inner product space H if

$$(e_j, e_k) \;=\; \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Lemma 4.3 (Bessel's inequality). *Let $e_1, e_2, \dots, e_N$ be an orthonormal set. Then for all $f \in H$*

$$\|f\|^2 \;\geqslant\; \sum_{k=1}^{N} |(f, e_k)|^2 \,. \tag{4.7}$$

Proof. Using the properties of inner product, we obtain

$$0 \;\leqslant\; \|f \;-\; \sum_{k=1}^{N}(f, e_k)\, e_k\|^2 \;=\; \big(f \;-\; \sum_{k=1}^{N}(f, e_k)\, e_k\,, f \;-\; \sum_{k=1}^{N}(f, e_k)\, e_k\big)$$
$$=\; \|f\|^2 \;-\; \sum_{k=-N}^{N} \big(f, (f, e_k)\, e_k\big) \;-\; \sum_{k=-N}^{N} \big((f, e_k)\, e_k, f\big)$$
$$+\; \sum_{k,m=-N}^{N} \big((f, e_k)\, e_k, (f, e_m)\, e_m\big) \;=\; \|f\|^2 \;-\; \sum_{k=-N}^{N} |(f, e_k)|^2 \,.$$

This implies (4.7). □

Remark. One can also prove that $\sum_{k=1}^{N}(f, e_k)\, e_k$ and $f - \sum_{k=1}^{N}(f, e_k)\, e_k$ are orthogonal, and then (4.7) follows from (4.6).

Theorem 4.4. *We have*

$$|(f, g)| \;\leqslant\; \|f\|\,\|g\| \qquad \text{(Cauchy–Schwarz inequality)}$$

and

$$\|f + g\| \;\leqslant\; \|f\| \;+\; \|g\| \qquad \text{(triangle inequality).}$$

Proof. The case $g = 0$ is trivial, so suppose $g \neq 0$. Then the vector $g/\|g\|$ by itself form an orthonormal set. Applying Bessel's inequality we obtain

$$\|f\|^2 \;\geqslant\; |(f, g/\|g\|)|^2 \;=\; \|g\|^{-2}\,|(f, g)|^2$$

which implies the Cauchy–Schwarz inequality follows.

By the Cauchy–Schwarz inequality

$$\|f + g\|^2 \;=\; \|f\|^2 \;+\; (f, g) \;+\; (g, f) \;+\; \|g\|^2 \;=\; \|f\|^2 \;+\; 2\,\mathrm{Re}(f, g) \;+\; \|g\|^2$$
$$\leqslant\; \|f\|^2 \;+\; 2\,|(f, g)| \;+\; \|g\|^2 \;\leqslant\; \|f\|^2 \;+\; 2\,\|f\|\,\|g\| \;+\; \|g\|^2 \,.$$

Thus, $\|f + g\|^2 \leqslant (\|f\| + \|g\|)^2$ which implies the triangle inequality.

Having proved the triangle inequality we can introduce

Definition 4.5. The distance between f and g in an inner product space H is given by $\|f - g\|$.

As usual, we say that f_k is a Cauchy sequence if $\|f_k - f_m\| \to 0$ as $k, m \to \infty$. An inner product space H is said to be a Hilbert space if it is complete, that is every Cauchy sequence has a limit in H.

Example. The standard example of a Hilbert space is $H = \mathbb{C}^n$ with $(f, g) = \sum_{k=1}^n f_k \, \bar{g}_k$.

The space L_2.
We are here concerned with a space of functions $f : (-\pi, \pi) \to \mathbb{C}^1$ with

$$(f, g) \;=\; \int_{-\pi}^{\pi} f(x) \, \bar{g}(x) \, dx \, . \tag{4.8}$$

We define $H = L_2(-\pi, \pi)$ as a set of functions f such that

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx \;<\; \infty$$

when the integral is interpreted in the most general possible sense (which actually means a Lebesgue integral, but we will need almost nothing from the general theory, provided we assume a few sensible looking properties and one density theorem).

The space $L_2(-\pi, \pi)$ includes all piecewise continuous functions on $[-\pi, \pi]$. It includes all uniform limits of such functions. It also includes functions like $f(x) = |x - \alpha|^{-\beta}$, $\beta < 1/2$, or $f(x) = \sin(x^{-1})$.

If f and g are continuous, then the integral in (4.8) can be understood in the usual sense. It is easy to see that for such functions the inner product (4.8) possesses all the necessary properties. Analogous results remain valid for the Lebesgue integral, and they imply (4.1)–(4.3) in the general case.

However, there are two problems in showing that $L_2(-\pi, \pi)$ is a Hilbert space.

(1) The condition that $\|f\|^2 = \int_{-\pi}^{\pi} |f(x)|^2 \, dx = 0$ implies $f = 0$ is not literally true. For example if f vanishes except for a finite number of points then $\|f\| = 0$. It turns out that $\|f\| = 0$ implies $f(x) = 0$ almost everywhere, i.e., except for x in a small set called a null set. All finite or countable sets are null sets, and we shall identify two functions whenever they differ only in a null set. This problem rarely causes difficulties.

(2) It can be shown that $L_2(-\pi, \pi)$ is indeed complete if we use an appropriate general definition of integration — Lebesgue integration, but the proof is hard and we shall take this result in faith.

A density result.
Our final result we cannot prove rigorously concerns the space C^∞ of infinitely differentiable (smooth) periodic functions on $[-\pi, \pi]$.

Definition 4.6. A subset $H_0 \subset H$ is said to be dense in H if for any $f \in H$ and any $\varepsilon > 0$ there exists $g \in H_0$ such that $\|f - g\| \leqslant \varepsilon$.

Theorem 4.7. *The set C^∞ is dense in $L_2(-\pi, \pi)$.*

While this cannot be proved for an arbitrary $f \in L_2(-\pi, \pi)$ without a proper justification of the Lebesgue integral, we can prove it for a large number of more elementary functions f, enough to make it very plausible.

Indeed, any step function f can be uniformly approximated by smooth functions, which coincide with f outside a finite number of arbitrarily small intervals containing all the points of jumps and the points $-\pi$ and π. From here it follows that any step function can be approximated by the smooth functions in $L_2(-\pi, \pi)$. Since piecewise continuous functions are uniformly approximated by step functions, these functions can also be approximated in $L_2(-\pi, \pi)$ by the smooth functions. Moreover, any function which is uniformly approximated by step functions is approximated by smooth functions in $L_2(-\pi, \pi)$.

5. Complete orthonormal sets in L_2

Properties of complete orthonormal sets.

Let H be an abstract Hilbert space.

Definition 5.1. We say that an orthonormal set $\{e_k\}_{k=-\infty}^{\infty} \subset H$ is complete if for any $f \in H$ there are constants α_k such that

$$\lim_{N\to\infty} \|f - \sum_{k=-N}^{N} \alpha_k\, e_k\| \;=\; 0\,. \tag{5.1}$$

Lemma 5.2. *If $\{e_k\}_{k=-\infty}^{\infty}$ is a complete orthonormal set then (5.1) implies $\alpha_k = (f, e_k)$ for all k.*

Proof. For large enough N we have

$$\begin{aligned}\varepsilon \;\geqslant\; \|f - \sum_{k=-N}^{N} \alpha_k\, e_k\|\, \|e_m\| \;\geqslant\; |\,(f - \sum_{k=-N}^{N} \alpha_k\, e_k\,,\, e_m)\,| \\ =\; |\,(f, e_m) \;-\; \sum_{k=-N}^{N} \alpha_k\,(e_k, e_m)\,| \;=\; |\,(f, e_m) \;-\; \alpha_m\,|\end{aligned}$$

where $\varepsilon \to 0$ as $N \to \infty$. Hence $\alpha_m = (f, e_m)$. $\square$

Thus, if $\{e_k\}$ is complete then $f = \sum_{k=-\infty}^{\infty} (f, e_k)\, e_k$ is a norm convergent series for all $f \in H$.

Definition 5.3. We call $\alpha_k = (f, e_k)$ the Fourier coefficients of f with respect to the complete orthonormal set $\{e_k\}$.

Theorem 5.4. *The following statements about the orthonormal set $\{e_k\}$ are equivalent:*

1. *$\{e_k\}$ is complete;*
2. *$\|f\|^2 = \sum_{k=-\infty}^{\infty} |(f, e_k)|^2$ for all $f \in H$;*
3. *if $(f, e_k) = 0$ for all k then $f = 0$;*
4. *the set of finite linear combinations of e_k is dense in H.*

Proof. The implication (1) ⇒ (4) is obvious.

(4) ⇒ (3)
If (4) is fulfilled then for any $f \in H$ there exist constants β_k such that

$$\|\, f - \sum_{k=-N}^{N} \beta_k\, e_k\,\| \;\leqslant\; \frac{1}{2}\,\|f\|\,.$$

If $(f, e_k) = 0$ then we obtain

$$\frac{1}{4}\,\|f\|^2 \;\geqslant\; \|\, f - \sum_{k=-N}^{N} \beta_k\, e_k\,\|^2 \;=\; \|f\|^2 \;+\; \sum_{k=-N}^{N} |\beta_k|^2$$

which implies $f = 0$ (and $\beta_k = 0$).

(3) $\Rightarrow$ (2)
By Bessel's inequality $\sum_{k=-\infty}^{\infty} |\alpha_k|^2 \leqslant \|f\|^2$ (where α_k are the Fourier coefficients of f). Let $g_N = \sum_{k=-N}^{N} \alpha_k\, e_k$. Then for $1 \leqslant N < M < \infty$

$$\|g_M - g_N\|^2 \;=\; \| \sum_{N+1 \leqslant |k| \leqslant M} \alpha_k\, e_k \|^2 \;=\; \sum_{N+1 \leqslant |k| \leqslant M} |\alpha_k|^2 \;\to\; 0$$

as $M, N \to \infty$, so g_N is a Cauchy sequence. Completeness of H implies $g_N \to g$ for some $g \in H$. Now

$$(g, e_m) \;=\; \lim_{N\to\infty} (g_N, e_m) \;=\; \lim_{N\to\infty} \Big(\sum_{k=-N}^{N} \alpha_k\, e_k \,,\, e_m \Big) \;=\; \alpha_m \;=\; (f, e_m)\,.$$

Thus $(f - g, e_m) = 0$ for all m, and by (3) $f = g$. Finally,

$$\|f\|^2 = \|g\|^2 = \lim_{N\to\infty} \|g_N\|^2 = \lim_{N\to\infty} \sum_{k=-N}^{N} |\alpha_k|^2 = \sum_{k=-\infty}^{\infty} |(f, e_k)|^2\,.$$

(2) $\Rightarrow$ (1) We have already shown that

$$\| f - \sum_{k=-N}^{N} \alpha_k\, e_k \|^2 \;=\; \|f\|^2 \;-\; \sum_{k=-N}^{N} |\alpha_k|^2$$

(see the proof of Bessel's inequality). Now from (2) it follows that the right hand side vanishes as $N \to \infty$. This implies (1). □

We see that for a complete orthonormal set $\{e_k\}$ Bessel's inequality $\|f\|^2 \geqslant \sum_{k=-\infty}^{\infty} |(f, e_k)|^2$ becomes

$$\|f\|^2 \;=\; \sum_{k=-\infty}^{\infty} |(f, e_k)|^2 \tag{5.1}$$

which is also called Parseval's formula.

The relationship between the function and its coefficients can be taken even further. Theorem 5.4 immediately implies

Corollary 5.5. *The sequence $\{\alpha_k\}$ is the sequence of Fourier coefficients of some $f \in H$ if and only if $\sum |\alpha_k|^2 < \infty$, and if this holds then f is unique and given by $f = \sum \alpha_k\, e_k$. If f and g have Fourier coefficients α_k and β_k respectively, then $f + g$ has coefficients $\alpha_k + \beta_k$, $c\,f$ has coefficients $c\,\alpha_k$ (where c is a constant), and*

$$(f, g) \;=\; \sum_k \alpha_k\, \bar{\beta}_k\,.$$

Completeness of $e_k(x) = (2\pi)^{-1/2}\, e^{ikx}$ in $L_2(-\pi,\pi)$.

We have to prove one of the above criteria for this particular orthonormal set, and we choose (4).

Let $f \in L_2(-\pi,\pi)$ and $\varepsilon > 0$. By our basic density lemma there exists $g \in C^\infty$ such that $\|f - g\| \leqslant \varepsilon/2$. We have already proved that the Fourier series of g converges uniformly to g. So there is a partial sum

$$\sum_{k=-N}^{N} c_k\, e^{ikx} \;=\; \sum_{k=-N}^{N} (2\pi)^{1/2} c_k\, e_k(x)$$

such that

$$|\, g(x) - \sum_{k=-N}^{N} (2\pi)^{1/2} c_k\, e_k(x)\,| \;\leqslant\; (2\pi)^{-1/2}\varepsilon/2$$

for all x. Then

$$\|\, g - \sum_{k=-N}^{N} (2\pi)^{1/2} c_k\, e_k\, \| \;\leqslant\; \varepsilon/2$$

and

$$\|\, f - \sum_{k=-N}^{N} (2\pi)^{1/2} c_k\, e_k\, \| \;\leqslant\; \varepsilon$$

as required for (4). Thus, we have proved

Theorem 5.6. *If $f \in L_2(-\pi,\pi)$ and*

$$c_k \;=\; (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x)\,\overline{e_k(x)}\, dx \;=\; (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-ikx}\, f(x)\, dx$$

then the finite sums $f_N(x) = \sum_{k=-N}^{N} c_k\, e^{ikx}$ converges to f in the mean square sense as $N \to \infty$, that is

$$\int_{-\pi}^{\pi} |f(x) - f_N(x)|^2\, dx \;\to\; 0\,, \qquad N \to \infty\,.$$

This is not the same as uniform or pointwise convergence, and indeed is rather a weak sense of convergence, but nevertheless is highly important because of its extreme generality.

Now, in view of Corollary 5.5, we have a one-to-one correspondence between functions $f \in L_2(-\pi,\pi)$ and their Fourier coefficients, and we can use this fact to prove various results.

Convolution in $L_2(-\pi,\pi)$.

The convolution of two functions $f, g \in L_2(-\pi,\pi)$ is defined by the same formula

$$f * g(x) \;=\; \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy\,,$$

here we assume that f and g are extended to periodic functions on $\mathbb{R}^1$.

Theorem 5.7. *If $f, g \in L_2(-\pi,\pi)$ then $h = f * g$ is continuous and periodic with Fourier coefficients*

$$(h, e_k) \;=\; (2\pi)^{1/2}\,(f, e_k)\,(g, e_k)\,,$$

where $e_k(x) = (2\pi)^{-1/2}\,e^{ikx}$.

Proof. Let $f_N(x) = \sum_{k=-N}^{N}(f, e_k)\,e_k,\;\; g_N(x) = \sum_{k=-N}^{N}(g, e_k)\,e_k$, then $f_N \to f$ and $g_N \to g$ in the mean square sense. Also

$$f_N * g_N \;=\; \sum_{k=-N}^{N}\sum_{m=-N}^{N}(f, e_k)\,(g, e_m)\,e_k * e_m \;=\; \sum_{k=-N}^{N}\gamma_k\,e_k$$

where $\gamma_k = (2\pi)^{1/2}\,(f, e_k)\,(g, e_k)$, since

$$e_k * e_m \;=\; (2\pi)^{-1}\,e^{ikx}\int_{-\pi}^{\pi} e^{i(m-k)y}\,dy \;=\; \begin{cases} (2\pi)^{1/2}\,e_k, & k = m,\\ 0, & k \neq m.\end{cases}$$

Now

$$\begin{aligned}\sum_{k=-N}^{N}|\gamma_k| \;&=\; (2\pi)^{1/2}\sum_{k=-N}^{N}|(f, e_k)|\,|(g, e_k)|\\ &\leqslant\; (2\pi)^{1/2}\left(\sum_{k=-N}^{N}|(f, e_k)|^2\right)^{1/2}\left(\sum_{k=-N}^{N}|(g, e_k)|^2\right)^{1/2} \;\leqslant\; (2\pi)^{1/2}\,\|f\|\,\|g\|\,.\end{aligned}$$

Thus, $\sum_{k=-\infty}^{\infty}|\gamma_k| < \infty$. This implies that if $f_N * g_N \to h$ pointwise then $\sum_{k=-N}^{N}\gamma_k\,e_k \to h$ uniformly, and so h is continuous and periodic.

It remains to prove that $f_N * g_N \to h$ pointwise. We denote $f_x(y) = f(x-y)$, $f_{x,N}(y) = f_N(x-y)$, and write

$$f * g(x) \;=\; \int_{-\pi}^{\pi} f(x-y)\,g(y)\,dy \;=\; (f_x, \bar g)\,,$$

$$f_N * g_N(x) \;=\; \int_{-\pi}^{\pi} f_N(x-y)\,g_N(y)\,dy \;=\; (f_{x,N}, \bar g_N)\,.$$

Then

$$\begin{aligned}|f * g(x) - f_N * g_N(x)| \;&=\; |\,(f_x, \bar g) - (f_{x,N}, \bar g_N)\,|\\ &=\; |\,(f_x, \bar g) - (f_{x,N}, \bar g) + (f_{x,N}, \bar g) - (f_{x,N}, \bar g_N)\,|\\ \leqslant\; |\,(f_x - f_{x,N}, \bar g)\,| \;+\; |(f_{x,N}, \bar g - \bar g_N)\,| &\leqslant\; \|f_x - f_{x,N}\|\,\|\bar g\| \;+\; \|f_{x,N}\|\,\|\bar g - \bar g_N\|\,.\end{aligned}$$

By Theorem 5.6 the right hand side vanishes as $N \to \infty$, so $f_N * g_N(x) \to h(x)$.

Now let us return to the convolution equation

$$f + f * h \;=\; g\,,$$

where $h, g \in L_2(-\pi,\pi)$ and we are looking for a solution $f \in L_2(-\pi,\pi)$. We can apply the same method as before (see Section 3). However, even not solving the equation, we see from above that a solution f is continuous and periodic if and only if g is continuous and periodic. Further connections between regularity properties of f and g, h can be obtained by similar methods.

Example. Let us try to find all the periodic continuous solutions to the equation

$$-f'' = f * f .$$

By Theorem 5.7 we have $k^2\,\alpha_k = (2\pi)^{1/2}\,\alpha_k^2$. So either $\alpha_k = 0$ or $\alpha_k = (2\pi)^{-1/2}\,k^2$. Since the Fourier series is convergent, only a finite number of α_k can be non-zero. Therefore each solution is a trigonometrical polynomial.

6. Fourier transform in $L_1(\mathbb{R}^1)$

We now start to look at the generalization of all our above results to non-periodic functions $f : \mathbb{R}^1 \to \mathbb{C}^1$. It turns out that the analogue of the set $\{\alpha_k\}$ of Fourier coefficients

$$\alpha_k \; = \; (2\pi)^{-1/2} \int_{-\pi}^{\pi} e^{-ikx} \, f(x) \, dx$$

is a new function $\hat{f} : \mathbb{R}^1 \to \mathbb{C}^1$ defined by

$$\hat{f}(\xi) \; = \; \mathcal{F}_{x\to\xi} f \; = \; (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} \, f(x) \, dx. \tag{6.1}$$

Further on we shall mostly deal with integrals over $\mathbb{R}^1$, and then we shall write $\int$ instead of $\int_{-\infty}^{\infty}$.

Some function spaces.

The first problem is to choose a suitable class of functions f, and there are two choices we shall consider at first:

(1) $C_0^\infty(\mathbb{R}^1)$ which consists of all infinitely differentiable functions with compact supports;
(2) $L_1(\mathbb{R}^1)$ which consists of the functions f such that $\int |f(x)| \, dx < \infty$.

Later on we shall deal also with the space $L_2(\mathbb{R}^1)$ which consists of the functions f such that $\int |f(x)|^2 \, dx < \infty$.

The space $L_2(\mathbb{R}^1)$ has the inner product $(f, g) = \int f(x) \, \bar{g}(x) \, dx$ and the norm $\|f\|_2 = \sqrt{(f, f)}$. The space $L_1(\mathbb{R}^1)$ has the norm $\|f\|_1 = \int |f(x)| \, dx$. Both norms satisfy

$$\|c\, f\| = |c| \, \|f\| \,, \qquad \|f + g\| \leqslant \|f\| + \|g\| \,, \qquad \|f\| \geqslant 0 \,.$$

Moreover, $\|f\|_2 = 0$ if and only if $f = 0$ almost everywhere (i.e., outside a null set), and $\|f\|_1 = 0$ if and only if $f = 0$ almost everywhere. In both cases we shall identify the functions which only differ on a null set.

If f is equal to zero outside some interval (a, b) then the Cauchy-Schwarz inequality (Theorem 4.4) with g being the characteristic function of (a, b) implies

$$\|f\|_1 = \int |f(x)| \, dx = (|f|, g) \leqslant \|g\|_2 \|f\|_2 = \sqrt{b - a} \, \|f\|_2 \,.$$

Therefore every L_2-function with compact support belongs to $L_1(\mathbb{R}^1)$.

We can consider $L_1(\mathbb{R}^1)$ and $L_2(\mathbb{R}^1)$ as the metric spaces with $\operatorname{dist}(f, g) = \|f - g\|_1$ and $\operatorname{dist}(f, g) = \|f - g\|_2$ respectively. Both spaces are complete, so we say that $L_1(\mathbb{R}^1)$ is a Banach space and $L_2(\mathbb{R}^1)$ is a Hilbert space. Completeness depends upon using Lebesgue integration.

One has $C_0^\infty(\mathbb{R}^1) \subset L_p(\mathbb{R}^1)$, $p = 1, 2$, but neither of $L_1(\mathbb{R}^1)$ and $L_2(\mathbb{R}^1)$ contains the other. This causes some complication compared with the periodic case where $L_2(-\pi, \pi) \subset L_1(-\pi, \pi)$ so that we could concentrate exclusively on the former.

We shall need the following (unproved) result.

Lemma 6.1. *The space $C_0^\infty(\mathbb{R}^1)$ is dense in $L_1(\mathbb{R}^1)$ and $L_2(\mathbb{R}^1)$.*

Fourier transform in $L_1(\mathbb{R}^1)$.

If $f \in L_1(\mathbb{R}^1)$ then the integral (6.1) converges and defines a function $\hat{f} : \mathbb{R}^1 \to \mathbb{C}^1$.

Theorem 6.2. *If $f \in L_1(\mathbb{R}^1)$ then $\hat{f}$ is a bounded continuous function.*

Proof. We have

$$\begin{aligned} |\hat{f}(\xi)| &= (2\pi)^{-1/2} \left| \int e^{-ix\xi} f(x)\, dx \right| \\ &\leqslant (2\pi)^{-1/2} \int |e^{-ix\xi} f(x)|\, dx = (2\pi)^{-1/2} \int |f(x)|\, dx = (2\pi)^{-1/2} \|f\|_1 \end{aligned}$$

which proves the boundness of $\hat{f}(\xi)$.

Continuity of $\hat{f}(\xi)$ is an elementary corollary of so-called Lebesgue's dominated convergence theorem, but we shall adopt a different method.

For all $f, g \in L_1(\mathbb{R}^1)$ we have

$$\begin{aligned} |\hat{f}(\xi) - \hat{g}(\xi)| &= (2\pi)^{-1/2} \left| \int e^{-ix\xi} (f(x) - g(x))\, dx \right| \\ &\leqslant (2\pi)^{-1/2} \int |f(x) - g(x)|\, dx = (2\pi)^{-1/2} \|f - g\|_1 . \qquad (6.2) \end{aligned}$$

By Lemma 6.1, given $\varepsilon > 0$ we can find $g \in C_0^\infty(\mathbb{R}^1)$ such that $\|f - g\|_1 \leqslant (2\pi)^{1/2}\, \varepsilon/3$. Then, in view of (6.2), for all ξ

$$|\hat{f}(\xi) - \hat{g}(\xi)| \leqslant \varepsilon/3 .$$

By the mean value theorem

$$e^{-ix\xi_1} - e^{-ix\xi_2} = -ix\, e^{-ix\widetilde{\xi}} (\xi_1 - \xi_2)$$

for some $\widetilde{\xi} \in [\xi_1, \xi_2]$. Therefore

$$\begin{aligned} |\hat{g}(\xi_1) - \hat{g}(\xi_2)| &= (2\pi)^{-1/2} \left| \int (e^{-ix\xi_1} - e^{-ix\xi_2})\, g(x)\, dx \right| \\ &\leqslant (2\pi)^{-1/2} \int |e^{-ix\xi_1} - e^{-ix\xi_2}|\, |g(x)|\, dx \\ &= (2\pi)^{-1/2} \int |x|\, |\xi_1 - \xi_2|\, |g(x)|\, dx \quad \leqslant \varepsilon/3 \end{aligned}$$

provided $|\xi_1 - \xi_2| \leqslant \sqrt{2\pi} \left(\int |x|\, |g(x)|\, dx \right)^{-1} \varepsilon/3$. Thus, we obtain

$$|\hat{f}(\xi_1) - \hat{f}(\xi_2)| \leqslant |\hat{f}(\xi_1) - \hat{g}(\xi_1)| + |\hat{f}(\xi_2) - \hat{g}(\xi_2)| + |\hat{g}(\xi_1) - \hat{g}(\xi_2)| \leqslant \varepsilon$$

for small enough $|\xi_1 - \xi_2|$. □

Theorem 6.3. *If $f_k \in L_1(\mathbb{R}^1)$, $f \in L_1(\mathbb{R}^1)$ and $\|f - f_k\|_1 \to 0$ then $\hat{f}_k \to \hat{f}$ uniformly.*

Proof. By (6.2) $|\hat{f}(\xi) - \hat{f}_k(\xi)| \leqslant \|f - f_k\|_1$ which implies the uniform convergence.

Theorem 6.4. *If* $f \in L_1(\mathbb{R}^1)$ *then* $\lim_{\xi\to\pm\infty} \hat f(\xi) = 0$.

Proof. This result does not follow from Lebesgue's dominated convergence theorem, but the method adopted above still works.

Let $\varepsilon > 0$ and $g \in C_0^\infty(\mathbb{R}^1)$ satisfy $\|f-g\| \leqslant (2\pi)^{1/2}\,\varepsilon/2$ so that $|\hat f(\xi)-\hat g(\xi)| \leqslant \varepsilon/2$ for all ξ. Integrating by parts we obtain

$$\begin{aligned}\hat g(\xi) &= (2\pi)^{-1/2}\int e^{-ix\xi}\, g(x)\, dx \\ &= -(2\pi)^{-1/2}\int (-i\xi)^{-1}\, e^{-ix\xi}\, g'(x)\, dx \;=\; (2\pi)^{-1/2}\,(i\xi)^{-1}\int e^{-ix\xi}\, g'(x)\, dx\,.\end{aligned}$$

Therefore

$$|\hat g(\xi)| \;\leqslant\; (2\pi)^{-1/2}\,|\xi|^{-1}\int |g'(x)|\, dx \;\leqslant\; \varepsilon/2$$

if $|\xi|$ is large enough. Thus,

$$|\hat f(\xi)| \;\leqslant\; |\hat f(\xi) - \hat g(\xi)| \;+\; |\hat g(\xi)| \;\leqslant\; \varepsilon$$

for sufficiently large $|\xi|$. □

Sine and cosine Fourier transforms.

The functions $\int \sin(x\xi)\, f(x)\, dx$ and $\int \cos(x\xi)\, f(x)\, dx$ are called the sine and cosine Fourier transforms of f respectively. Clearly, if f is even then its sine Fourier transform is equal to zero, and if f is odd then its cosine Fourier transform is equal to zero.

Theorem 6.4 immediately implies

Corollary 6.5. *If* $f \in L_1(\mathbb{R}^1)$ *then its sine and cosine Fourier transforms are bounded continuous functions which vanish at* $\pm\infty$.

Convolution in $L_1(\mathbb{R}^1)$.

We define the convolution of two functions $f, g \in L_1(\mathbb{R}^1)$ by

$$f * g(x) \;=\; \int f(x-y)\, g(y)\, dy\,.$$

Theorem 6.6. *If* $f, g \in L_1(\mathbb{R}^1)$ *then* $f * g \in L_1(\mathbb{R}^1)$ *and* $f * g = g * f$ *almost everywhere.*

Proof. Obviously, $|f * g(x)| \leqslant \int |f(x-y)\, g(y)|\, dy$. Changing variables $x = z + y$ we obtain

$$\int |f * g(x)|\, dx \;\leqslant\; \iint |f(x-y)\, g(y)|\, dy\, dx \;=\; \iint |f(z)\, g(y)|\, dy\, dz \;=\; \|f\|_1\, \|g\|_1\,,$$

so $f * g \in L_1(\mathbb{R}^1)$. The equality

$$f * g(x) \;=\; \int f(x-y)\, g(y)\, dy \;=\; \int g(x-z)\, f(z)\, dz \;=\; g * f(x)$$

is a matter of changing variables $y = x - z$. □

Theorem 6.7. *If* $f, g \in L_1(\mathbb{R}^1)$ *then* $\widehat{f * g} = (2\pi)^{1/2}\, \hat{f}\, \hat{g}$.

Proof. We have

$$\begin{aligned}
\mathcal{F}_{x\to\xi}(f * g) &= (2\pi)^{-1/2} \int e^{-ix\xi}\,(f * g)(x)\,dx \\
&= (2\pi)^{-1/2} \iint e^{-ix\xi}\, f(x-y)\, g(y)\, dy\, dx \\
&= (2\pi)^{-1/2} \iint e^{-i(y+z)\xi}\, f(z)\, g(y)\, dy\, dz \;=\; (2\pi)^{1/2}\, \hat{f}(\xi)\, \hat{g}(\xi)\,.
\end{aligned}$$

This proves the theorem. □

7. Schwartz space $\mathcal{S}(\mathbb{R}^1)$

Rapidly decreasing functions.

Definition 7.1. We say that $f \in \mathcal{S}(\mathbb{R}^1)$ if f is infinitely differentiable and for all $k, m = 0, 1, 2, \dots$, there exist constants $c_{k,m}$ such that

$$\sup_x |x^k f^{(m)}(x)| \leqslant c_{k,m}\,, \tag{7.1}$$

where $f^{(m)} = d^m f/dx^m$.

Obviously, $\mathcal{S}(\mathbb{R}^1)$ is a linear space. If $f \in \mathcal{S}(\mathbb{R}^1)$ then for all k and m

$$|f^{(m)}(x)| \leqslant \widetilde{c}_{k,m}\,(1+|x|)^{-k}$$

with some constants $\widetilde{c}_{k,m}$. The functions from $\mathcal{S}(\mathbb{R}^1)$ are called the rapidly decreasing functions.

Example 7.1. The function $f(x) = e^{-x^2}$ is rapidly decreasing.

Lemma 7.2. *If $f \in \mathcal{S}(\mathbb{R}^1)$ then $f^{(p)} \in \mathcal{S}(\mathbb{R}^1)$ and $x^p f \in \mathcal{S}(\mathbb{R}^1)$ for all $p = 1, 2, \dots$*

Proof. It is sufficient to prove the lemma for $p = 1$; then the general result is obtained by induction in p.

From (7.1) it follows that

$$|x^k \frac{d^m}{dx^m} f'| = |x^k \frac{d^{m+1}}{dx^{m+1}} f| \leqslant c_{k,m+1}\,,$$

$$|x^k \frac{d^m}{dx^m}(xf)| = |\,x^{k+1}\frac{d^m}{dx^m} f + m\,x^k \frac{d^{m-1}}{dx^{m-1}} f\,| \leqslant c_{k+1,m} + m\,c_{k,m-1}\,.$$

Therefore $f \in \mathcal{S}(\mathbb{R}^1)$ implies $f' \in \mathcal{S}(\mathbb{R}^1)$ and $xf \in \mathcal{S}(\mathbb{R}^1)$. □

It is easy to see that

$$C_0^\infty(\mathbb{R}^1) \subset \mathcal{S}(\mathbb{R}^1) \subset \left(L_1(\mathbb{R}^1) \cap L_2(\mathbb{R}^1)\right)\,.$$

Since $C_0^\infty(\mathbb{R}^1)$ is dense in $L_1(\mathbb{R}^1)$ and in $L_2(\mathbb{R}^1)$, so is $\mathcal{S}(\mathbb{R}^1)$.

Fourier transform in $\mathcal{S}(\mathbb{R}^1)$.

Since $\mathcal{S}(\mathbb{R}^1) \subset L_1(\mathbb{R}^1)$, the Fourier transform $\hat{f}$ of a function $f \in \mathcal{S}(\mathbb{R}^1)$ is well-defined, and it is a bounded continuous function (Theorem 6.2).

Lemma 7.3. *For all $f \in \mathcal{S}(\mathbb{R}^1)$*

$$(\hat{f})'(\xi) = -i\,\mathcal{F}_{x\to\xi}(xf)\,, \qquad \xi\,\hat{f}(\xi) = -i\,\mathcal{F}_{x\to\xi}(f')\,. \tag{7.2}$$

Proof. We have

$$(\hat{f})'(\xi) = (2\pi)^{-1/2}\int \frac{d}{d\xi}\left(e^{-ix\xi}\right) f(x)\,dx$$
$$= -i\,(2\pi)^{-1/2}\int e^{-ix\xi}\,xf(x)\,dx = -i\,\mathcal{F}_{x\to\xi}(xf)\,,$$

$$\xi\,\hat{f}(\xi) = (2\pi)^{-1/2}\int \xi\,e^{-ix\xi}\,f(x)\,dx = (2\pi)^{-1/2}\int i\,\frac{d}{dx}\left(e^{-ix\xi}\right) f(x)\,dx$$
$$= -i\,(2\pi)^{-1/2}\int e^{-ix\xi}\,f'(x)\,dx = -i\,\mathcal{F}_{x\to\xi}(f')\,.$$

□

Corollary 7.4. *If $f \in \mathcal{S}(\mathbb{R}^1)$ then $\hat{f} \in \mathcal{S}(\mathbb{R}^1)$.*

Proof. The equalities (7.2) imply that

$$\frac{d^m}{d\xi^m}\,\hat{f}(\xi) \;=\; (-i)^m\,\mathcal{F}_{x\to\xi}(x^m f)\,, \qquad \xi^k\,\hat{f}(\xi) \;=\; (-i)^k\,\mathcal{F}_{x\to\xi}(f^{(k)})\,.$$

Therefore $\hat{f}$ is infinitely differentiable, and

$$\xi^k\,\frac{d^m}{d\xi^m}\,\hat{f}(\xi) \;=\; (-i)^{m+k}\,\mathcal{F}_{x\to\xi}\left((x^m f)^{(k)}\right)\,. \tag{7.3}$$

By Lemma 7.2

$$(x^m f)^{(k)} \;=\; \frac{d^k}{dx^k}\,(x^m f)\;\in \mathcal{S}(\mathbb{R}^1)\;\subset\;L_1(\mathbb{R}^1)\,,$$

and by Theorem 6.2 the functions (7.3) are bounded for all k, m. $\square$

Example 7.5. Let us calculate the Fourier transform of the function $f(x) = \exp\left(-x^2/2\right)$. This function is a solution of the differential equation

$$f'(x) \;=\; -x\,f(x)\,. \tag{7.4}$$

In view of (7.2), applying the Fourier transform to (7.4) we obtain

$$i\xi\,\hat{f}(\xi) \;=\; -i\,(\hat{f})'(\xi)\,.$$

Now we see that

$$\left(\frac{\hat{f}(x)}{f(x)}\right)' \;=\; \frac{\hat{f}'(x)\,f(x)-\hat{f}(x)\,f'(x)}{f^2(x)} \;=\; \frac{-x\,\hat{f}(x)\,f(x)+x\,\hat{f}(x)\,f(x)}{f^2(x)} \;=\; 0\,,$$

which implies $\hat{f}(x) = c_0\,f(x) = c_0\,\exp\left(-x^2/2\right)$ with some constant $c_0 \geqslant 0$. Finally,

$$c_0^2 \;=\; \left(\hat{f}(0)\right)^2 \;=\; (2\pi)^{-1}\left(\int e^{-x^2/2}\,dx\right)^2 \;=\; (2\pi)^{-1}\iint e^{-(x^2+y^2)/2}\,dx\,dy$$
$$=\;(2\pi)^{-1}\int_0^\infty\int_{\mathbf{S}^1} e^{-r^2/2}\,r\,d\theta\,dr \;=\; \int_0^\infty e^{-r^2/2}\,r\,dr \;=\; \frac{1}{2}\int_0^\infty e^{-s/2}\,ds \;=\; 1\,,$$

so $\hat{f}(x) = f(x) = \exp\left(-x^2/2\right)$.

Inverse Fourier transform.
We shall need the following lemma.

Lemma 7.6. *Let $T : \mathcal{S}(\mathbb{R}^1) \to \mathcal{S}(\mathbb{R}^1)$ be a linear map commuting with multiplication by x and differentiation, that is*

$$T(xf) \;=\; x\,Tf\,, \qquad (Tf') \;=\; (Tf)'\,, \forall\, f(x) \in \mathcal{S}(\mathbb{R}^1)\,.$$

Then $Tf(x) = c\,f(x)$ where c is some constant.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^1)$ and $f(x_0) = 0$. Then by Taylor's formula $f(x) = (x-x_0)\,g(x)$ with some $g \in C^\infty(\mathbb{R}^1)$. We have $g(x) = (x-x_0)^{-1}\,f(x)$ for $x \neq x_0$, so $g \in \mathcal{S}(\mathbb{R}^1)$. Now we obtain

$$Tf(x_0) \;=\; T\left((x-x_0)\,g(x)\right)\big|_{x=x_0} \;=\; (x-x_0)\,Tg(x)\big|_{x=x_0} \;=\; 0\,.$$

Since T is a linear map, this implies that $Tf_1(x_0) = Tf_2(x_0)$ if $f_1(x_0) = f_2(x_0)$. Therefore for any function f the value of Tf at x_0 depends only on $f(x_0)$.

Let $f_0 \in \mathcal{S}(\mathbb{R}^1)$ be a function such that $f_0(x_0) = 1$, and let $z_0 = Tf(x_0)$. Since T is a linear map, for an arbitrary function $f \in \mathcal{S}(\mathbb{R}^1)$ we have $Tf(x_0) = z_0\,f(x_0)$.

Let φ be the function defined by $\varphi(x_0) = z_0$, $\forall\, x_0 \in \mathbb{R}^1$. Then by the preceding $Tf(x) = \varphi(x)f(x)$. Since $Tf(x) \in \mathcal{S}(\mathbb{R}^1)$, the function φ is differentiable. Now

$$\varphi f' \;=\; (Tf') \;=\; (Tf)' \;=\; (\varphi f)' \;=\; \varphi f' \,+\, \varphi' f\,, \qquad \forall f \in \mathcal{S}(\mathbb{R}^1),$$

implies $\varphi' \equiv 0$, i.e., φ is identically equal to some constant. □

Corollary 7.7. *If $f \in \mathcal{S}(\mathbb{R}^1)$ then*

$$f(x) \;=\; (2\pi)^{-1/2} \int e^{ix\xi}\,\hat f(\xi)\,d\xi\,. \tag{7.5}$$

Proof. Let us define

$$Tf(x) \;=\; (2\pi)^{-1/2} \int e^{ix\xi}\,\hat f(\xi)\,d\xi.$$

In view of (7.2) we have

$$x\,Tf(x) \;=\; (2\pi)^{-1/2} \int x\,e^{ix\xi}\,\hat f(\xi)\,d\xi$$

$$=\; (2\pi)^{-1/2} \int -i\,\frac{d}{d\xi}\left(e^{ix\xi}\right)\,\hat f(\xi)\,d\xi \;=\; (2\pi)^{-1/2} \int i\,e^{ix\xi}\,(\hat f)'(\xi)\,d\xi \;=\; T(xf)(x)\,,$$

$$(Tf')(x) \;=\; (2\pi)^{-1/2} \int e^{ix\xi}\,\widehat{f'}(\xi)\,d\xi$$

$$=\; (2\pi)^{-1/2} \int (i\xi)\,e^{ix\xi}\,\hat f(\xi)\,d\xi \;=\; (2\pi)^{-1/2} \int \frac{d}{dx}\left(e^{ix\xi}\right)\,\hat f(\xi)\,d\xi \;=\; (Tf)'(x)\,.$$

Therefore T satisfies the conditions of Lemma 7.6 and

$$c\,f(x) \;=\; (2\pi)^{-1/2} \int e^{ix\xi}\,\hat f(\xi)\,d\xi$$

with some constant c. Taking $f(x) = \exp\left(-x^2/2\right)$ we obtain

$$(2\pi)^{-1/2} \int e^{ix\xi}\,\hat f(\xi)\,d\xi \;=\; (2\pi)^{-1/2} \int e^{ix\xi}\,f(\xi)\,d\xi \;=\; \hat f(-x) \;=\; f(x)$$

which implies $c = 1$. □

The map

$$\mathcal{F}^{-1}_{\xi\to x} \;:\; f(\xi) \;\to\; (2\pi)^{-1/2} \int e^{ix\xi}\,f(\xi)\,d\xi$$

is called the inverse Fourier transform. Clearly, $\mathcal{F}^{-1}_{\xi\to x} f \;=\; \hat f(-x)$, so $\mathcal{F}^{-1}$ maps $\mathcal{S}(\mathbb{R}^1)$ onto $\mathcal{S}(\mathbb{R}^1)$. Thus, we have proved

Theorem 7.8. *The Fourier transform maps $\mathcal{S}(\mathbb{R}^1)$ one-to-one onto $\mathcal{S}(\mathbb{R}^1)$, and the inverse transformation is given by (7.5).*

8.1. Pointwise inversion of the L_1-Fourier transform

We have proved that for "good" functions f

$$f(x) \;=\; (2\pi)^{-1/2}\int e^{ix\xi}\,\hat f(\xi)\,d\xi\,.$$

The main difficulty with the L_1-Fourier transform is that $f\in L(\mathbb{R}^1)$ does not imply $\hat f\in L(\mathbb{R}^1)$, so the integral does not converge. Even the simple function

$$f(x) \;=\; \begin{cases} 1, & |x|\leqslant 1,\\ 0, & |x|>1\end{cases}$$

is a counterexample (in this case $\hat f(\xi)=2(2\pi)^{-1/2}(i\xi)^{-1}\sin\xi$).

Theorem 8.1. *Let $f\in L_1(\mathbb{R}^1)$ and f be piecewise continuous. If f has finite left and right derivatives at x then*

$$\frac{f(x+0)+f(x-0)}{2} \;=\; (2\pi)^{-1/2}\lim_{R\to\infty}\int_{-R}^{R} e^{ix\xi}\,\hat f(\xi)\,d\xi\,. \tag{8.1}$$

Proof. Since $\hat f$ is continuous (Theorem 6.2), each integral over a finite interval in the right hand side of (8.1) certainly makes sense. The structure of the proof is exactly the same as for the Fourier series.

We put $f_R(x)$ for the integral over $(-R,R)$ in the right hand side of (8.1). Then, by Fubini's theorem,

$$\begin{aligned} f_R(x) \;&=\; (2\pi)^{-1}\int_{-R}^{R}\int e^{-iy\xi}\,f(y)\,dy\,e^{ix\xi}\,d\xi \;=\; (2\pi)^{-1}\int_{-R}^{R}\int e^{i(x-y)\xi}\,f(y)\,dy\,d\xi\\ &=\; (2\pi)^{-1}\int \frac{e^{i(x-y)R}-e^{-i(x-y)R}}{i(x-y)}\,f(y)\,dy \;=\; \pi^{-1}\int f(x-y)\,\frac{\sin yR}{y}\,dy\,. \end{aligned}$$

Since the function $(\sin yR)/y$ is even, we obtain

$$\begin{aligned} \pi^{-1}\int f(x-y)\,\frac{\sin yR}{y}\,dy \;&=\; \frac{2}{\pi}\int_0^\infty \frac{f(x+y)+f(x-y)}{2}\,\frac{\sin yR}{y}\,dy\\ &=\; \frac{2}{\pi}\,\frac{f(x+0)+f(x-0)}{2}\int_0^\pi \frac{\sin yR}{y}\,dy\\ &+\;\frac{2}{\pi}\int_0^\pi\left(\frac{f(x+y)-f(x+0)}{2y}+\frac{f(x-y)-f(x-0)}{2y}\right)\sin yR\,dy\\ &+\;\frac{2}{\pi}\int_\pi^\infty \frac{f(x+y)+f(x-y)}{2y}\,\sin yR\,dy\,. \end{aligned} \tag{8.2}$$

Since $\sin yR=(e^{iyR}-e^{-iyR})/2i$ and the function

$$g(y) \;=\; \begin{cases} (f(x+y)+f(x-y))/2y, & y\geqslant\pi,\\ 0, & y<\pi\end{cases}$$

is from $L_1(\mathbb{R}^1)$, the third term in the right hand side of (8.2) goes to zero by Theorem 6.4. The second term goes to zero by the Riemann–Lebesgue lemma (see (2.5)). For the first term we are left with proving that

$$\frac{2}{\pi}\int_0^\pi \frac{\sin yR}{y}\,dy \;\to\; 1$$

as $R\to\infty$. This is a standard formula from complex analysis, but it also follows from the fact that the right hand side of (8.1) goes to $(f(x+0)+f(x-0))/2=f(x)$ for all $f\in\mathcal{S}(\mathbb{R}^1)$ (Theorem 7.8). $\square$

8.2. L_2-Fourier transform

Problems with the L_2-Fourier transform.
If $f \in L_2(\mathbb{R}^1)$ then the definition

$$\hat{f}(\xi) \;=\; (2\pi)^{-1/2} \int e^{-ix\xi}\, f(x)\, dx$$

does not always make sense, and it is not obvious that

$$(2\pi)^{-1/2} \lim_{R\to\infty} \int_{-R}^{R} e^{-ix\xi}\, f(x)\, dx$$

exists for all $f \in L_2(\mathbb{R}^1)$ either. Nevertheless the following theorem suggests that a good L_2-Fourier transform does exist.

Theorem 8.2 (Parseval's formula). *If $f, g \in \mathcal{S}(\mathbb{R}^1)$ then*

$$(f, g) \;=\; (\hat{f}, \hat{g})\,. \tag{8.3}$$

Proof. By Fubini's theorem for all $f, h \in \mathcal{S}(\mathbb{R}^1)$ we have

$$(\hat{f}, h) \;=\; (2\pi)^{-1/2} \iint e^{-ix\xi}\, f(x)\, dx\, \bar{h}(\xi)\, d\xi$$
$$=\; (2\pi)^{-1/2} \iint \overline{e^{ix\xi} h(\xi)}\, d\xi\, f(x)\, dx \;=\; (f, \mathcal{F}^{-1} h)\,,$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Taking $h = \hat{g}$ we obtain (8.3). □

From (8.3) it follows that

$$\|\hat{f}\|^2_{L_2(\mathbb{R}^1)} \;=\; \|f\|^2_{L_2(\mathbb{R}^1)} \tag{8.4}$$

so we can expect that $\hat{f}$ is well defined as a function from $L_2(\mathbb{R}^1)$ for all $f \in L_2(\mathbb{R}^1)$.

Definition of the L_2-Fourier transform.
Let $f \in L_2(\mathbb{R}^1)$. Then, since the space $\mathcal{S}(\mathbb{R}^1)$ is dense in $L_2(\mathbb{R}^1)$, there exists a sequence of functions $f_k \in \mathcal{S}(\mathbb{R}^1)$ such that $\|f - f_k\|_{L_2(\mathbb{R}^1)} \to 0$ as $k \to \infty$. Then $\|f_m - f_k\|_{L_2(\mathbb{R}^1)} \to 0$ as $k, m \to \infty$, i.e., $\{f_k\}$ is a Cauchy sequence in $L_2(\mathbb{R}^1)$. By (8.4) we obtain that $\|\hat{f}_m - \hat{f}_k\|_{L_2(\mathbb{R}^1)} \to 0$ as $k, m \to \infty$, which means that $\{\hat{f}_k\}$ is also a Cauchy sequence in $L_2(\mathbb{R}^1)$. Since the space $L_2(\mathbb{R}^1)$ is complete, the sequence $\{\hat{f}_k\}$ has a limit $g \in L_2(\mathbb{R}^1)$. Now we define $\hat{f} = g$.

We have to prove, of course, that our definition is correct. The function $g \in L_2(\mathbb{R}^1)$ is uniquely determined by $\hat{f}_k$ and, consequently, it is determined by f_k. The only problem is that g may be different if we take another sequence h_k convergent to f. However, if f_k and h_k converge to f then

$$\|\hat{f}_k - \hat{h}_k\|_{L_2(\mathbb{R}^1)} \;=\; \|f_k - h_k\|_{L_2(\mathbb{R}^1)} \;\to\; 0$$

as $k \to \infty$, so the sequences $\{\hat{f}_k\}$ and $\{\hat{h}_k\}$ have the same limit.

Properties of the L_2-Fourier transform.
It follows directly from the definition that (8.4) remains valid for all $f \in L_2(\mathbb{R}^1)$. This means that the Fourier transform preserves the norm (and so the metric) in $L_2(\mathbb{R}^1)$. Such transformations of a Hilbert space are said to be isometric. Since the inner product is continuous, Parceval's formula holds for all $f, g \in L_2(\mathbb{R}^1)$ as well. This result can also be proved as follows.

Lemma 8.3. *If A is a linear isometric transformation of the Hilbert space H then $(Af, Ag) = (f, g)$ for all $f, g \in H$.*

Proof. From the properties of the inner product it follows that

$$(f,g) \;=\; \frac{1}{4}\,\|f+g\|^2 \;-\; \frac{1}{4}\,\|f-g\|^2 \;+\; \frac{i}{4}\,\|f+ig\|^2 \;-\; \frac{i}{4}\,\|f-ig\|^2\,, \tag{8.5}$$

which immediately implies the lemma. $\square$

Remark. The equality (8.5) is very useful and is of interest in itself. It shows, in particular, that the inner product $(\cdot,\cdot)$ in the Hilbert space is uniquely determined by the norm $\|\cdot\|$.

It is not obvious that the L_2-Fourier transform of $f \in L_1(\mathbb{R}^1) \cap L_2(\mathbb{R}^1)$ coincides with the L_1-Fourier transform of f. We shall deduce this result from the following lemma.

Lemma 8.4. *Let $f \in L_2(\mathbb{R}^1)$. Define*

$$f_R(x) \;=\; \begin{cases} f(x), & |x| \leqslant R, \\ 0, & |x| > R, \end{cases}$$

and let

$$\widetilde{f}_R \;=\; (2\pi)^{-1/2} \int e^{-ix\xi}\, f_R(x)\,dx \;=\; (2\pi)^{-1/2} \int_{-R}^{R} e^{-ix\xi}\, f(x)\,dx$$

be the L_1-Fourier transform of f_R. Then $\|\hat f - \widetilde{f}_R\|_{L_2(\mathbb{R}^1)} \to 0$ as $R \to \infty$, where $\hat f$ is the L_2-Fourier transform of f.

Proof. Let $\hat f_R$ be the L_2-Fourier transforms of the functions f_R. From the definition of integrals over an unbounded interval it follows that $\|f - f_R\|_{L_2(\mathbb{R}^1)} \to 0$ as $R \to \infty$. In view of Parseval's formula $\|\hat f - \hat f_R\|_{L_2(\mathbb{R}^1)} \to 0$, so we only need to prove that $\hat f_R = \widetilde{f}_R$.

Let $h_k \in \mathcal{S}(\mathbb{R}^1)$ and $h_k \to f_R$ in $L_2(\mathbb{R}^1)$. Obviously, we can choose h_k in such a way that $\operatorname{supp} h_k \subset (-R-1, R+1)$ for all k. Let χ be the characteristic function of the interval $(-R-1, R+1)$. Then, by the Cauchy-Schwarz inequality,

$$\|f_R - h_k\|_1 = \int |f_R(x) - h_k(x)|\chi(x)\,dx \leqslant \|\chi\|_2 \|f_R - h_k\|_2 \to 0\,.$$

as $k \to \infty$. Now Theorem 6.3 implies that $\hat h_k \to \widetilde{f}_R$ uniformly. On the other hand, the sequence $\{\hat h_k\}$ is mean square convergent to $\hat f_R$. Since uniform convergence implies mean square convergence on each bounded set, we obtain that $\hat f_R = \widetilde{f}_R$ on each bounded set, so $\hat f_R = \widetilde{f}_R$ everywhere. $\square$

Corollary 8.5. *If $f \in L_1(\mathbb{R}^1) \cap L_2(\mathbb{R}^1)$ then the L_2-Fourier transform $\hat{f}$ coincides with the L_1-Fourier transform of f.*

Proof. Let $\widetilde{f}_R$ be the L_1-Fourier transform of f_R. Then

$$\begin{aligned} &| \, (2\pi)^{-1/2} \int e^{-ix\xi} \, f(x) \, dx - \widetilde{f}_R(\xi) \, | \\ &\qquad = \; (2\pi)^{-1/2} \, | \int_{|x| \geqslant R} e^{-ix\xi} \, f(x) \, dx \, | \\ &\qquad\qquad \leqslant \; (2\pi)^{-1/2} \int_{|x| \geqslant R} |f(x)| \, dx \, . \end{aligned}$$

Since $\int |f(x)| \, dx < \infty$, the integral on the right hand side vanishes as $R \to \infty$. This implies that $\widetilde{f}_R$ is uniformly convergent to the L_1-Fourier transform of f as $R \to \infty$. Now in the same way as in the proof of Lemma 8.4 we obtain that the L_1-Fourier transform of f coincides with $\hat{f}$.

For the inverse Fourier transform $\mathcal{F}^{-1}$ in $\mathcal{S}(\mathbb{R}^1)$ we have

$$\mathcal{F}^{-1}_{\xi \to x} f \; = \; \hat{f}(-x) \, .$$

Therefore all the results we have proved for the L_2-Fourier transform $\mathcal{F}$ remain valid for $\mathcal{F}^{-1}$. Namely,

(1) the inverse Fourier transform $\mathcal{F}^{-1}$ in $\mathcal{S}(\mathbb{R}^1)$ can be extended to $L_2(\mathbb{R}^1)$ so that

$$(\mathcal{F}^{-1} f, \mathcal{F}^{-1} g) \; = \; (f, g) \, , \qquad \forall \, f, g \in L_2(\mathbb{R}^1) \qquad \text{(Parceval's formula);}$$

(2) if $f \in L_2(\mathbb{R}^1)$ then $\mathcal{F}^{-1}_{\xi \to x} f$ coincides with the mean square limit

$$(2\pi)^{-1/2} \lim_{R \to \infty} \int_{-R}^{R} e^{ix\xi} \, f(\xi) \, d\xi \, ;$$

(3) if $f \in L_1(\mathbb{R}^1) \cap L_2(\mathbb{R}^1)$ then the inverse L_2-Fourier transform $\mathcal{F}^{-1}_{\xi \to x} f$ coincides with

$$(2\pi)^{-1/2} \int e^{ix\xi} \, f(\xi) \, d\xi \, .$$

Moreover, we have

$$\mathcal{F}^{-1} \hat{f} \; = \; f$$

for all $f \in L_2(\mathbb{R}^1)$. Indeed, by Parceval's formula

$$\|\mathcal{F}^{-1} f\|_{L_2(\mathbb{R}^1)} \; = \; \|f\|_{L_2(\mathbb{R}^1)} \, , \qquad \forall \, f \in L_2(\mathbb{R}^1).$$

Given $f \in L_2(\mathbb{R}^1)$, we choose $f_k \in \mathcal{S}(\mathbb{R}^1)$ such that $\|f - f_k\|_{L_2(\mathbb{R}^1)} \to 0$. Then

$$\begin{aligned} \|\mathcal{F}^{-1} \hat{f} - f\|_{L_2(\mathbb{R}^1)} \; &= \; \lim_{k \to \infty} \|\mathcal{F}^{-1} \hat{f} - f_k\|_{L_2(\mathbb{R}^1)} \\ &= \; \lim_{k \to \infty} \|\mathcal{F}^{-1} \hat{f} - \mathcal{F}^{-1} \hat{f}_k\|_{L_2(\mathbb{R}^1)} \; = \; \lim_{k \to \infty} \|\hat{f} - \hat{f}_k\|_{L_2(\mathbb{R}^1)} \; = \; 0 \, . \end{aligned}$$

9. Sobolev spaces H^s

Definition 9.1. We say that $f \in H^s(\mathbb{R}^1)$ with $s \geqslant 0$ if $f \in L_2(\mathbb{R}^1)$ and $(1+\xi^2)^{s/2}\hat{f}(\xi) \in L_2(\mathbb{R}^1)$, where $\hat{f}$ is the Fourier transform of f.

We define the inner product $(\cdot,\cdot)_s$ and the norm $\|\cdot\|_s$ in $H^s(\mathbb{R}^1)$ by

$$(f,g)_s \;=\; \int (1+\xi^2)^s\,\hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,d\xi \tag{9.1}$$

and $\|f\|_s = \sqrt{(f,f)_s}$ respectively. The inner product (9.1) satisfies all the necessary conditions (it is proved in the same way as for the inner product in $L_2(\mathbb{R}^1)$). In view of Parceval's formula, $H^0(\mathbb{R}^1) = L_2(\mathbb{R}^1)$.

We have

$$(f,g)_s \;=\; (\hat{f}_s, \hat{g}_s)$$

where

$$\hat{f}_s(\xi) = (1+\xi^2)^{s/2}\hat{f}(\xi)\,, \qquad \hat{g}_s(\xi) = (1+\xi^2)^{s/2}\hat{g}(\xi)\,,$$

and $(\cdot,\cdot)$ denotes the inner product in $L_2(\mathbb{R}^1)$. From here it follows that the space $H^s(\mathbb{R}^1)$ is complete. Indeed, if $\{f_k\}$ is a Cauchy sequence in $H^s(\mathbb{R}^1)$ then the functions

$$(\widehat{f_k})_s(\xi) \;=\; (1+\xi^2)^{s/2}\hat{f}_k(\xi) \tag{9.2}$$

form a Cauchy sequence in $L_2(\mathbb{R}^1)$. Since $L_2(\mathbb{R}^1)$ is complete, the sequence (9.2) has a limit $\hat{f}_s \in L_2(\mathbb{R}^1)$. Then $\{f_k\}$ converges in $H^s(\mathbb{R}^1)$ to the function $f \in H^s(\mathbb{R}^1)$ defined by

$$\hat{f}(\xi) = (1+\xi^2)^{-s/2}\hat{f}_s(\xi)\,.$$

Obviously, $\mathcal{S}(\mathbb{R}^1) \subset H^s(\mathbb{R}^1) \subset H^{s-\varepsilon}(\mathbb{R}^1)$ for all $s,\varepsilon \geqslant 0$. Moreover, $\mathcal{S}(\mathbb{R}^1)$ is dense in $H^s(\mathbb{R}^1)$ for all $s \geqslant 0$. Indeed, since $\mathcal{S}(\mathbb{R}^1)$ is dense in $L_2(\mathbb{R}^1)$, for any $f \in H^s(\mathbb{R}^1)$ there exists a sequence of functions $(\widehat{f_k})_s \in \mathcal{S}(\mathbb{R}^1)$ such that

$$\|\hat{f}_s - (\widehat{f_k})_s\|_{L_2(\mathbb{R}^1)} \;\to\; 0.$$

Then $\|f - f_k\|_s \to 0$, where f_k are the $\mathcal{S}$-functions defined by

$$\hat{f}_k(\xi) = (1+\xi^2)^{-s/2}(\widehat{f_k})_s\,.$$

Derivatives of the functions from H^s.

We shall need the following lemma.

Lemma 9.2. *Let $f \in L_2(\mathbb{R}^1)$. Assume that f is piecewise continuously differentiable and that $f' \in L_2(\mathbb{R}^1)$. Then $\widehat{f'}(\xi) = i\xi\,\hat{f}(\xi)$.*

Remark 9.3. The corresponding result for $L_1(\mathbb{R}^1)$ is obvious. Indeed, if $f \in L_1(\mathbb{R}^1)$ is differentiable and $f' \in L_1(\mathbb{R}^1)$, then

$$f(-R) \;=\; \int_{-\infty}^{-R} f'(x)\,dx \;\to\; 0\,, \quad f(R) \;=\; \int_{R}^{\infty} f'(x)\,dx \;\to\; 0$$

as $R \to \infty$. Therefore

$$\widehat{f'}(\xi) \;=\; (2\pi)^{-1/2} \int e^{-ix\xi}\, f'(x)\, dx \;=\; (2\pi)^{-1/2} \lim_{R\to\infty} \int_{-R}^{R} e^{-ix\xi}\, f'(x)\, dx$$

$$= \;(2\pi)^{-1/2} \lim_{R\to\infty} e^{-ix\xi}\, f(x)\big|_{-R}^{R} - (2\pi)^{-1/2} \lim_{R\to\infty} \int f(x)\, \frac{d\left(e^{-ix\xi}\right)}{dx}\, dx \;=\; i\xi\, \hat f(\xi)\,.$$

Proof of Lemma 9.2. If $\varphi \in \mathcal{S}(\mathbb{R}^1)$ then the derivative $(f\varphi)'$ lies in L_1 and

$$(f\varphi)(-R) \;=\; \int_{-\infty}^{-R} (f\varphi)'(x)\, dx \;\to\; 0\,, \quad (f\varphi)(R) \;=\; \int_{R}^{\infty} (f\varphi)'(x)\, dx \;\to\; 0$$

as $R \to \infty$. Therefore $\int f'(x)\, \overline{\varphi(x)}\, dx = -\int f(x)\, \overline{\varphi'(x)}\, dx$, and from Parceval's formula and Lemma 7.3 it follows that

$$(\widehat{f'}, \hat\varphi) \;=\; (f', \varphi) \;=\; -(f, \varphi') \;=\; -(\hat f, \widehat{\varphi'}) \;=\; -(\hat f, i\xi\, \hat\varphi) \;=\; (i\xi\, \hat f, \hat\varphi)$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^1)$. Since $\mathcal{S}(\mathbb{R}^1)$ is dense in $L_2(\mathbb{R}^1)$, the equality

$$(\widehat{f'}, \hat\varphi) \;=\; (i\xi\, \hat f, \hat\varphi) \tag{9.3}$$

holds for all $\varphi \in L_2(\mathbb{R}^1)$. In particular, if $\hat\varphi = \widehat{f'}(\xi) - i\xi\, \hat f(\xi)$ then (9.3) takes the form

$$(\, \widehat{f'}(\xi) - i\xi\, \hat f(\xi)\,,\, \widehat{f'}(\xi) - i\xi\, \hat f(\xi)\,) \;=\; \|\widehat{f'}(\xi) - i\xi\, \hat f(\xi)\|^2_{L_2(\mathbb{R}^1)} \;=\; 0\,,$$

so $\widehat{f'}(\xi) = i\xi\, \hat f(\xi)$. □

Proposition 9.4. *Let $f \in L_2(\mathbb{R}^1)$ and m be a positive integer. Assume that for all $k \leqslant m$ the derivatives $d^k f/dx^k$ exist, are piecewise continuous and lie in $L_2(\mathbb{R}^1)$. Then $f \in H^m(\mathbb{R}^1)$.*

Proof. By Lemma 9.2 $(i\xi)^k \hat f(\xi) = \mathcal{F}_{x\to\xi}(d^k f/dx^k) \in L_2(\mathbb{R}^1)$. Therefore

$$(1 + i\xi)^m \hat f(\xi) \in L_2(\mathbb{R}^1)$$

and

$$\|(1+\xi^2)^{m/2} \hat f\|^2_{L_2(\mathbb{R}^1)} \;=\; \int (1+\xi^2)^m |\hat f(\xi)|^2\, d\xi \;=\; \|(1+i\xi)^m \hat f\|^2_{L_2(\mathbb{R}^1)} \;<\; \infty\,.$$

□

Proposition 9.5. *Let $f \in H^{m+1/2+\varepsilon}$, where $\varepsilon > 0$ and m is a non-negative integer. Then for all $k \leqslant m$ the derivatives $d^k f/dx^k$ exist and are continuous.*

Proof. By the Schwarz inequality

$$\int |\,\xi^k\, \hat f(\xi)\,|\, d\xi \;=\; \int \left(|\xi|^k\, (1+\xi^2)^{-(m+1/2+\varepsilon)/2}\right) \left((1+\xi^2)^{(m+1/2+\varepsilon)/2}\, |\hat f(\xi)|\right) d\xi$$

$$\leqslant\; \left(\int \xi^{2k}\, (1+\xi^2)^{-(m+1/2+\varepsilon)}\, d\xi\right)^{1/2} \left(\int (1+\xi^2)^{m+1/2+\varepsilon}\, |\hat f(\xi)|^2\, d\xi\right)^{1/2}.$$

The second integral in the right hand side coincides with the $H^{m+1/2+\varepsilon}$-norm of f and is finite. If $k \leqslant m$ then the first integral is also finite, so $\xi^k \hat{f}(\xi) \in L_1(\mathbb{R}^1)$.

By the inverse Fourier formula

$$f(x) \;=\; (2\pi)^{-1/2} \int e^{ix\xi}\, \hat{f}(\xi)\, d\xi\,.$$

As $\xi^k \hat{f} \in L_1(\mathbb{R}^1)$, we can take k derivatives with respect to x under the integral sign. This implies that the derivatives

$$d^k f/dx^k \;=\; (2\pi)^{-1/2} \int e^{ix\xi}\, (i\xi)^k\, \hat{f}(\xi)\, d\xi \tag{9.4}$$

exist for all $k = 0, 1, \ldots, m$. By Theorem 6.2 these derivatives are continuous. □

Generalized derivatives.

Propositions 9.4 and 9.5 show that the functions $f \in H^s(\mathbb{R}^1)$ are becoming smoother and smoother as $s \to \infty$. When $s = 0$, we can say only that $f \in L_2(\mathbb{R}^1) = H^0(\mathbb{R}^1)$. If $s > 1/2$ then the functions $f \in H^s(\mathbb{R}^1)$ are continuous, if $s > 3/2$ then $f \in H^s(\mathbb{R}^1)$ are continuously differentiable, etc.

Proposition 9.4 suggests that all the functions $f \in H^1(\mathbb{R}^1)$ are differentiable with the derivatives $f' \in L_2(\mathbb{R}^1)$. However, generally speaking that is not true.

Example. Let $\varphi \in \mathcal{S}(\mathbb{R}^1)$, $f(x) = \varphi(x)$ as $x \neq 0$ and $f(0) = \varphi(0)+1$. Then $\hat{f} = \hat{\varphi}$, so $f \in H^s(\mathbb{R}^1)$ for all s. However, f is not differentiable at $x = 0$.

In the example above the function f is differentiable everywhere outside $\{x = 0\}$. The value of the function at one fixed point does not affect its Fourier transform, so $\hat{f}$ coincides with the Fourier transform of the smooth function. More generally, if f is differentiable "almost everywhere"(i.e., outside a very small set) then $\hat{f}$ behaves like if f is differentiable. From this observation it is clear that the classical definition of the derivative is too restrictive for our purposes.

Definition 9.6. If $f \in H^m$ with non-negative integer m then the functions $f^{(k)} \in L_2(\mathbb{R}^1)$ defined by

$$\mathcal{F}_{x\to\xi} f^{(k)} \;=\; (i\xi)^k\, \hat{f}(\xi)\,, \qquad k = 1, 2, \ldots, m,$$

are said to be the generalized derivatives of f.

In view of Proposition 9.5, if $f \in H^m$ then the derivatives $d^k f/dx^k$ of order $k \leqslant m-1$ are defined in the classical sense, and by (9.4) these classical derivatives coincide with the corresponding generalized derivatives. The generalized derivative of order m is another matter, it is well defined even if the classical derivative does not exist.

Example 9.7. Let f be a continuous function. Assume that there exist real numbers $a_1 < a_2 < \cdots < a_m$ such that f is continuously differentiable on the intervals

$$(a_1, a_2)\,,\; (a_2, a_3)\,,\; \ldots\; (a_{m-1}, a_m)$$

and $f(x) = 0$ as $x \leqslant a_1$ or $x \geqslant a_m$. Then the first generalized derivative of f coincides with

$$g(x) \;=\; \begin{cases} f'(x), & x \in (a_k, a_{k+1})\,, \quad k = 1, \dots, m-1, \\ 0, & x \leqslant a_1 \text{ or } x \geqslant a_m. \end{cases}$$

Indeed, integrating by parts we obtain

$$\begin{aligned} \hat{g}(\xi) \;&=\; \sum_{k=1}^{m-1} \int_{a_k}^{a_{k+1}} e^{-ix\xi}\, f'(x)\, dx \\ &=\; \sum_{k=1}^{m-1} \big(e^{-ix\xi}\, f(x)\big)\Big|_{a_k}^{a_{k+1}} \;+\; \sum_{k=1}^{m-1} \int_{a_k}^{a_{k+1}} i\xi\, e^{-ix\xi}\, f(x)\, dx \\ &=\; \sum_{k=1}^{m-1} \big(e^{-ix\xi}\, f(x)\big)\Big|_{a_k}^{a_{k+1}} \;+\; i\xi\, \hat{f}(\xi)\,. \end{aligned}$$

Since f is continuous and $f(a_1) = f(a_m) = 0$, we have

$$\big(e^{-ix\xi}\, f(x)\big)\Big|_{a_k}^{a_{k+1}} \;=\; 0$$

and, consequently, $\hat{g}(\xi) = i\xi\, \hat{f}(\xi)$.

Obviously, $(1+\xi^2)^{m/2} \hat{f} \in L_2(\mathbb{R}^1)$ with non-negative integer m if and only if $\xi^k \hat{f} \in L_2(\mathbb{R}^1)$, $k = 0, 1, \dots, m$. Therefore, using the notion of generalized derivatives, we can give another definition of the Sobolev space $H^m(\mathbb{R}^1)$.

Definition 9.8. If m is a non-negative integer then $f \in H^m(\mathbb{R}^1)$ means that $f \in L_2(\mathbb{R}^1)$ and for all $k \leqslant m$ there exist the generalized derivatives $f^{(k)} \in L_2(\mathbb{R}^1)$.

Differential equations with constant coefficients.

Let us consider the differential equation

$$\sum_{k=0}^{m} \widetilde{c}_k\, \frac{d^k f}{dx^k} \;=\; g$$

on $\mathbb{R}^1$, where $\widetilde{c}_k$ are some constants and $g \in L_2(\mathbb{R}^1)$ is a given function. We rewrite this equation in the form

$$\sum_{k=0}^{m} c_k\, D^k f \;=\; g\,, \tag{9.5}$$

where $c_k = i^k\, \widetilde{c}_k$ and

$$D^k f \;=\; (-i)^k\, \frac{d^k f}{dx^k}\,.$$

Of course, the left hand side of (9.5) is well defined only if the derivatives of f exist. We shall consider $D^k f$ in (9.5) as the generalized derivatives, and then we have to assume that $f \in H^m(\mathbb{R}^1)$. The solutions from $H^m(\mathbb{R}^1)$ are called the generalized solutions of the differential equation (9.5). If a generalized solution f

appears to be sufficiently smooth, then the generalized derivatives coincide with the usual ones and f is also a classical solution. However, the equation (9.5) may not have any classical solutions but only generalized ones.

From the definition of generalized derivatives it follows that $\widehat{D^k f}(\xi) = \xi^k \hat{f}(\xi)$. Applying the Fourier transform, we obtain from (9.5)

$$P(\xi)\,\hat{f}(\xi) \;=\; \hat{g}(\xi)\,, \tag{9.6}$$

where

$$P(\xi) \;=\; \sum_{k=0}^{m} c_k\,\xi^k\,.$$

Therefore $\hat{f}(\xi)$ must coincide with $\hat{g}(\xi)/P(\xi)$. If $P(\xi)$ is nowhere equal to zero, then $(1+\xi^2)^{m/2}/P(\xi)$ is uniformly bounded. This implies $\mathcal{F}^{-1}\,(\hat{g}/P) \in H^m(\mathbb{R}^1)$, so

$$f \;=\; \mathcal{F}^{-1}\,(\hat{g}/P) \tag{9.7}$$

is the required generalized solution.

If $P(\xi) = 0$ for some $\xi \in \mathbb{R}^1$ then the equation (9.5) may not have any H^m-solutions. Indeed, if there exist a solution $f \in H^m(\mathbb{R}^1)$ then, in view of (9.6), g must be equal to zero at the points where P is equal to zero.

Definition 9.9. The polynomial P is called the symbol of the differential operator $\sum c_k\, D^k$.

We have seen that the solutions of differential equations can be constructed in terms of the symbols of corresponding differential operators. It is not always easy to derive from (9.7) an explicit formula for f. However, we can often obtain some information about f even not trying to calculate the inverse Fourier transform.

Example. Let $D^{2p}f + f = g$ where $g \in L_2(\mathbb{R}^1)$. Then by the preceding

$$\xi^k\,\hat{f}(\xi) \;=\; \xi^k\,(1+\xi^{2p})^{-1}\,\hat{g}(\xi)\,, \qquad k = 0,1,\dots,2p\,. \tag{9.8}$$

Since $\hat{g} \in L_2(\mathbb{R}^1)$ and $|\,\xi^k\,(1+\xi^{2p})^{-1}\,| \leqslant 1$ as $k \leqslant 2p$, the equality (9.8) implies that for all $k \leqslant 2p$ the generalized derivatives $f^{(k)}$ exist and

$$\|f^{(k)}\|_{L_2(\mathbb{R}^1)} \;\leqslant\; \|g\|_{L_2(\mathbb{R}^1)}\,.$$

Appendix

Definition A.1. A set K is said to be closed if it contains all its limit points, i.e., if $x_1, x_2, ... \in K$ and $x_k \to x$ imply $x \in K$.

Definition A.2. A set K is open if for any point $x_0 \in K$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $x \in K$.

Lemma A.3. *The complement of an open set in $\mathbb{R}^n$ is closed, and the complement of a closed set is open.*

Definition A.4. A set K is said to be compact if any sequence of points $x_1, x_2, \ldots$ in K contains a subsequence $x_{i_1}, x_{i_2}, \ldots$ which converges to some point $x \in K$.

Lemma A.5. *A subset K of the Euclidian space $\mathbb{R}^n$ is compact if and only if K is bounded and closed.*

Uniform continuity.

Definition A.6. A function f is called uniformly continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_1 - x_2| \leqslant \delta$ implies $|f(x_1) - f(x_2)| \leqslant \varepsilon$.

Obviously, a uniformly continuous function is continuous.

Lemma A.7. *Any continuous function defined on a compact set is uniformly continuous.*

Convergence of series.

Definition A.8. Let a_k be complex numbers. We say that the series $\sum_{k=-\infty}^{\infty} a_k$ converges to a and write $\sum_{k=-\infty}^{\infty} a_k = a$ if $\sum_{k=-N}^{N} a_k \to a$ as $N \to \infty$. We write $\sum_{k=-\infty}^{\infty} a_k < \infty$ if $\sum_{k=-\infty}^{\infty} a_k = a$ for some (finite) a.

Lemma A.9. *If $\sum_{k=-\infty}^{\infty} a_k < \infty$ then $a_k \to 0$ as $k \to \pm\infty$.*

Lemma A.10. *If $|a_k| \leqslant b_k$ and $\sum_{k=-\infty}^{\infty} b_k < \infty$ then $\sum_{k=-\infty}^{\infty} a_k < \infty$.*

Lemma A.11. *If $\frac{|a_{k+1}|}{|a_k|} \leqslant q < 1$ for sufficiently large k then $\sum_{k=-\infty}^{\infty} a_k < \infty$.*

Example. Let $a_k = |k|^{-\alpha}$ with some real α as $k \neq 0$. Then $\sum_{k=-\infty}^{\infty} a_k < \infty$ if and only if $\alpha > 1$.

Uniform convergence.

Definition A.12. We say that a sequence of functions g_N converges uniformly to a function f as $N \to \infty$ if for any $\varepsilon > 0$ there exists N_ε such that $N > N_\varepsilon$ implies $|f(x) - g_N(x)| \leqslant \varepsilon$ for all x.

Definition A.13. We say that a series $\sum_{k=-\infty}^{\infty} f_k$ converges uniformly to f if the sequence $g_N = \sum_{k=-N}^{N} f_k$ converges uniformly to f.

Lemma A.14 (Weierstrass test). *If a series $\sum_{k=-\infty}^{\infty} f_k$ converges to f pointwise and $\sum_{k=-\infty}^{\infty} \sup_x |f_k(x)| < \infty$ then the series converges to f uniformly.*

Theorem A.15. *If a series* $\sum_{k=-\infty}^{\infty} f_k$ *converges to* f *uniformly and the functions* f_k *are continuous then* f *is continuous.*

Theorem A.16. *A uniformly convergent series can be integrated term by term.*

Polar coordinates.

The polar coordinates (r, φ) in $\mathbb{R}^2$ are defined by

$$x_1 = r \sin \varphi\,,, \qquad x_2 = r \cos \varphi\,.$$

The Jacobian of the change of variables $(x_1, x_2) \to (r, \varphi)$ is equal to r, so

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1\, dx_2 = \int_0^{\infty} \int_0^{2\pi} f(r, \varphi)\, r\, d\varphi\, dr\,.$$