SET THEORY

We use some notions from set theory, which can be found in the online lecture notes http://www.mth.kcl.ac.uk/∼ysafarov/Lectures/CM221/jhanotes.pdf on pages 1–4. In particular,

$\mathbb{N}$ is the set of positive integer numbers, $\mathbb{N} = \{1, 2, \ldots\}$;

$\mathbb{Z}$ is the set of integer numbers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$;

$\mathbb{Q}$ is the set of rational numbers, $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}\}$;

$\mathbb{R}$ is the set of real numbers;

$\mathbb{C}$ is the set of complex numbers.

The symbol ∞ is a shorthand for "infinity". It is not a proper number.

COMPLEX NUMBERS

We define $\mathbb{C}$ as the set of ordered pairs $z = (x, y)$ with $x, y \in \mathbb{R}$ and the algebraic operations

$$\begin{aligned} z_1 + z_2 = (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ z_1 \times z_2 = (x_1, y_1) \times (x_2, y_2) &= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)\,. \end{aligned}$$

The numbers x and y are said to be the real and imaginary parts of z and are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$.

The set $\mathbb{C}$ can be identified with $\mathbb{R}^2$. We have $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ and $(x_1, 0) \times (x_2, 0) = (x_1\, x_2, 0)$. Therefore the above defined addition and multiplication in $\mathbb{C}$ become the usual addition and multiplication in the space or real numbers, if we restrict them to the set of complex numbers with zero imaginary parts. Thus, we can consider $\mathbb{R}$ as a subset (and a subalgebra) of $\mathbb{C}$. Further of we write x instead of $(x, 0)$.

Denote $i = (0, 1)$. From the definition of multiplication it follows that $i^2 = -1$. Thus, i can be thought of as a square root of one.

A complex number $z = (x, y)$ can be written as $x + i\,y$, where x and y are the usual (Euclidean) coordinates of the corresponding point on the plane $\mathbb{R}^2$. In the polar coordinates, every nonzero $z \in \mathbb{C}$ is defined by a pair (r, θ) where $r = \sqrt{x^2 + y^2}$ and θ is defined by $\cos\theta = r^{-1}x$. The numbers r and θ are called, respectively, the *modules* and the *argument* of the complex number z, and are denoted by $|z|$ and $\arg z$.

The argument is not uniquely defined: one can take $\theta + 2\pi k$ with any $k \in \mathbb{Z}$ instead of θ. We can choose $k \in \mathbb{Z}$ in such a way that $-\pi < \theta + 2\pi k \leqslant \pi$. This number $\theta + 2\pi k$ is said to be the *principal value* of the argument and is denoted by $\operatorname{Arg} z$.

Define $e^{i\theta} = \cos\theta + i\sin\theta$. Then every nonzero complex number z can be written in the form $|z|\, e^{i\arg z}$. We have $z_1\, z_2 = |z_1|\,|z_2|\, e^{i(\arg z_1 + \arg z_2)}$ which implies, in particular, that $|z_1\, z_2| = |z_1|\,|z_2|$.

By definition, $|z_1 - z_2|$ is the Euclidean distance between z_1 and z_2. In particular, $|z|$ is the distance from z to the origin. The standard triangle inequality (proved in Euclidean geometry) implies that $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

The complex number $\bar z = x - iy$ is called the (complex) conjugate to z. We have $\bar z = |z|\, e^{-i\arg z}$ and, consequently, $z\,\bar z = |z|^2$. A direct calculation shows that $\overline{z_1\, z_2} = \bar z_1\, \bar z_2$, $\overline{z^{-1}} = (\bar z)^{-1}$ and $\overline{(z_1 + z_2)} = \bar z_1 + \bar z_2$, $z + \bar z = 2\mathrm{Re}\, z$ and $z - \bar z = 2i\,\mathrm{Im}\, z$.

TOPOLOGICAL PROPERTIES OF THE COMPLEX PLANE

All the topological definitions and results mentioned in this section remain valid for a general space Ω equipped with a nonnegative symmetric "distance function" $\rho(u,v)$, such that $\rho(u,v) = 0$ if and only if $u = v$ and $\rho(u,v) \leqslant \rho(u,w) + \rho(w,v)$, where $u, v, w \in \Omega$, The only difference is that, in all definitions, one has to replace $|u - v|$ with $\rho(u,v)$. Proofs and detailed discussions can be found in
http://www.mth.kcl.ac.uk/∼ysafarov/Lectures/Past/321.pdf

Definition. Let z_0 be a fixed point of $\mathbb{C}$ and $r > 0$. The set of points $z \in \mathbb{C}$ such that $|z - z_0| < r$ is said to be the *open disc* of radius r centred at z_0.

We denote this open disc by $\mathcal{D}(z_0, r)$, that is, $\mathcal{D}(z_0, r) = \{z \in \mathbb{C} \,:\, |z - z_0| < r\}$. A closed disc is the set $\mathcal{D}(z_0, r) = \{z \in \mathbb{C} \,:\, |z - z_0| \leqslant r\}$.

Definition. A set $\Omega \subseteq \mathbb{C}$ is said to be *bounded* if it is a subset of a disc $\mathcal{D}(z_0, r)$.

Definition. A set $\Omega \subseteq \mathbb{C}$ is said to be *open* if for each $z_0 \in A$ there exists $r > 0$ such that $\mathcal{D}(z_0, r) \subseteq A$. A set $\Omega \subseteq \mathbb{C}$ is said to be *closed* if its complement in $\mathbb{C}$ is open.

One can easily see that an open disc is open and a closed disc is closed.

Theorem. A set $\Omega \subseteq \mathbb{C}$ is closed if and only if the limit of every convergent sequence of points $z_n \in \Omega$ also belongs to Ω.

Recall that a sequence $\{z_1, z_2, z_3, \ldots\}$ converges to a limit z if $|z - z_n| \underset{n\to\infty}{\to} 0$.

Definition. A set $A \subseteq \mathbb{C}$ is said to be *disconnected* if there exists a pair of disjoint open sets $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ such that $A \subseteq \Omega_1 \bigcup \Omega_2$ and each of these set contains at least one element of A. A set is *connected* if it is not disconnected.

Definition. Let $z_0, z_1 \in \mathbb{C}$. A path from z_0 to z_1 is a continuous map $\gamma : [0,1] \mapsto \mathbb{C}$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$.

Here continuity means that $|\gamma(t_1) - \gamma(t_2)| \to 0$ as $|t_1 - t_2| \to 0$. The continuous path γ can be thought of as a continuous line joining z_0 and z_1.

Definition. A set $A \subseteq \mathbb{C}$ is said to be path-connected if every two points $z_0, z_1 \in A$ can be joined by a continuous path with values in A.

Theorem. Every path-connected set is connected. An open connected set is path-connected. A general (not open) connected set may not be path-connected.

THE RIEMANN SPHERE

We wish to add the point ∞ to the complex plane $\mathbb{C}$. It can be done formally, by saying that the extended plane $\hat{\mathbb{C}} = \mathbb{C} \bigcup \infty$. We then provide the set $\hat{\mathbb{C}}$ with a topological structure as follows.

Let $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$ centred at the origin. Let us identify the complex plane with the horizontal plane $\{x_3 = 0\}$ in $\mathbb{R}^3$, and assume that $\{x_2 = x_3 = 0\}$ is the real line. Then, for any point $z \in \mathbb{C}$, the straight line going from the "north pole" $(0, 0, 1)$ through this point cuts the sphere $\mathbb{S}^2$ in a unique point. Denote the coordinates of this point by $x_k(z)$. Clearly, the point $(x_1(z), x_2(z), x_3(z))$ uniquely determines z. If $z = x + iy$ then

$$x_1(z) = \frac{2x}{|z|^2 + 1}, \quad x_2(z) = \frac{2y}{|z|^2 + 1}, \quad x_3(z) = \frac{|z|^2 - 1}{|z|^2 + 1}$$

and

$$x = \frac{x_1(z)}{1 - x_3(z)}, \quad y = \frac{x_2(z)}{1 - x_3(z)}.$$

If $|z| \to \infty$ then $x_1(z) \to 0$, $x_2(z) \to 0$ and $x_3(z) \to 1$. Therefore we can assume that the north pole corresponds to the point ∞ on $\hat{\mathbb{C}}$. The map

$$(x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right)$$

from $\mathbb{S}^2$ onto $\hat{\mathbb{C}}$ is called the *stereographic projection*.

The sphere $\mathbb{S}^2$ provided with the usual Euclidean distance is a metric space, so we can speak about open subsets of $\mathbb{S}^2$ and convergent sequences in $\mathbb{S}^2$.

Definition. We shall say that a subset of $\hat{\mathbb{C}}$ is open if its inverse image under the stereographic projection is open. We shall say that a sequence of points $z_n \in \hat{\mathbb{C}}$ converges to $z \in \hat{\mathbb{C}}$ if the inverse images of z_n converge to the inverse image of z in $\mathbb{S}^2$.

Another possible way to define the topological structure on $\hat{\mathbb{C}}$ is to consider sets $\{z \in \mathbb{C} : |z| > R\}$ as open discs centred at ∞ and to use the formal definitions of open and closed sets given in the previous section. This approach is equivalent to the one described above.

MÖBIUS TRANSFORMATIONS

Definition. Let $a, b, c, d \in \mathbb{C}$ and $ad \neq bc$. The map

$$z \mapsto T(z) = \frac{az + b}{cz + d}$$

is called the Möbius transformation.

Remark. If $a \neq 0$, $c \neq 0$ and $ad = bc$ then the image of T is a one point set because

$$T(z) = \frac{az + b}{cz + d} = \frac{ac(az + b)}{ac(cz + d)} = \frac{a(acz + bc)}{c(acz + ad)} = \frac{a}{c} \quad \text{for all} \quad z \in \mathbb{C}.$$

Exercise. What happens if $ad = bc$, and $a = 0$ or $c = 0$?

Exercise. Show that the composition of two Möbius transformations is again a Möbius transformation.

Lemma. The Möbius transformation $T(z) = \frac{az+b}{cz+d}$ is invertible. It's inverse is given by $T^{-1}(z) = \frac{-dz+b}{cz-a}$.

Proof is by direct calculation.

Theorem. Every Möbius transformation $T(z) = \frac{az+b}{cz+d}$ can be represented as the composition of the following four basic transformations:

(1) translation $z \mapsto z + z_0$ with $a \in \mathbb{C}$,

(2) dilatation $z \mapsto rz$ with $r \in \mathbb{R}$ and $r > 0$,

(3) rotation $z \mapsto e^{i\theta} z$ with $\theta \in \mathbb{R}$,

(4) inversion $z \mapsto z^{-1}$.

Proof. First of all, let us note that the multiplication by a complex number z_0 is the composition of a dilation and a rotation because $z_0 = r\, e^{i\theta}$ with $r, \theta \in \mathbb{R}$ and $r > 0$.

Let $c = 0$. Then the condition $ad \neq bc$ implies that $a \neq 0$ and $d \neq 0$. Thus we have $T(z) = \frac{a}{d}\, z + \frac{b}{d}$. Since $d \neq 0$, the addition of the factor $\frac{b}{d}$ is a translation and the multiplication by $\frac{a}{d}$ is the composition of a dilation and a rotation.

Let $c \neq 0$. Denote $k = (bc - ad)/c$. Then $az + b = c^{-1}\,[a(cz + d) + ck]$ and

$$T(z) \;=\; \frac{a}{c} \;+\; \frac{k}{cz+d}\,.$$

This is the composition of

(1) the combination of dilation and rotation $z \mapsto c\, z$,

(2) the translation $cz \mapsto cz + d$,

(3) the inversion $cz + d \mapsto (cz + d)^{-1}$,

(4) the combination of of dilation and rotation $(cz + d)^{-1} \mapsto k\,(cz + d)^{-1}$,

(5) $k\,(cz + d)^{-1} \mapsto \frac{a}{c} + k\,(cz + d)^{-1}$.

Corollary. The Möbius transformation preserves the set of circles and straight lines.

Proof. Clearly, translations, dilations and rotations map lines into lines and circles into circles. Therefore it is sufficient to prove the corollary for the inversion. It can be done using Cartesian form of equations defining the lines and circles (see in books or lecture notes).

COMPLEX SEQUENCES

A sequence is a collection of complex numbers $w_0, w_1, w_2, \ldots$ "labelled" by nonnegative integers. We shall denote such a sequence by $\{w_n\}$, where $n = 1, 2, \ldots$ and w_n are the elements of the sequence. Traditionally, one assumes that a sequence has infinitely many members (a finite collection is sometimes called a "finite sequence"). There is no requirement for the members of a sequence to be distinct numbers, some of them may coincide. If all w_n coincide with the same complex number, the sequence is called constant.

Definition. We say that the sequence of complex numbers w_n converges to a limit $w \in \mathbb{C}$ if $|w - w_n| \to 0$ as $n \to \infty$. We say that $\{w_n\}$ is a Cauchy sequence if $|w_m - w_n| \to 0$ as $m, n \to \infty$.

One can easily show that, for any pair of convergent sequences $\{w_n\}$, $\{v_n\}$ and all $a, b \in \mathbb{C}$, we have $\lim_{n\to\infty}(aw_n + bv_n) = a \lim_{n\to\infty} w_n + b \lim_{n\to\infty} v_n$.

Theorem. A sequence of complex numbers converges if and only if it is a Cauchy sequence.

This is a consequence of the fact that the space of complex numbers is complete which follows from the completeness axiom for real numbers, see
http://www.mth.kcl.ac.uk/∼ysafarov/Lectures/Past/321.pdf

COMPLEX SERIES

Let $w_0, w_1, w_2, \ldots$ be a sequence of complex numbers. We say that the (formal) infinite series $\sum_{n=0}^{\infty} w_n$ converges if the sequence of its partial sums $S_m = \sum_{n=0}^{m} w_n$ converges to a limit $w \in C$ as $m \to \infty$. Then we write $w = \sum_{n=0}^{\infty} w_n$ and call w the sum of the series.

Remark. If the series $\sum_{n=0}^{\infty} w_n$ converges then $|w_n| \to 0$ as $n \to \infty$. Indeed, by the above S_m is a Cauchy sequence. Therefore $|w_m| = |S_m - S_{m-1}| \to 0$ as $m \to \infty$.

There are divergent (not convergent) series satisfying the condition $\lim_{n\to\infty} |w_n| = 0$. Even if all partial sums S_N lie in a bounded set, the series may well diverge.

Remark. Let $\sum_{n=0}^{\infty} w_n$ and $\sum_{n=0}^{\infty} v_n$ be convergent series. Then, for all $a, b \in \mathbb{C}$, the series $\sum_{n=0}^{\infty}(aw_n + bv_n)$ converges and

$$\sum_{n=0}^{\infty}(aw_n + bv_n) \;=\; a\sum_{n=0}^{\infty} w_n + b\sum_{n=0}^{\infty} v_n$$

(this follows from the definition and the corresponding result for sequences).

Definition. If $\sum_{n=0}^{\infty} |w_n| < \infty$, the series $\sum_{n=0}^{\infty} w_n$ is said to be absolutely convergent.

The following result is less obvious.

Theorem. Every absolutely convergent series converges.

Proof. Assume that the sum $\sum_{n=0}^{\infty} |w_n|$ is finite. Let $m > k$. Applying the triangle inequality, we obtain

$$|S_m - S_k| = |\sum_{k+1}^{m} w_n| \leqslant \sum_{k+1}^{m} |w_n| \leqslant \sum_{k+1}^{\infty} |w_n| .$$

Since $\sum_{n=0}^{\infty} |w_n| < \infty$, the sum in the right hand side goes to 0 as $k \to \infty$. This implies that the partial sums S_m form a Cauchy sequence.

Recall that the upper limit of is a sequence of real numbers t_n is $\limsup t_n = \lim_{k\to\infty} (\sup_{n>k} t_n)$. The limit exists as $s_k = \sup_{n>k} t_n$ form a nonincreasing sequence. The upper limit coincides with the largest accumulation point of the sequence $\{t_n\}$.

Theorem ($\mathbf{n^{th}}$-Root Test). Let $\limsup |w_n|^{1/n} = L$. Then the series $\sum_{n=0}^{\infty} w_n$ converges absolutely if $L < 1$ and diverges if $L > 1$.

Proof. Let $s_k = \sup_{n>k} |w_n|^{1/n}$.

Assume first that $L < 1$ and choose an arbitrary r such that $L < r < 1$. Since $s_k \to L$ as $k \to \infty$, we have $s_k < r$ for all sufficiently large k. This means that $|w_n| < r^n$ for all sufficiently large n. It follows that the sum $\sum_{n=0}^{\infty} |w_n|$ is finite by comparison with the convergent geometric series $\sum_{n=0}^{\infty} r^n$.

Assume now that $L > 1$. Then $s_k \geqslant 1$ for all sufficiently large k. It follows from the definition of supremum that there exists $m_k \geqslant k$ such that $|w_{m_k}|^{1/m_k} \geqslant 1$ and, consequently $|w_{m_k}| \geqslant 1$. Since m_k go to infinity as $k \to \infty$, we see that the sequence $\{|w_n|\}$ does not converge to zero. By the above, this implies that the series does not converge.

Theorem (Ratio Test). Assume that $w_n \neq 0$ and that the sequence $|w_{n+1}|/|w_n|$ converges. Let $\lim |w_{n+1}|/|w_n| = L$. Then the series $\sum_{n=0}^{\infty} w_n$ converges absolutely if $L < 1$ and diverges if $L > 1$.

Proof. Assume first that $L < 1$ and choose an arbitrary r such that $L < r < 1$. Since $|w_{n+1}|/|w_n| \to L$ as $n \to \infty$, we have $|w_{m+1}| < r\,|w_m|$ for all sufficiently large m. It follows that

$$\sum_{n=m}^{\infty} |w_n| \;=\; \sum_{n=0}^{\infty} |w_{n+m}| \;\leqslant\; \sum_{n=0}^{\infty} r^n\,|w_m| \;=\; (1-r)^{-1}\,|w_m|$$

provided that m is large enough. This implies that the series is absolutely convergent.

Assume now that $L > 1$. Then $|w_{n+1}| \geqslant |w_n|$ for all sufficiently large n. It follows that $|w_n| \not\to 0$ and, consequently, the series diverges.

POWER SERIES

A complex power series is a series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $a_n, z, z_0 \in \mathbb{C}$. The absolute convergence of such a power series is determined by the convergence of the real series $\sum_{n=0}^{\infty} |a_n| r^n$ where r denotes $|z - z_0|$. In particular, the power series absolutely converges for $z = z_0$ (then $w = 0$).

Comparison test. If $0 \leqslant t_n \leqslant s_n$ and $\sum_{n=0}^{\infty} s_n < \infty$ then $\sum_{n=0}^{\infty} t_n < \infty$.

Let us denote by Ω the set of all nonnegative numbers r such that the sequence $\{a_0, a_1 r, a_2 r^2, a_3 r^3 \ldots\}$ is bounded. This set Ω is not empty as it contains $r = 0$. Denote by $\hat{R}$ its supremum (the least upper bound). Note that $\hat{R}$ may be equal to zero (if the above sequence is unbounded for all $r > 0$). If the set Ω is unbounded (or, in other words, if the sequence is bounded for all $r \geqslant 0$) then we define $\hat{R} = \infty$.

Main Lemma. If $\hat{R} > 0$ and $|z - z_0| < \hat{R}$ then the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is absolutely convergent. If $|z - z_0| > \hat{R}$ then the series diverges.

Proof. Assume first that $|z - z_0| < \hat{R}$. By the definition of $\hat{R}$, there exists a positive number r, lying arbitrarily close to $\hat{R}$, such that the set $\{a_0, a_1 r, a_2 r^2, a_3 r^3 \ldots\}$ is bounded. Let us choose r in such a way that $r > |z - z_0|$. Then there exists r_1 satisfying $|z - z_0| < r_1 < r$. Now we estimate

$$| a_n (z - z_0)^n | \ < \ |a_n| r_1^n \ = \ |a_n| r^n (r_1/r)^n .$$

Since the set $\{a_0, a_1 r, a_2 r^2, a_3 r^3 \ldots\}$ is bounded, there is a constant L such that $|a_n| r^n \leqslant L$ for all n (in other words, this set lies in a disc of radius L centred at zero). The above inequality implies that $| a_n (z - z_0)^n | < L (r_1/r)^n$. Therefore the series $\sum_{n=0}^{\infty} | a_n (z - z_0)^n |$ converges by comparison with the geometric progression.

Assume now that $|z - z_0| > \hat{R}$. Since $\hat{R}$ is the supremum of the set Ω, the set $\{a_0, a_1 r, a_2 r^2, a_3 r^3 \ldots\}$ is unbounded for each $r > \hat{R}$. In particular, this is true for $r = |z - z_0|$. This means that the set of numbers $| a_n (z - z_0)^n |$ is unbounded, which implies that $| a_n (z - z_0)^n |$ do not converge to zero. As we have shown before, in this case the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ diverges. The proof is complete.

Definition. The number $\hat{R}$ is called the *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

Remark. The lemma says nothing about the behaviour of the series for $|z - z_0| = \hat{R}$. This depends on finer properties of the coefficients a_n; the series may diverge or converge at those points.

Example. The series $\sum_{n=0}^{\infty} z^n$ is absolutely convergent for $|z| < 1$. It diverges for $|z| \geqslant 1$ because then $z^n \not\to 0$.

Example. The series $\sum_{n=0}^{\infty} n^{-2} z^n$ is absolutely convergent for $|z| \leqslant 1$ and is divergent for $|z| > 1$.

Example. The series $\sum_{n=0}^{\infty} n^{-1} z^n$ is absolutely convergent for $|z| < 1$ and is divergent for $|z| > 1$. It converges at all points z such that $|z| = 1$ with the exception of $z = 1$.

Corollary. $\hat{R} = \left(\limsup |a_n|^{1/n}\right)^{-1}$.

Proof. Denote $C = \limsup |a_n|^{1/n}$.

Let $w_n = a_n(z - z_0)^n$. Then $L := \limsup |w_n|^{1/n} = C\,|z - z_0|$. Clearly, $L < 1$ if and only $|z - z_0| < C^{-1}$, and $L > 1$ if and only $|z - z_0| > C^{-1}$. Therefore, in view of the n-th Root Test and the Main Lemma, $\hat{R} = C^{-1}$.

Corollary. Assume that $a_n \neq 0$. If the sequence $|a_n|/|a_{n+1}|$ converges then $\hat{R} = \lim |a_n|/|a_{n+1}|$.

Proof. Denote $C = \lim |a_{n+1}|/|a_n|$.

Let $w_n = a_n(z - z_0)^n$. Then $L := |w_{n+1}|/|w_n| = C\,|z - z_0|$ We have $L < 1$ if and only $|z - z_0| < C^{-1}$, and $L > 1$ if and only $|z - z_0| > C^{-1}$. Therefore, in view of the Ratio Test and the Main Lemma, $\hat{R} = C^{-1} = \lim |a_n|/|a_{n+1}|$.

COMPLEX-VALUED FUNCTIONS

A complex-valued function of complex variable is a mapping from $\mathbb{C}$ or a subset of $\mathbb{C}$ into $\mathbb{C}$. The function f may not be defined on the whole plane; for example, $f(z) = \frac{1}{z}$ is not defined at $z = 0$ (unless it is regarded as a mapping from $\mathbb{C}$ into $\hat{C}$). The function f is real-valued if its range lies in $\mathbb{R}$, considered as a subset of $\mathbb{C}$.

Definition. A function f is said to be continuous at z_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z_0) - f(z)| < \varepsilon$ whenever $|z - z_0| < \delta$. The function f is said to be continuous on a set Ω if it is continuous at every point $z_0 \in \Omega$.

The function f is continuous at z_0 if and only if $|f(z_n) - f(z)| \to 0$ as $z_n \to z$ for all sequences $\{z_n\}$ and all points z lying in the domain of definition of f. A detailed discussion of various definitions of continuity can be found in
http://www.mth.kcl.ac.uk/∼ysafarov/Lectures/Past/321.pdf .

Example. $\operatorname{Re} z$, $\operatorname{Im} z$ and $|z|$ are continuous real-valued functions on $\mathbb{C}$. The function $\bar{z}$ is a continuous complex-valued function.

Example. A polynomial $P(z) = \sum_{n=0}^{m} a_n(z - z_0)^n$, where a_n and z_0 are fixed complex numbers, is a continuous function on $\mathbb{C}$.

Example. The series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ defines a function on the disc $\mathcal{D}(z_0, \hat{R})$ where $\hat{R}$ is the radius of convergence.

UNIFORM CONVERGENCE

Let $\sum_{n=0}^{\infty} a_n\,(z-z_0)^n = f(z)$ and $f_m(z) = \sum_{n=0}^{m} a_n\,(z-z_0)^n$. Then $f_m(z) \to f(z)$ for each $z \in \mathcal{D}(z_0, \hat{R})$ where $\hat{R}$ is the radius of convergence (see Week 2).

Definition. We say that a sequence of functions f_m converges to f *uniformly* on a set $\Omega \subset \mathbb{C}$ if $\sup_{z\in\Omega} |f(z) - f_m(z)| \to 0$ as $m \to \infty$. The series $\sum_{n=0}^{\infty} a_n\,(z-z_0)^n$ converges uniformly to its sum $f(z)$ if the sequence of partial sum f_m converges to f uniformly.

In other words, uniform convergence on a set Ω means that

$$|f(z) - f_m(z)| \leqslant C_m\,, \qquad \forall z \in \Omega\,,$$

where C_m are some constants such that $C_m \to 0$ as $m \to \infty$.

Theorem. The series $\sum_{n=0}^{\infty} a_n\,(z-z_0)^n$ is uniformly convergent on any disc $\mathcal{D}(z_0, r)$ with $r < \hat{R}$, where $\hat{R}$ is the radius of convergence.

Proof of the above theorem uses some notions of the theory of metric spaces. In short, one defines the distance $\operatorname{dist}(f,g) = \sup_z |f(z) - g(z)|$ on the space of functions and then applies the triangle inequality for this distance instead of the triangle inequality for the modules. See
http://www.mth.kcl.ac.uk/~ysafarov/Lectures/Past/321.pdf
for details.

Theorem. The function $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is continuous function on the disc $\mathcal{D}(z_0, \hat{R})$ where $\hat{R}$ is the radius of convergence.

Proof. The partial sums f_m are polynomials and, therefore, are continuous functions. If $z_n \to z$ then, for all sufficiently large values of n, the point z_n lies in the disc $\mathcal{D}(z, r)$ on which the series is uniformly convergent. For these n, we have

$$\begin{aligned} |f(z) - f(z_n)| \;&\leqslant\; |f(z) - f_m(z)| + |f_m(z) - f_m(z_n)| + |f(z_n) - f_m(z_n)| \\ &\leqslant\; 2\,C_m + |f_m(z) - f_m(z_n)|\,, \end{aligned}$$

where C_m are some constants converging to 0 as $m \to \infty$. Given a positive ε, we can find a positive integer m such that $C_m < \varepsilon/3$. Then we can choose n_ε in such a way that $|f_m(z) - f_m(z_n)| < \varepsilon/3$ for all $n \geqslant n_\varepsilon$. Thus, for all $n \geqslant n_\varepsilon$ we have $|f(z) - f(z_n)| < \varepsilon$. Since ε is an arbitrary positive number, this shows that $|f(z) - f(z_n)| \to 0$ as $z_n \to z$. The proof is complete.

Identity theorem for power series. Suppose that the power series $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$ both converge absolutely on the disc $\mathcal{D}(z_0, \hat{R})$. Suppose, further, that there is a sequence $\{\zeta_k\}$ in $\mathcal{D}(z_0, \hat{R})$ such that $\zeta_k \to z_0$ as $k \to \infty$ and $f(\zeta_k) = g(\zeta_k)$ for all k. Then $a_n = b_n$ for all n, that is, $f = g$.

Proof. Let $c_n = a_n - b_n$, so that $f(z) - g(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$. Since the both series converge absolutely, so does the series with the coefficients c_n. Suppose that there are nonzero coefficients in this series, and let c_m be the first nonzero coefficient. Then $f(z) - g(z) = (z - z_0)^m\, h(z)$ where $h(z) = \sum_{n=m}^{\infty} c_n(z - z_0)^{n-m}$. By the above, the function h is continuous. Since $h(\zeta_k) = 0$ and $\zeta_k \to z_0$, we have $h(z_0) = 0$. This implies that $c_m = 0$, which is a contradiction.

COMPLEX DIFFERENTIATION

Let $f(z)$ be a complex-valued function defined on an open set $\Omega \subseteq \mathbb{C}$.

Definition. We say that a function $f(z)$ is differentiable at z_0 there exists a complex number ζ_0 such that for each $\varepsilon >$ there exists $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \zeta_0 \right| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta\,.$$

The number ζ_0 is said to be the derivative of f at the point z_0 and is denoted by $f'(z_0)$. The function f is said to be differentiable if it is differentiable at each point of the set Ω. Then its derivative f' is a complex-valued function on Ω.

Remark. In other words, the functions is differentiable if $\frac{f(z)-f(z_0)}{z-z_0} \to \zeta_0$ as $z \to z_0$. This is equivalent to the statement that $\frac{f(z_n)-f(z_0)}{z_n-z_0} \to \zeta_0$ for each sequence $\{z_n\}$ such that $z_n \to z_0$ and $z_n \neq z_0$.

Example. The polynomial $f(z) = \sum_{n=0}^{m} a_n(z - z_0)^n$ is a differentiable function on the whole complex plane. Its derivative is $f'(z) = \sum_{n=1}^{m} n a_n(z - z_0)^{n-1}$. In particular, the constant function is differentiable and its derivative is identically equal to zero.

Theorem. If f is differentiable at each point $z \in \Omega$ then f is continuous on Ω.

Proof. From the definition of the derivative it follows that, for every $\varepsilon_1 > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < (|f'(z_0)| + \varepsilon_1)\,|z - z_0| \quad \text{whenever} \quad |z - z_0| < \delta.$$

If ε is another positive number, we can choose ε_1 and the corresponding δ in such a way that $(|f'(z_0)| + \varepsilon_1)\,\delta < \varepsilon$. Then the above condition implies that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Therefore the function f is continuous.

There exist continuous functions which are nowhere differentiable.

Example. The functions $\operatorname{Re} z$, $\operatorname{Im} z$, $\bar{z}$ and $|z|$ are nowhere differentiable. For instance, if $\theta = \arg(z - z_0)$ then

$$\frac{\operatorname{Re} z - \operatorname{Re} z_0}{z - z_0} = \frac{|z - z_0|\,\cos\theta}{|z - z_0|\,(\cos\theta + i\sin\theta)} = (1 + i\ \tan\theta)^{-1}\,.$$

We have $z \to z_0$ if and only if $|z - z_0| \to 0$. The above formula shows that the limit does not exist, as the right hand side depends on the argument of $z - z_0$.

Theorem. Let f and g be differentiable functions on an open set $\Omega \subset \mathbb{C}$. Then

(1) $(\alpha f + \beta g)' = \alpha f' + \beta g'$,

(2) $(fg)' = f'g + fg'$,

(3) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ at the points z where $g(z) \neq 0$ (Quotient Rule).

Proof is the same as for functions of one real variable.

Theorem (Chain Rule). Let f and g be differentiable functions such that f is defined on an open set Ω and g is defined on an open set containing the image $f(\Omega)$. Then the composition $g \circ f(z) = g(f(z))$ is also differentiable and $(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0)$.

Proof. We have

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \, \frac{f(z) - f(z_0)}{z - z_0} .$$

Let $z \to z_0$. Then the second ration in the right hand side converges to $f'(z_0)$. Also, in the denominator of the first fraction $f(z) \to f(z_0)$ because the function f is continuous. Now, by the definition of the derivative, the first ration converges to $g'(f(z_0))$. This proves the theorem.

CAUCHY–RIEMANN EQUATIONS

Let f be a complex-valued function on an open set $\Omega \subseteq \mathbb{C}$. Denote $u(z) = \operatorname{Re} f(z)$ and $v(z) = \operatorname{Im} f(z)$, so that $f = u + iv$. We shall consider u and v as real-valued functions of two real variables $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$, so that $f(z) = u(x, y) + iv(x, y)$.

Theorem. If f is differentiable at $z_0 = x_0 + iy_0$ then u and v satisfy the following *Cauchy–Riemann equations*

$$u_x(x_0, y_0) = v_y(x_0, y_0)\,, \qquad u_y(x_0, y_0) = -v_x(x_0, y_0)\,,$$

where u_x , v_x and u_y , v_y are the derivatives with respect to the first and second variable respectively.

Proof. Assume that f is differentiable at z_0. Then $\frac{f(z) - f(z_0)}{z - z_0}$ converges to the limit $f'(z_0)$ as $z \to z_0$. In particular, this is true if we take $z = z_0 + \varepsilon$, where $\varepsilon \in \mathbb{R}$ and $\varepsilon \to 0$. Since

$$\frac{f(z_0 + \varepsilon) - f(z_0)}{(z_0 + \varepsilon) - z_0} = \frac{u(x_0 + \varepsilon, y_0) - u(x_0, y_0)}{\varepsilon} + i\, \frac{v(x_0 + \varepsilon, y_0) - v(x_0, y_0)}{\varepsilon} ,$$

letting $\varepsilon \to 0$, we see that $f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$, Similarly, taking $z = z_0 + i\varepsilon$ with $\varepsilon \in \mathbb{R}$ and letting $\varepsilon \to 0$, we obtain $f'(z_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0)$. Thus we have $u_x(x_0, y_0) + i\, v_x(x_0, y_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0)$, which is equivalent to the Cauchy–Riemann equations.

It is convenient to denote

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\,\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\right).$$

The notation is motivated by the equalities $x = (z+\bar{z})/2$ and $y = (z-\bar{z})/2i$. If we consider x and y as functions of z and $\bar{z}$, then the formal differentiation of the identity $f(x,y) = f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$ leads to the above equalities.

Note that the Cauchy–Riemann equations are equivalent to each of the following equations

$$\frac{\partial f}{\partial \bar{z}} = 0\,, \qquad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\,, \qquad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial (iy)}\,.$$

Remark. Generally speaking, the Cauchy–Riemann equations at a fixed point z_0 do not imply that the function is differentiable at the point z_0. For example, the function $f(x+iy) = |xy|^{1/2}$ satisfies the Cauchy–Riemann equations at the point $z_0 = 0$. However, if $s \in \mathbb{R}$ and $s > 0$ then

$$\lim_{s\to 0}\frac{f(s+is)-f(0)}{s+is} = \frac{1}{1+i} \neq \frac{1}{1-i} = \lim_{s\to 0}\frac{f(s-is)-f(0)}{s-is}\,.$$

ANALYTIC FUNCTIONS

Definition. A complex-valued function f defined on an open set $\Omega \subseteq \mathbb{C}$ is said to be *analytic* (or *holomorphic*) in Ω if it is differentiable at every point $z_0 \in \Omega$. We shall denote the linear space of analytic in Ω functions by $H(\Omega)$.

Functions from $H(\mathbb{C})$ are called *entire* functions.

Example. A polynomial is an entire function.

Example. The function $f(z) = \frac{az+b}{cz+d}$ is an analytic function in the punctured complex plane $\mathbb{C} \setminus \{z = -\frac{d}{c}\}$.

PROPERTIES OF ANALYTIC FUNCTIONS

Theorem. Let Ω be an open set. If f is continuously differentiable and satisfies the Cauchy–Riemann equations at every point $z \in \Omega$ then $f \in H(\Omega)$.

Proof. Let us fix $z_0 \in \Omega$. We need to show that $f'(z) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. Since Ω is open, we have $z \in \Omega$ whenever z is sufficiently close to z_0, so that $f(z)$ in the right hand side is well defined.

Let $z - z_0 = s + it$. Then, by the mean value theorem,

$$\begin{aligned}
\frac{f(z)-f(z_0)}{z-z_0} &= \frac{f(x+s,y+t)-f(x,y)}{s+it} \\
&= \frac{f(x+s,y+t)-f(x,y+t)}{s}\,\frac{s}{s+it} + \frac{f(x,y+t)-f(x,y)}{t}\,\frac{t}{s+it} \\
&= \frac{s}{s+it}\, f_x(x+s^*,y+t) + \frac{t}{s+it}\, f_y(x,y+t^*)\,,
\end{aligned}$$

where $|s^*| \leqslant s$ and $|t^*| \leqslant t$. Since the partial derivatives of f are continuous,

$$f_x(x+s^*,y+t) \to f_x(x,y) \quad \text{and} \quad f_y(x,y+t^*) \to f_y(x,y)$$

as $z \to z_0$. On the other hand, $\left|\frac{s}{s+it}\right| \leqslant 1$ and $\left|\frac{t}{s+it}\right| \leqslant 1$. Therefore

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{s}{s+it}\, f_x(x,y) + \frac{t}{s+it}\, f_y(x,y) + R(z)\,,$$

where $R(z)$ is a function such that $R(z) \to 0$ as $z \to z_0$. Now, by the Cauchy–Riemann equations, $f_y = i f_x$, which implies that the right hand side coincides with

$$\frac{s}{s+it}\, f_x(x,y) + \frac{t}{s+it}\, i f_x(x,y) + R(z) = f_x(x,y) + R(z)$$

and converges to $f_x(x,y)$ as $z \to z_0$.

Theorem. Let Ω be an open connected set, and let $f \in H(\Omega)$. If $f'(z) = 0$ for all $z \in \Omega$ then the function f is constant.

Proof. The Cauchy–Riemann equations imply that all the partial derivatives of f are equal to zero. It follows that f is constant on every horizontal and every vertical line segment lying in Ω. Since every two points in an open disc can be joined by a path consisting of one horizontal and one vertical line segment, we see that the function f is constant on every open disc $\mathcal{D}(z,r)$ lying in Ω.

Let us fix a point $z_0 \in \Omega$ and denote $f(z_0) = a$. Consider all open discs $\mathcal{D} \subset \Omega$ such that $f = a$ on $\mathcal{D}$. Let Ω_1 be the union of all these discs. Clearly, Ω_1 is a subset of Ω. This subset is open because for every point $z' \in \mathcal{D}$ there is a smaller disc $\mathcal{D}(z',r')$ lying in $\mathcal{D}$.

Assume that $f(z_1) \neq a$ at some point $z_1 \in \Omega$. Since the differentiable function f is continuous (see Week 3), we have $f(z) \neq a$ for all z lying in a sufficiently small disc about z_1. Let us consider all open discs $\mathcal{D} \subset \Omega$ such that $f(z) \neq a$ for each

$z \in \mathcal{D}$. Denote the union of these discs by Ω_2. For the same reasons as above, Ω_2 is an open subset of Ω. Clearly, $\Omega_1 \bigcap \Omega_2 = \varnothing$ and $\Omega_1 \bigcup \Omega_2 = \Omega$. Since the set Ω is connected, one of these sets must be empty (otherwise we would have obtained a disconnection of Ω. The set Ω_1 is not empty because it contains z_0. Thus, $\Omega_2 = \varnothing$ and $\Omega_1 = \Omega$, which implies that $f = a$ on the whole set Ω.

Lemma. If a real-valued function f is differentiable at z_0 then $f'(z_0) = 0$.

Proof. Assume that $\frac{f(z)-f(z_0)}{z-z_0} \to \zeta_0$ as $z \to z_0$, and denote $\operatorname{Re}(z - z_0) = t$, $\operatorname{Im}(z - z_0) = s$, so that $z - z_0 = t + is$. Taking $s = 0$ and letting $t \to 0$, we see that $\frac{f(z)-f(z_0)}{t} \to \zeta_0$ as $t \to 0$. Since f is real-valued, this implies that $\operatorname{Im}\zeta_0 = 0$. Taking $t = 0$ and letting $s \to 0$, we see that $\frac{f(z)-f(z_0)}{is} \to \zeta_0$ as $s \to 0$, which implies that $\operatorname{Re}\zeta_0 = 0$.

Combining the above results, we immediately obtain

Theorem. If Ω is a connected set then every real-valued analytic in Ω function is identically equal to constant.

Corollary. Let Ω be a connected set and f be an analytic in Ω function such that $|f(z)|$ is a constant function on Ω. Then f is identically equal to a constant.

Proof. If $|f| = 0$ then $f = 0$. Assume that $|f| = a \neq 0$. Then $|f|^2 = f\bar{f} = a^2$ and, consequently, $\bar{f} = \frac{a^2}{f}$ is an analytic function in Ω. This implies that the real-valued functions $\operatorname{Re} f = \frac{1}{2}(f + \bar{f})$ and $\operatorname{Im} f = \frac{1}{2i}(f - \bar{f})$ are also analytic. By the above, both these functions are constant.

FUNCTIONS DEFINED BY POWER SERIES

Let $\hat{R}$ be the radius of convergence of the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. We know that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a continuous function on $\mathcal{D}(z_0, \hat{R})$.

Theorem. If $\hat{R} > 0$ then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is analytic in the open disc $\mathcal{D}(z_0, \hat{R})$. Its derivative is given by $f'(z) = \sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1}$, where the "derived" series $\sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1}$ has the same radius of convergence $\hat{R}$.

Proof. The proof proceeds in several steps.

Step 1. In view of the chain rule, it is sufficient to prove the theorem assuming that $z_0 = 0$, that is, for the function $g(z) = f(z + z_0) = \sum_{n=0}^{\infty} a_n z^n$ defined on $\mathcal{D}(0, \hat{R})$.

Step 2. The series $\sum_{n=0}^{\infty} b_n z^n$ and $\sum_{n=0}^{\infty} b_{n+k} z^n$ have the same radius of convergence (here k is a fixed positive integer).

Indeed, the radius of convergence $\hat{R}$ of the series $\sum_{n=0}^{\infty} b_n z^n$ is the supremum of the set of positive numbers r such that the sequence $\{b_0, b_1 r, b_2 r^2, \ldots\}$ is bounded (see Week 2). The sequence $\{b_k, b_{k+1} r, b_{k+2} r^2 \ldots\}$ is obtained by removing a finite number of terms and multiplying the remaining terms by r^{-k}. Therefore it is bounded if and only if $\{b_0, b_1 r, b_2 r^2, \ldots\}$ is bounded. Finally, the supremum of the set of $r > 0$ for which $\{b_k, b_{k+1}r, b_{k+2}r^2 \ldots\}$ is bounded is the radius of convergence of $\sum_{n=0}^{\infty} b_{n+k}\, z^n$.

Step 3. The series $\sum_{n=0}^{\infty} b_n\, z^n$ and $\sum_{n=0}^{\infty} n^k\, b_n\, z^n$ have the same radius of convergence (here k is a fixed positive integer).

Let Ω' be the set of positive integers r such that the sequence $\{n^k\, b_n r^n\}_{n=0,1,2,\ldots}$ is bounded, and let Ω be the set of positive integers r such that the sequence $\{b_n r^n\}_{n=0,1,2,\ldots}$ is bounded. Then $\sup \Omega'$ (the least upper bound of Ω') is the radius of convergence of $\sum_{n=0}^{\infty} n^k\, b_n\, z^n$, and $\sup \Omega$ is the radius of convergence of $\sum_{n=0}^{\infty} b_n\, z^n$. Clearly, $\sup \Omega' \leqslant \sup \Omega$. On the other hand, $n^k (r_0/r)^n \to 0$ as $n \to \infty$ for each $r_0 < r$. Therefore $n^k\, b_n r_0^n \leqslant b_n r^n$ for all sufficiently large n. This implies that Ω' contains all positive integers r_0 which are strictly smaller that any $r \in \Omega$. Thus, $\sup \Omega' = \sup \Omega$.

Step 4. In view of the previous two results, the derived series $\sum_{n=1}^{\infty} n\, a_n\, z^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, z^n$ has the same radius of convergence $\hat{R}$. In particular, this implies that the series $\sum_{n=1}^{\infty} n\, a_n\, z^{n-1}$ is absolutely convergent whenever $|z| < \hat{R}$ (see Week 2), that is, $\sum_{n=1}^{\infty} n\, |a_n|\, r^{n-1} < \infty$ for all $r < \hat{R}$.

Step 5. Let us denote $g_k(z) = \sum_{n=1}^{k-1} a_n\, z_n$ and $\tilde{g}_k(z) = \sum_{n=k}^{\infty} a_n\, z_n$, so that $g = g_k + \tilde{g}_k$. Define also $h(z) = \sum_{n=1}^{\infty} n\, a_n\, z^{n-1}$, $h_k(z) = \sum_{n=1}^{k-1} n\, a_n\, z^{n-1}$ and $\tilde{h}_k(z) = \sum_{n=k}^{\infty} n\, a_n\, z^{n-1}$. Then, obviously, $h = h_k + \tilde{h}_k$ and $h_k = g_k'$.

We are going to show that the derivative $g'(\zeta)$ exists for every $\zeta \in \mathcal{D}(0, \hat{R})$ and coincides with $h(\zeta)$. By definition, this means that $\left|\frac{g(z)-g(\zeta)}{z-\zeta} - h(\zeta)\right| \to 0$ as $z \to \zeta$. Let us estimate

$$\left|\frac{g(z) - g(\zeta)}{z - \zeta} - h(\zeta)\right| \;\leqslant\; \left|\frac{g_k(z) - g_k(\zeta)}{z - \zeta} - h_k(\zeta)\right| + \left|\frac{\tilde{g}_k(z) - \tilde{g}_k(\zeta)}{z - \zeta}\right| + \left|\tilde{h}_k(\zeta)\right| .$$

Since $h_k = g_k'$, the first term in the right hand side converges to zero as $z \to \zeta$ for each fixed k. Thus it is sufficient to show that the other two terms can be made arbitrarily small by choosing large k.

Step 6. Since ζ does not lie on the boundary of the disc $\mathcal{D}(0, \hat{R})$, we can find $r < \hat{R}$ such that $|\zeta| < r$. Also, $|z| < r$ provided that z is sufficiently close to ζ. Further on we shall always be assuming that $|\zeta| < r$ and $|z| < r$. Recall that the series $\sum_{n=1}^{\infty} n\, |a_n|\, r^{n-1}$ converges. From the definition of convergence of a series, it follows that $\sum_{n=k}^{\infty} n\, |a_n|\, r^{n-1} \to 0$ as $k \to \infty$.

Step 7. In order to estimate the second term in the right hand side, let us note that $\frac{\tilde{g}_k(z)-\tilde{g}_k(\zeta)}{z-\zeta} = \sum_{n=k}^{\infty} a_n\, \frac{z^n-\zeta^n}{z-\zeta}$. By direct calculation,

$$\frac{z^n - \zeta^n}{z - \zeta} = z^{n-1} + z^{n-2}\zeta + z^{n-3}\zeta^2 + \ldots z\zeta^{n-2} + \zeta^{n-1}$$

and, consequently, $\left|\frac{z^n-\zeta^n}{z-\zeta}\right| \leqslant n\, r$. Therefore

$$\left|\frac{\tilde{g}_k(z) - \tilde{g}_k(\zeta)}{z - \zeta}\right| \;\leqslant\; \sum_{n=k}^{\infty} |a_n| \left|\frac{z^n - \zeta^n}{z - \zeta}\right| \;\leqslant\; r \sum_{n=k}^{\infty} n\, |a_n|\, r^{n-1} \underset{k\to\infty}{\to} 0\,.$$

For the third term in the right hand side, we have

$$|\tilde{h}_k(\zeta)| \;\leqslant\; \sum_{n=k}^{\infty} |n\, a_n\, z^{n-1}| \;\leqslant\; \sum_{n=k}^{\infty} n\, |a_n|\, r^{n-1} \underset{k\to\infty}{\longrightarrow} 0\,.$$

Thus we can make $\left|\frac{\tilde{g}_k(z)-\tilde{g}_k(\zeta)}{z-\zeta}\right| + |\tilde{h}_k(\zeta)|$ smaller than any given positive number by choosing a sufficiently large k. As was explained in Step 5, this proves that $g'(\zeta) = h(\zeta)$.

Remark. Steps 2 and 3 could be slightly simplified by using a suitable formula for $\hat{R}$ (see Week 2).

Corollary. The function $f(z) = \sum_{n=0}^{\infty} a_n\,(z-z_0)^n$ is infinitely differentiable in $\mathcal{D}(z_0, \hat{R})$ and its mth derivative is given by

$$f^{(m)}(z) \;=\; \sum_{n=m}^{\infty} n(n-1)\dots(n-(m-1))\, a_n\,(z-z_0)^{n-m}\,.$$

Proof is by induction in m.

PRODUCT OF POWER SERIES

Theorem. Let $f(z) = \sum_{n=0}^{\infty} a_n\,(z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n\,(z-z_0)^n$, and assume that both series converge on an open disc $\mathcal{D}(z_0,\hat{R})$. Then $f(z)\,g(z) = \sum_{n=0}^{\infty} c_n\,(z-z_0)^n$, where $c_n = \sum_{k=0}^{n} a_k\,b_{n-k}$ and the series converges on the same disc $\mathcal{D}(z_0,\hat{R})$.

Proof. Since $f(z) = \lim\limits_{m\to\infty} \sum_{n=0}^{m} a_n\,(z-z_0)^n$ and $g(z) = \lim\limits_{m\to\infty} \sum_{n=0}^{m} b_n\,(z-z_0)^n$, we have

$$f(z)\,g(z) = \lim_{m\to\infty} \left(\sum_{n=0}^{m} a_n\,(z-z_0)^n\right)\left(\sum_{n=0}^{m} b_n\,(z-z_0)^n\right)$$

The product in the right hand side coincides with the finite sum $\sum_{n=0}^{2m} c_{n,m}(z-z_0)^n$ where $c_{n,m} = \sum_{j,k} a_j\,b_k$ and the sum is taken over all j,k such that $j+k=n$, $j \leqslant m$ and $k \leqslant m$. Note that for $n \leqslant m$ the first condition implies the other two. Therefore, $c_{n,m} = c_n$ whenever $n \leqslant m$.

By the above $f(z)\,g(z) = \lim_{m\to\infty} \sum_{n=0}^{2m} c_{n,m}\,(z-z_0)^n$, but we need to prove that

$$f(z)\,g(z) = \sum_{n=0}^{\infty} c_n\,(z-z_0)^n = \lim_{m\to\infty} \sum_{n=0}^{2m} c_n\,(z-z_0)^n\,.$$

Clearly, the latter is true if the difference

$$\sum_{n=0}^{2m} c_{n,m}\,(z-z_0)^n - \sum_{n=0}^{2m} c_n\,(z-z_0)^n = \sum_{n=m+1}^{2m} (c_{n,m}-c_n)\,(z-z_0)^n$$

tends to zero as $m\to\infty$.

Define $d_n = \sum_{k=0}^{n} |a_k|\,|b_{n-k}|$. We have $|c_n| = |\sum_{k=0}^{n} a_k b_{n-k}| \leqslant d_n$ and, similarly,

$$|c_{n,m}| = \left|\sum_{j,k} a_j b_k\right| \leqslant \sum_{j,k} |a_j|\,|b_k| \leqslant \sum_{j+k=n} |a_j|\,|b_k| = d_n\,,$$

where the first two sums are taken over j,k such that $j+k=n$, $j\leqslant m$ and $k \leqslant m$. Also, since the series are absolutely convergent of every closed disc of radius $r<\hat{R}$ centred at z_0, we have for all $r<\hat{R}$

$$\sum_{n=0}^{\infty} |d_n|\,r^n \;\leqslant\; \sum_{n=0}^{\infty}\sum_{k=0}^{n} |a_k|\,|b_{n-k}|\,r^n \;\leqslant\; \left(\sum_{k=0}^{\infty} |a_k|\,r^k\right)\left(\sum_{j=0}^{\infty} |b_j|\,r^j\right) < \infty$$

This inequality shows that the series $\sum_{n=0}^{\infty} |d_n|\,r^n$ converges, which implies that $\sum_{n=m+1}^{\infty} |d_n|\,r^n \to 0$ as $m\to\infty$. If $z\in\mathcal{D}(z_0,\hat{R})$ then $|z-z_0|\leqslant r$ for some $r<\hat{R}$ and we have

$$|\sum_{n=m+1}^{2m} (c_{n,m}-c_n)\,(z-z_0)^n| \leqslant \sum_{n=m+1}^{2m} |c_{n,m}-c_n)|\,|z-z_0|^n \leqslant 2\sum_{n=m+1}^{\infty} d_n\,r^n \underset{m\to\infty}{\to} 0\,.$$

THE EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

By definition,

$$\exp z \;=\; \sum_{n=0}^{\infty} \frac{z^n}{n!} \;=\; 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots\,,$$

$$\sin z \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n\, z^{2n+1}}{(2n+1)!} \;=\; z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

$$\cos z \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n\, z^{2n}}{(2n)!} \;=\; 1 - \frac{z^2}{2!} + \frac{z^4}{5!} - \frac{z^6}{6!} + \dots$$

All the above series have radius of convergence $\hat{R} = +\infty$ because the sequence $\{r^n (n!)^{-1}\}$ converges to zero as $n \to \infty$ and is therefore bounded for each $r \in \mathbb{R}$. Thus, exp, sin and cos are entire functions.

The following identities are evident:
$\exp 0 = \cos 0 = 1\,,\ \sin 0 = 0\,,\ \sin(-z) = -\sin z\,,\ \cos(-z) = \cos z\,,$
$\overline{\exp z} = \exp \bar{z}\,,\ \overline{\sin z} = \sin \bar{z}\,,\ \overline{\cos z} = \cos \bar{z}$ (because $\overline{z^k} = (\bar{z})^k$),
$\exp(iz) = \cos z + i \sin z\,,\ \cos z = \frac{\exp(iz)+\exp(-iz)}{2}\,,\ \sin z = \frac{\exp(iz)-\exp(-iz)}{2i}\,.$
Since we can differentiate power series term by term, $\exp' z = \exp z$, $\sin' z = \cos z$ and $\cos' z = -\sin z$.

The following results are less obvious.

Lemma.
(1) $\exp(z+w) = \exp z\, \exp w$;
(2) $\exp(-z) = \frac{1}{\exp z}$ (and, consequently, $\exp z \neq 0$ for all $z \in \mathbb{C}$);
(3) $\exp(x+iy) = \exp x\,(\cos y + i \sin y)$ for all $x, y \in \mathbb{C}$ (and, in particular, for all $x, y \in \mathbb{R}$).

Proofs.
(1) Let $w \in \mathbb{C}$ be a fixed number and $f(z) = \exp(z+w)\, \exp(-z)$. Differentiating f and applying the formulae for the derivative of the product and the chain rule, we see that $f'(z) = 0$ for all $z \in \mathbb{C}$. Thus f is identically equal to a constant. To find this constant, we can take $z = 0$ which yields $f(z) = f(w)$. Now, putting $a = z + w$ and $b = -z$, we obtain $\exp a\, \exp b = \exp(a+b)$ (where a and b can be arbitrary complex numbers).
(2) Taking $w = -z$ in (1), we obtain $\exp z\, \exp(-z) = 1$, which means that $\exp(-z) = \frac{1}{\exp z}$.
(3) By (1), $\exp(x+iy) = \exp x\, \exp(iy)$. The identity $\exp(iy) = (\cos x + i \sin y)$ is obvious from the series expansions.

Remark. Let $e = \exp 1$, that is, $e = \sum_{n=0}^{\infty} \frac{1}{n!} \;=\; 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots$ Then $\exp n = e^n$ for all $n \in \mathbb{Z}$. One often writes e^z for $\exp z$.

TRIGONOMETRIC FORMULAE

If x then, by the above, $\cos x = \mathrm{Re}\, \exp(ix)$ and $\sin x = \mathrm{Im}\, \exp(ix)$. Using these identities and the elementary properties of the exponential function, one can easily prove all standard trigonometric formulae for functions of real variable. For example, if $x, y \in \mathbb{R}$ then $\cos(x+y) = \mathrm{Re}\, e^{ix+iy} = \mathrm{Re}\,(e^{ix}e^{iy})$ and

$$\mathrm{Re}\,(e^{ix}e^{iy}) = \mathrm{Re}\,(e^{ix})\,\mathrm{Re}\,(e^{iy}) - \mathrm{Im}\,(e^{ix})\,\mathrm{Im}\,(e^{iy}) = \cos x\,\cos y - \sin x\,\sin y\,.$$

On the other hand, the theorem on the product of power series together with the identity theorem for power series (Week 3) imply that any equality involving linear combinations of the functions $(\sin z)^k\,(\cos z)^m$ with some nonnegative integers k and m holds for all $z \in \mathbb{C}$ whenever it is true for all $z \in \mathbb{R}$. In particular, the standard trigonometric formulae (including the famous identity $\sin^2 z + \cos^2 z = 1$) remain valid for $z \in \mathbb{C}$.

THE ARGUMENT REVISITED

We have declared that any complex number z can be written in the form $z = r\,e^{i\theta}$ where $\theta = \arg z$ and, by definition, $e^{i\theta} = \cos\theta + i\sin\theta$. Now we see that this definition is justified if $e^{i\theta}$, $\sin\theta$ and $\cos\theta$ are given by the corresponding power series. From the properties of the exponential function it follows that $\overline{e^{i\theta}} = e^{-i\theta} = (e^{i\theta})^{-1}$. Therefore $|e^{i\theta}| = 1$. Also, the standard results from the theory of functions of real variable imply that $e^{i\theta} = 1$ if and only if $\sin\theta = 0$ and $\cos\theta = 1$, that is, if $\theta = 2\pi k$ with $k \in \mathbb{Z}$.

THE COMPLEX LOGARITHM

For real positive z, we define $\ln z$ by the formula $e^{\ln z} = z$. Let us set

$$\ln z \;=\; \ln|z| + i\arg z \qquad \text{for all } z \neq 0.$$

Then the same identity holds for complex $z \neq 0$.

Recall that $\arg z$ is defined modulo $2\pi k$ where $k \in \mathbb{Z}$. Therefore $\ln z$ is defined modulo $2i\pi k$. The value of $\arg z$ lying in $(-\pi, \pi]$ is called the principal value and is denoted by $\mathrm{Arg}\, z$. Similarly, $\ln|z| + i\mathrm{Arg}\, z$ is called the principal value of the logarithm and is denoted by $\mathrm{Ln}\, z$.

The function $\mathrm{Ln}\, z$ is continuous at every point lying outside the negative half-line $\{z \in \mathbb{C} \,:\, \mathrm{Im}\, z = 0\,, \mathrm{Re}\, z \leqslant 0\}$. However, it is not defined at the origin and has a jump at the open negative half-line $\{z \in \mathbb{C} \,:\, \mathrm{Im}\, z = 0\,, \mathrm{Re}\, z < 0\}$. The former is inevitable, but one can try to fix the latter problem by putting $\arg z = \mathrm{Arg}\, z + 2\pi k(z/|z|)$ where k is an integer-valued function on the unit circle $\mathbb{S}$ about the origin, which jumps by -1 when we pass through the point -1 in the clockwise direction. Then $\arg z$ is continuous at the negative half-line. However, such an integer-valued function inevitably has other jumps, and so does the logarithm defined by the above formula. This shows that it is impossible to define logarithm as a continuous function on the punctured complex plane $\mathbb{C} \setminus \{0\}$.

Remark. One can think of $\mathrm{Ln}\, z$ as a smooth multi-valued function on $\mathbb{C}$, or as a proper function defined on a more complicated underlying set (the union of infinitely many copies of $\mathbb{C}$ without the negative real half-line).

Definition. Let θ be an arbitrary point on the unit circle $\mathbb{S}$, and let $k(z/|z|)$ be the function which takes an integer value m on the arc going from θ to -1 in the clockwise direction and is identically equal to $m-1$ on the remaining part of $\mathbb{S}$. Then the function $\ln|z| + i\mathrm{Arg}\, z + 2i\pi k(z/|z|)$ is called a branch of the complex logarithm.

Usually, a branch of the complex logarithm is also denoted by $\ln z$. In has a jump on the half-line originating from the origin and passing through the point θ, and is continuous outside this half-line.

Remark. The function $\ln|z| + i\mathrm{Arg}\, z + 2i\pi m$ with $m \in \mathbb{Z}$ is also called a branch of the complex logarithm.

Theorem.

(1) $e^{\ln z} = z$.

(2) $\ln(e^z) = z + 2i\pi n$ with some $n \in \mathbb{Z}$.

(3) $\ln(z_0 z) = \ln(z_0) + \ln z + 2i\pi n$ with some $n \in \mathbb{Z}$.

(4) $\ln(z^{-1}) = -\ln z + 2i\pi n$ with some $n \in \mathbb{Z}$.

(5) Let $\ln z$ be a branch of the logarithm which is continuous on an open set $\Omega \subset \mathbb{C}$. Then $\ln z$ is analytic in Ω and $(\ln z)' = z^{-1}$.

Proof.

(1) immediately follows from the definition.

(2) $e^{\ln(e^z)} = e^z$, that is, $z - \ln(e^z) = 2i\pi n$.

(3) $e^{\ln(z_0 z)} = z z_0 = e^{\ln(z_0)}\, e^{\ln(z_0)} = e^{\ln(z_0)+\ln(z_0)}$. Therefore $\ln(z_0 z) = \ln(z_0) + \ln(z_0)$ modulo $2i\pi n$.

(4) $e^{\ln(z^{-1})} = z^{-1} = (e^{\ln z})^{-1} = e^{-\ln z}$, that is, $\ln(z^{-1}) = -\ln z + 2i\pi n$.

(5) We have

$$\frac{\ln z - \ln z_0}{z - z_0} = \frac{\ln z - \ln z_0}{e^{\ln z} - e^{\ln z_0}} = \left(\frac{e^{\ln z} - e^{\ln z_0}}{\ln z - \ln z_0}\right)^{-1} .$$

Since $\ln z$ is a continuous branch, $z \to z_0$ implies that $\ln z \to \ln z_0$. Since the exponential function is continuous, $\frac{e^{\ln z} - e^{\ln z_0}}{\ln z - \ln z_0}$ converges to its derivative at the point $\ln z_0$ as $\ln z \to \ln z_0$. Thus,

$$\lim_{z\to z_0} \frac{\ln z - \ln z_0}{z - z_0} = (\exp'(\ln z_0))^{-1} = (\exp(\ln z_0))^{-1} = z_0^{-1} .$$

This completes the proof.

Remark. A continuous branch of logarithm on an open set Ω exists if and only if there is a closed half-line originating from the origin which does not intersect Ω.

COMPLEX POWERS

Definition. If $z, w \in \mathbb{C}$, we define $z^w = \exp(w \ln z)$.

Clearly, z^w depends on the choice of the branch of the logarithm and may, generally speaking, take infinitely many values. However, if $w = p/q$ is a real rational number then there are only finitely many values of z^w, corresponding to $\arg w = \operatorname{Arg} w + 2\pi k$ with $k = 1, 2, \ldots, q$.

Note that our definition is consistent with the traditional one when $w = m \in \mathbb{Z}$. Indeed, the product of m copies of $z = e^{\ln z}$ coincides with $z = e^{m \ln z}$, and $z^{-m} = (e^{m \ln z})^{-1} = e^{-m \ln z}$.

Finally, there is no logical contradiction with the notation e^z for the exponential function. Indeed, if we take the principal branch $\ln z = \operatorname{Ln} z$ then $\operatorname{Ln} e = 1$ and $\exp(z \operatorname{Ln} e) = e^z$.

COMPLEX INTEGRATION I

In complex analysis, it is sometimes more convenient to define a path as a continuous map from a nondegenerate interval $[a, b]$ into $\mathbb{C}$ (rather than from the unit interval $[0, 1]$). From now on, we shall be using this definition.

The convention is to take anticlockwise direction as positive (because the polar angle is increasing). The image of the map γ is said to be the *trace* of γ and is denoted by $\operatorname{tr} \gamma$.

Example. $\gamma_1(t) = e^{2i\pi t}$ and $\gamma_2(t) = e^{-4i\pi t}$ with $t = [0, 1]$ have the same trace, the unit circle around the origin. But γ_1 runs once anticlockwise around the origin, and γ_2 goes round twice clockwise.

Definition. A path is said to be *simple* if it does not cross itself (that is, $\gamma(t_1) \neq \gamma(t_2)$ for distinct values $t_1, t_2 \in (a, b)$. The possibility $\gamma(a) = \gamma(b)$ is not excluded; if this is the case, the path is called *closed.*

Definition. If $\gamma : [a, b] \to \mathbb{C}$ is a path, then the map $\tilde{\gamma} : [a, b] \to \mathbb{C}$ given by $\tilde{\gamma}(t) = \gamma(a + b - t)$ is called the *reverse* path.

Obviously, $\tilde{\gamma}$ is "γ in the opposite direction", in particular, $\operatorname{tr} \tilde{\gamma} = \operatorname{tr} \gamma$.

Definition. A path is said to be *smooth* (or continuously differentiable) if the derivative $\gamma'(t) = x'(t) + iy'(t)$ exists and is continuous, with the left and right derivatives, respectively, at the points b and a.

Definition. A *contour* is a piecewise smooth path; that is, a path $\gamma : [a, b] \to \mathbb{C}$ for which there exists a finite collection of numbers $a = a_1 < a_2 < \cdots < a_{n-1} < a_n = b$ such that $\gamma : [a_k, a_{k+1}] \to \mathbb{C}$ are smooth paths.

Remark. The trace of a smooth path may not look like a smooth curve.For example, the path $\gamma(t) = \cos^3(2\pi t) + i \sin^3(2\pi t)$, $t \in [0, 1]$, is smooth. However, its trace has sharp angles at the points ± 1 and $\pm i$.

Definition. The length of a contour γ is $\int_a^b |\dot{\gamma}(t)| \, \mathrm{d}t$, where $\dot{\gamma}$ is the derivative.

This definition is justified by the following observation. If the trace of a path $\gamma : [a, b] \to \mathbb{C}$ is composed from line segments joining points $z_j = \gamma(t_j)$, $t_1 < t_2 < \dots$, then its length is

$$\sum_j |z_{j+1} - z_j| = \sum_j \left| \frac{z_{j+1} - z_j}{t_{j+1} - t_j} \right| (t_{j+1} - t_j) = \sum_j |\dot\gamma(t_j)| \, (t_{j+1} - t_j) = \int_a^b |\dot\gamma(t)| \, \mathrm{d}t \,.$$

A general contour γ can be approximated by such "piecewise linear" paths with $\gamma(t_j)$ being the end points of the line segments. If the distance between t_{j+1} and t_j goes to 0, then $\frac{z_{j+1}-z_j}{t_{j+1}-t_j}$ converges to the derivative $\gamma(t_j)$ (or $\gamma(t_{j+1})$ which is almost the same if $(t_{j+1} - t_j)$ is small). Therefore

$$\sum_j |z_{j+1} - z_j| = \sum_j \left| \frac{z_{j+1} - z_j}{t_{j+1} - t_j} \right| (t_{j+1} - t_j) = \sum_j |\dot\gamma(t_j)| \, (t_{j+1} - t_j)$$

converges to $\int_a^b |\dot\gamma(t_j)| \, \mathrm{d}t$ as $(t_{j+1} - t_j) \to 0$. Thus this integral should be regarded as the length of the contour γ.

Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path and $z_1, z_2, \dots$ be some points on its trace $\mathrm{tr}\gamma$ such that $z_j = \gamma(t_j)$. Consider the sum

$$\sum_j f(z_j)(z_{j+1} - z_j) = \sum_j f(\gamma(t_j)) \, \frac{\gamma(t_{j+1}) - \gamma(t_j)}{t_{j+1} - t_j} \, (t_{j+1} - t_j) \,.$$

If $(t_{j+1} - t_j)$ are small then the right hand side is approximately equal to

$$\sum_j f(\gamma(t_j)) \, \dot\gamma(t_j) \, (t_{j+1} - t_j) \ \approx \ \int_a^b f(\gamma(t)) \, \dot\gamma(t) \, \mathrm{d}t \,.$$

On the other hand, in the right hand side $z_{j+1} - z_j$ is approximately equal to the length of the part γ joining z_{j+1} and z_j, and $f(z_j) \approx f(z)$ for all z lying on this part of the path. This suggests the following definition.

Definition. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path and f be a continuous complex-valued function on a neighbourhood of $\mathrm{tr}\, \gamma$. Then $\int_\gamma f(z) \, \mathrm{d}z := \int_a^b f(\gamma(t)) \, \dot\gamma(t) \, \mathrm{d}t$. If γ is a contour composed of smooth paths $\gamma_k : [a_k, a_{k+1}] \to \mathbb{C}$ then $\int_\gamma f(z) \, \mathrm{d}z := \sum_k \int_{a_k}^{a_{k+1}} f(\gamma(t)) \, \dot\gamma_k(t) \, \mathrm{d}t$.

If $\tilde\gamma$ is the reverse path then $\int_{\tilde\gamma} f(z) \, \mathrm{d}z = -\int_\gamma f(z) \, \mathrm{d}z$.

Example. Let $\gamma(t) = e^{it}$ for $t \in [a, b]$ and $f(z) = z^{-1}$. Then

$$\int_\gamma f(z) \, \mathrm{d}z = \int_\gamma \frac{\mathrm{d}z}{z} = \int_a^b e^{-it} \, ie^{it} \, \mathrm{d}t = i(b - a) \,.$$

In particular, if $a = 0$ and $b = 2\pi$ then $\int_\gamma f(z) = 2\pi i$.

Example. Let γ be the line segment $a + t(b - a)$ where $t \in [0, 1]$. Then, changing variables $\tau = a + t(b - a)$, we obtain $\int_\gamma f(z) \, \mathrm{d}z = \int_0^1 f(a + t(b - a)) \, \mathrm{d}t = \int_a^b f(\tau) \, \mathrm{d}\tau$.

Obviously, $\int_\gamma (\alpha f(z) + \beta g(z))\,\mathrm{d}z = \alpha \int_\gamma f(z)\,\mathrm{d}z + \beta \int_\gamma g(z)\,\mathrm{d}z$ for all continuous functions f and g.

Lemma. The integral $\int_\gamma f(z)\,\mathrm{d}z$ does not depend on the choice of parametrization of the contour. More precisely, if $\gamma : [a,b] \to \mathbb{C}$, $\phi : [\alpha,\beta] \to [a,b]$ with $\phi(\alpha) = a$ and $\phi(\beta) = b$ then $\int_{\gamma_\phi} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z$ where $\gamma_\phi(t) = \gamma(\phi(t))$.

Proof. Clearly, it is sufficient to prove the lemma assuming that γ is a smooth path. Let $F(t)$ be a primitive of $f(\gamma(t))\,\dot\gamma(t)$, that is, a function satisfying $F'(t) = f(\gamma(t))\,\dot\gamma(t)$. Then, by the fundamental theorem of calculus,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\,\dot\gamma(t)\,\mathrm{d}t = F(b) - F(a)\,.$$

On the other hand, applying the chain rule, we obtain

$$\begin{aligned}F(b) - F(a) = F(\phi(\beta)) - F(\phi(\alpha)) &= \int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}s} F(\phi(s))\,\mathrm{d}s \\ = \int_\alpha^\beta F'(\phi(s))\,\phi'(s)\,\mathrm{d}s &= \int_\alpha^\beta f(\gamma(\phi(s)))\,\dot\gamma(\phi(s))\,\phi'(s)\,\mathrm{d}s \\ = \int_\alpha^\beta f(\gamma(\phi(s)))\,\frac{\mathrm{d}}{\mathrm{d}s}\gamma(\phi(s))\,\mathrm{d}s &= \int_\alpha^\beta f(\gamma_\phi(s))\,\dot\gamma_\phi(s)\,\mathrm{d}s = \int_{\gamma_\phi} f(z)\,\mathrm{d}z\,.\end{aligned}$$

Lemma (the basic estimate for integrals). $|\int_\gamma f(z)\,\mathrm{d}z| \leqslant |\gamma| \sup_{z\in \operatorname{tr}\gamma} |f(z)|\,,$ where $|\gamma|$ is the length of the contour γ.

Proof. As before, it is sufficient to prove the lemma assuming that $\gamma : [a,b] \mapsto \mathbb{C}$ is a smooth path. If we knew that $|\int_a^b g(t)\,\mathrm{d}t| \leqslant \int_a^b |g(t)|\,\mathrm{d}t$ for every continuous complex-valued function g then the required result would be obvious because we could estimate

$$|\int_\gamma f(z)\,\mathrm{d}z| = |\int_a^b f(\gamma(t))\,\dot\gamma(t)\,\mathrm{d}t| \leqslant \int_a^b |f(\gamma(t))\,\dot\gamma(t)|\,\mathrm{d}t \leqslant \sup_{z\in\operatorname{tr}\gamma} |f(z)| \int_a^b |\dot\gamma(t)|\,\mathrm{d}t$$

where $\int_a^b |\dot\gamma(t)|\,\mathrm{d}t = |\gamma|$ by definition of the length. Here we have used (and will use later in the proof) the known from real analysis fact that $\int_a^b g_1(t)\,\mathrm{d}t \leqslant \int_a^b g_2(t)\,\mathrm{d}t$ whenever $g_1 \leqslant g_2$.

Thus we only need to prove the estimate $|\int_a^b g(t)\,\mathrm{d}t| \leqslant \int_a^b |g(t)|\,\mathrm{d}t$ for complex-valued g. Using the polar decomposition, we see that $\int_a^b g(t)\,\mathrm{d}t = e^{i\theta} |\int_a^b g(t)\,\mathrm{d}t|$ where θ is an argument of $\int_a^b g(t)\,\mathrm{d}t$. We have

$$|\int_a^b g(t)\,\mathrm{d}t| = e^{-i\theta}\int_a^b g(t)\,\mathrm{d}t = \int_a^b \mathrm{Re}\,\left(e^{-i\theta} g(t)\right)\mathrm{d}t + i\int_a^b \mathrm{Im}\,\left(e^{-i\theta} g(t)\right)\mathrm{d}t\,.$$

The second integral in the right hand side is equal to zero because the left hand side is a real number. Since $\mathrm{Re}\,w \leqslant |w|$, the first integral in the right hand side is not greater than $\int_a^b |e^{-i\theta} g(t)|\mathrm{d}t = \int_a^b |g(t)|\mathrm{d}t$. Therefore $|\int_a^b g(t)\,\mathrm{d}t| \leqslant \int_a^b |g(t)|\mathrm{d}t$.

INTEGRATING UNIFORMLY CONVERGENT SERIES

Recall that a series $f(z) = \sum_{n=0}^{\infty} g_n(z)$ is said to be uniformly convergent on a set Ω if $\sup_{z\in\Omega} |f(z) - f_m(z)| \to 0$ as $m \to 0$ where f_m is the partial sum $f_m(z) = \sum_{n=0}^{m} g_n(z)$. The basic estimate for integrals immediately implies that $\int_\gamma f\,\mathrm{d}z = \sum_{n=0}^{\infty} \int_\gamma g_n\,\mathrm{d}z$ whenever the series converges uniformly on γ. Indeed, we have

$$\left|\int_\gamma f\,\mathrm{d}z - \sum_{n=0}^{m}\int_\gamma g_n \mathrm{d}z\right| = \left|\int_\gamma (f - f_m)\,\mathrm{d}z\right| \leqslant |\gamma|\,(\sup_{z\in\gamma} |f(z) - f_m(z)|) \underset{m\to\infty}{\to} 0.$$

In particular, we can integrate a power series $\sum_{n=0}^{\infty} a_n(z - z_0)$ term by term over any contour lying in the disc $\mathcal{D}(z_0, r)$ provided that r is strictly smaller than the radius of convergence (see Week 3).

PRIMITIVES

The fundamental theorem of calculus for functions of complex variable. If f is differentiable and its derivative f' is continuous in a neighbourhood of the trace $\operatorname{tr}\gamma$ of a contour $\gamma : [a, b] \to \mathbb{C}$ then $\int_\gamma f'(z)\,\mathrm{d}z = f(\gamma(b)) - f(\gamma(a))$.

Proof. If γ is a smooth path then, by the chain rule, $f'(\gamma(t))\,\dot\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))$. Therefore the theorem follows from the corresponding result for functions of one real variable.

Remark. Strictly speaking, we have established the chain rule only for the composition of two functions of complex variable (Week 3). However, the same proof works for $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))$.

Definition. Let f be a complex function defined on an open set $\Omega \subset \mathbb{C}$. A function F defined on the same set Ω is said to be a primitive of f if F is analytic in Ω and $F'(z) = f(z)$ for all $z \in \Omega$.

The fundamental theorem implies that $\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$ for any contour $\gamma : [a, b] \to \mathbb{C}$. In particular, if has a primitive then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for each closed contour γ.

Example. If $f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n$ and $\hat R$ is the radius of convergence then the primitives of f on the disc $\mathcal{D}(z_0, \hat R)$ are $F(z) = C + \sum_{n=0}^{\infty} a_n\,(n + 1)^{-1}\,(z - z_0)^{n+1}$ where C is an arbitrary complex constant. Indeed, the series defining F can be differentiated term by term (see Week 4), which implies that $F' = f$ on $\mathcal{D}(z_0, \hat R)$.

Example. If Ω is an open set which does not contain the origin then $F(z) = (1 - n)^{-1} z^{1-n}$ is a primitive of $f(z) = z^{-n}$ for all $n = 2, 3, \dots$

Remark. Note that the existence of a primitive depends not only on f but also on the set Ω. For instance, if Ω is an open disc which does not contain the origin then a continuous on Ω branch of the logarithm $\ln z$ is a primitive for $f(z) = z^{-1}$. However, the function z^{-1} does not have a primitive on any open set containing the unit circle $\mathbb{S}$ about the origin because $\int_{\mathbb{S}^1} z^{-1}\,\mathrm{d}z \neq 0$ (see above).

DIFFERENTIATION UNDER THE INTEGRAL SIGN

We shall need the following theorem (see Theorem 8.22 in the online lecture notes http://www.mth.kcl.ac.uk/∼ysafarov/Lectures/Past/321.pdf).

Theorem. Let $f(t,\tau)$ be a continuous function of two variables $t \in [a,b]$ and $\tau \in [\alpha,\beta]$. If the derivative $\frac{\partial}{\partial\tau} f$ exists and is continuous then $\int_a^b f(t,\tau)\,\mathrm{d}t$ is a differentiable function of the variable τ, such that its derivative is continuous and is given by $\frac{\mathrm{d}}{\mathrm{d}\tau}\int_a^b f(t,\tau)\,\mathrm{d}t = \int_a^b \frac{\partial}{\partial\tau} f(t,\tau)\,\mathrm{d}t$.

Remark. In particular, the theorem implies a similar result for contour integrals because, by definition, they can be written as integrals over some bounded intervals.

CAUCHY'S INTEGRAL FORMULA

Lemma. Let γ be the anticlockwise oriented circle about z_0. Then $\int_\gamma \frac{\mathrm{d}w}{w-z} = 2\pi i$ for all z lying in the open disc $\mathcal{D}$ bounded by γ.

Proof. If z is inside the open disc $\mathcal{D}$ then $\mathrm{Arg}\,(w-z)$ runs from $-\pi$ to π as w moves along the path γ, and $\mathrm{Arg}\,(w_1-z) \neq \mathrm{Arg}\,(w_2-z)$ for any two distinct points $w_1, w_2 \in \gamma$. Let $w_0 \in \gamma$ and $w_\varepsilon^\pm \in \gamma$ be defined by the equalities $\mathrm{Arg}\,(w_0-z) = \pi$ and $\mathrm{Arg}\,(w_\varepsilon^\pm - z) = \pm(\pi-\varepsilon)$. Clearly, w_0 lies between w_ε^+ and w_ε^-, and $w_\varepsilon^\pm \to w_0$ as $\varepsilon \to 0$. Denote by γ'_ε the part of the path γ starting at w_ε^- and going to w_ε^+. The principal branch of the logarithm $\mathrm{Ln}\,(w-z)$ is analytic in a neighbourhood of γ'_ε. Therefore it is a primitive of $(w-z)^{-1}$ in this neighbourhood, which implies that

$$\int_{\gamma'_\varepsilon} \frac{\mathrm{d}w}{w-z} = \mathrm{Ln}\,(w_\varepsilon^+ - z) - \mathrm{Ln}\,(w_\varepsilon^- - z)$$
$$= \ln|w_\varepsilon^+ - z| + i(\pi-\varepsilon) - \ln|w_\varepsilon^- - z| + i(\pi-\varepsilon)\,.$$

Obviously, the length of the arc γ''_ε going from w_ε^+ to w_ε^- tends to zero as $\varepsilon \to 0$. By the basic estimate,

$$\left|\int_\gamma \frac{\mathrm{d}w}{w-z} - \int_{\gamma'_\varepsilon} \frac{\mathrm{d}w}{w-z}\right| \;=\; \left|\int_{\gamma''_\varepsilon} \frac{\mathrm{d}w}{w-z}\right| \;\leqslant\; |\gamma''_\varepsilon| \sup_{w\in\gamma_r} (w-z||^{-1} \;\underset{\varepsilon\to 0}{\to}\; 0\,.$$

Therefore, letting $\varepsilon \to 0$, we obtain

$$\int_\gamma \frac{\mathrm{d}w}{w-z} \;=\; \lim_{\varepsilon\to 0}\int_{\gamma'_\varepsilon} \frac{\mathrm{d}w}{w-z} \;=\; \ln|w_0-z| + i\pi - \ln|w_0-z| + i\pi = 2\pi i\,.$$

Remark. Note that, under the conditions of the Lemma, we have $\int_\gamma (w-z)^n\,\mathrm{d}w = 0$ for all integer $n \neq -1$. Indeed, if $n \neq -1$ then the function $(w-z)^n$ has the primitive $(n+1)^{-1}(w-z)^{n+1}$ in a neighbourhood of γ.

Theorem (Cauchy's integral formula for a disk). Assume that f is analytic in an open disc $\mathcal{D}(z_0, \hat{R})$ and that f' is continuous. Let γ_r be an anticlockwise oriented circle about z_0 of radius $r < \hat{R}$. Then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, \mathrm{d}w \qquad \text{for all } z \text{ such that } |z - z_0| < r.$$

Proof. Let us fix a point z and define $g(\tau) = \int_\gamma \frac{f(z+\tau(w-z))}{w-z}\, \mathrm{d}w$. Differentiating under the integral sign, we obtain $g'(\tau) = \int_\gamma f'(z + \tau(w-z))\, \mathrm{d}w$. For each $\tau \neq 0$, the function $f'(z + \tau(w-z))$ has the primitive $F(w) = \tau^{-1} f(z + \tau(w-z))$ which is an analytic function in $\mathcal{D}(z_0, \hat{R})$ (as a function of the variable w). Therefore $g'(\tau) = 0$ for all $\tau \neq 0$. Since g' is continuous, we also have $g'(0) = 0$. Thus g is constant on the interval $[0, 1]$. This implies that

$$\int_\gamma \frac{f(w)}{w-z}\, \mathrm{d}w = g(1) = g(0) = f(z) \int_\gamma \frac{1}{w-z}\, \mathrm{d}w = 2\pi i\, f(z)\,.$$

Applying Cauchy's integral formula to the function $\tilde{f}(w) = (w-z)\, f(w)$, we obtain

Corollary (Cauchy's theorem for a disk). Under the conditions of the previous theorem $\int_\gamma f(w)\, \mathrm{d}w = 0$.

Remark. It will be shown later that the derivative of any analytic function is continuous and, therefore, we do not have to assume the continuity of f' in the above and further theorems. However, the proof of this result is rather complicated.

POWER EXPANSION FOR AN ANALYTIC FUNCTION

Theorem. Assume that f is analytic in the disc $\mathcal{D}(z_0, \hat{R})$ with continuous derivative f'. Let $r < \hat{R}$ and γ_r be the anticlockwise oriented circle around z_0 of radius r. Then $f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n$ for all $z \in \mathcal{D}(z_0, r)$, where $a_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-z_0)^{n+1}}$ and the series is absolutely and uniformly convergent in the disc $\mathcal{D}(z_0, r)$.

Proof. If $z \in \mathcal{D}(z_0, r)$ and $w \in \operatorname{tr} \gamma_r$ then $|z - z_0| < |w - z_0|$ and, by the geometric progression formula,

$$\sum_{n=0}^\infty \frac{(z-z_0)^n}{(w-z_0)^{n+1}} = \frac{1}{w-z}\,.$$

The series in the right hand side is absolutely convergent. The general theorem about absolutely convergent power series implies that it converges uniformly with respect to $w \in \operatorname{tr} \gamma_r$ (see the first page of notes on Week 3). Since $f(w)$ is bounded on $\operatorname{tr} \gamma_r$, the same is true for the series

$$\sum_{n=0}^\infty \frac{f(w)\,(z-z_0)^n}{(w-z_0)^{n+1}} = \frac{f(w)}{w-z}\,.$$

Integrating it term by term, we obtain

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-z)}\, \mathrm{d}w = \frac{1}{2\pi i} \int_{\gamma_r} \left(\sum_{n=0}^\infty \frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}} \right) \mathrm{d}w = \sum_{n=0}^\infty a_n (z - z_0)^n\,.$$

Now the Cauchy's integral formula implies that $f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n$. By the basic estimate,

$$|a_n| \;=\; \left|\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(w)}{(w-z_0)^{n+1}}\right| \;\leqslant\; (2\pi)^{-1}|\gamma_r|\,r^{-n-1}\sup_{w\in\gamma_r}|f(w)| = r^{-n}\sup_{w\in\gamma_r}|f(w)|\,.$$

Therefore the radius of convergence of the series $\sum_{n=0}^{\infty} a_n\,(z-z_0)^n$ is equal to r, which implies that it is absolutely and uniformly convergent in the disc $\mathcal{D}(z_0, r')$ for any $r' < r$.

Corollary. Under the conditions of the theorem, the n-th derivative $f^{(n)}$ satisfies

$$f^{(n)}(z_0) \;=\; n!\,a_n \;=\; \frac{n!}{2\pi i}\int_{\gamma_r}\frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w\,, \qquad \forall r < \hat{R}\,.$$

Proof is obtained by differentiating the series and putting $z = z_0$.

Corollary (Cauchy's estimate). Under the conditions of the theorem, $|f^{(n)}(z_0)| \;\leqslant\; n!\,r^{-n}\sup_{w\in\operatorname{tr}\gamma_r}|f(w)|$ for all $r < \hat{R}$.

Proof. The inequality immediately follows from the previous corollary and the basic estimate.

The first corollary shows that a_n do not depend on r and are equal to $f^{(n)}(z_0)/n!$. Thus we obtain Taylor's expansion expansion

$$f(z) \;=\; \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}\,(z-z_0)^n\,,$$

where the series is absolutely and uniformly convergent in the disc $\mathcal{D}(z_0, r)$ for any $r < \hat{R}$.

SOME COROLLARIES

Liouville's Theorem. A bounded entire function is constant.
Proof: see solutions to Exercise Sheet 5 or prove it yourselves.
Hint: apply Cauchy's estimate and let $r \to \infty$.

Fundamental Theorem of Algebra. Every nonconstant polynomial has a root in $\mathbb{C}$ (that is, vanishes at some point $z \in \mathbb{C}$).
Proof: see solutions to Exercise Sheet 5 or prove it yourselves.
Hint: this follows from Liouville's Theorem.

Theorem. Let f be analytic in an open connected set Ω with continuous derivative f'. If there exists a point $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all $n = 0, 1, 2, \ldots$ then f is identically equal to zero.

Proof. Let $f^{(n)}(z_0) = 0$ for all $n = 0, 1, 2, \ldots$ Then, by Taylor's expansion, $f = 0$ on an open disc about z_0. Let us take an arbitrary point $z \in \Omega$ and join it with z_0 by a path $\gamma : [a, b] \to \Omega$, such that $\gamma(0) = z_0$ and $\gamma(1) = z$. Let t_0 be the supremum of the set of $t \in [0, 1]$ such that $f^{(n)}(\gamma(t)) = 0$ for all n. Then, by continuity, $f^{(n)}(\gamma(t_0)) = 0$ for all n and $f = 0$ on a disc about $\gamma(t_0)$. This implies that $t_0 = 1$, that is, $f(z) = 0$.

Corollary. Let the conditions of the previous theorem be fulfilled, and let f is not identically zero. If $f(z_0) = 0$ at some point z_0 then there exist a positive integer m and an analytic in Ω function g such that $f(z) = (z - z_0)^m g(z)$ and $g(z_0) \neq 0$.

Proof. Let m be the smallest positive integer such that $f^{(m)}(z_0) \neq 0$. Taylor's expansion implies that the function $g(z) = (z - z_0)^{-m} f(z)$ is analytic in a disc about z_0. Since $(z - z_0)^{-m}$ is analytic outside this disc, g is analytic in Ω.

The number p is called the *multiplicity* of the root z_0. In other words, the corollary says that every root of a complex function has a finite multiplicity.

Corollary. Let the conditions of the previous theorem be fulfilled. Then the roots of the analytic function f are isolated (that is, the set of roots does not have an accumulation point) unless f is identically equal to zero.

Proof. Assume that there is a sequence of numbers z_k such that $f(z_k) = 0$ and $z_k \to z_0$ as $k \to \infty$. Expanding f into Taylor's series at z_0 and applying the identity theorem for series, we see that $f = 0$ on a disc about z_0. Then $f^{(n)}(z_0) = 0$ for all n and $f = 0$ everywhere on Ω.

Corollary. If P is an entire function such that $|P(z)| \leqslant C\,(|z| + 1)^n$ with some constants $C > 0$ and $n \in \mathbb{N}$ then $P(z)$ is a polynomial.

Proof is by induction in n. Assume that the result holds for all $n' < n$, and consider an entire function P satisfying the above estimate. Let $P(0) = a$. Then $P(z) - a$ is an entire function which satisfies the estimate $|P(z) - a| \leqslant (C + |a|)\,(|z| + 1)^n$. The point $z = 0$ is a root of the function $P(z) - a$. Therefore $P(z) - a = z^m\, P_a(z)$ where m is the multiplicity of the root and P_0 is another entire function. Since $|z^m\, P_0(z)| \leqslant (C + |a|)\,(|z| + 1)^n$, we have $|P_0(z)| \leqslant C_0\,(|z| + 1)^{n-m}$ with some other constant C_0. By the induction assumption, P_0 is a polynomial, and thus $P(z) = z^m\, P_0(z) + a$ is also a polynomial.

Corollary. A polynomial $P(z)$of degree n can be written as the product

$$P(z) \;=\; c\,(z-z_1)^{m_1}(z-z_2)^{m_2}\dots(z-z_k)^{m_k}\,,$$

where z_j are the roots of P of multiplicity m_j, and is a constant.

Proof. Assume that the result holds for all $n' < n$, and consider a polynomial P of degree n. By Fundamental Theorem of Algebra, it has at least one root z_1. Then $P(z) = (z-z_1)^{m_1}P_1(z)$ where m_1 is the multiplicity of the root z_1 and P_1 is an entire function. Clearly, $|P_1(z)| \leqslant C\,(|z|+1)^n$ with some constant C and, consequently, P_1 is a polynomial. Applying the induction assumption to P_1, we see that P can also be represented as required.

The maximum modules principle. Let f be analytic in some open connected set Ω. If $|f|$ attains the maximum value at some point $z\in\Omega$ then f is constant.

Proof. Assume that $|f|$ attains the maximum value M at some point $a\in\Omega$. If $\mathcal{D}(a,r)\subset\Omega$ then, by Cauchy's formula, $f(a) = \frac{1}{2\pi i}\int_{\gamma_r}\frac{f(z)\,\mathrm{d}z}{z-a}$ where γ_r is the boundary of $\mathcal{D}(a,r)$. Since $|\int g| \leqslant \int |g|$ (see Week 6) and $|z-a| = r$ for all $z\in\gamma_r$, this implies that

$$M \;=\; |f(a)| \;\leqslant\; \frac{1}{2\pi}\int_{\gamma_r}\frac{|f(z)|\,\mathrm{d}z}{|z-a|} \;\leqslant\; \frac{1}{2\pi r}\int_{\gamma_r}|f(z)|\,\mathrm{d}z\,.$$

Note that $|\gamma_r| = 2\pi r$, so that the right hand side coincides with the average of $|f|$ over γ_r.

The above estimate implies that $|f(z)| = M$ for all $z\in\gamma_r$. Indeed, if $|f(w)| < M$ for some $w\in\gamma_r$ then, by continuity, $|f|\leqslant M-\varepsilon$ with some $\varepsilon>0$ on an open arc γ' containing w. If γ'' is the remaining part of γ_r then, by the basic estimate,

$$\frac{1}{2\pi r}\int_{\gamma_r}|f(z)|\,\mathrm{d}z = \frac{1}{2\pi r}\left(\int_{\gamma'}|f(z)|\,\mathrm{d}z + \int_{\gamma''}|f(z)|\,\mathrm{d}z\right) \leqslant (M-\varepsilon)\,\frac{|\gamma'|}{2\pi r}| + M\,\frac{|\gamma''|}{2\pi r}$$

which is a contradiction because $|\gamma'|+|\gamma''| = |\gamma| = 2\pi r$ and, consequently, the right hand side is strictly smaller that M.

Thus we see that $|f| = M$ on any circle in Ω about any point $a\in\Omega$, at which $|f(a)| = M$. Consequently, $|f| = M$ on any disc about a lying in Ω. This implies that the set of points $a\in\Omega$ such that $|f(a)| = M$ is open. Denote this set by Ω_1.

On the other hand, due to the continuity of the real valued function $|f|$, if $|f(a)|\neq M$ then $|f(z)|\neq M$ for all z lying in a sufficiently small disc about a. This implies that the set $\Omega_2 = \{a\in\Omega : |f(a)|\neq M\}$ is also open. Clearly, $\Omega_1\bigcap\Omega_2 = \emptyset$ and $\Omega_1\bigcup\Omega_2 = \Omega$. Since Ω is connected, one of these sets must be empty (otherwise we would obtain a disconnection of Ω). The set Ω_1 is not empty because, by our assumption, $|f|$ attains the maximum value at some point. Thus $\Omega_2 = \emptyset$, that is, $|f(z)| = M$ for all $z\in\Omega$. Therefore, by the corollary from Week 4, the function f is constant.

HOMOTOPY OF PATHS AND CAUCHY'S THEOREM

Let $\gamma_0,\gamma_1 : [0,1]\Omega$ be two closed paths in an open set $\Omega\in\mathbb{C}$. We say that the paths are homotopic in Ω if there exists a continuous function $\Gamma : [0,1]\times[0,1]\to\Omega$ such that for each $\Gamma(t,s)$ is a closed path for each fixed $s \in [0,1]$ with $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$. The homotopy G can be thought of as a continuous transformation γ_0 into γ_1. If two paths a homotopic in Ω, we write $\gamma_0 \sim \gamma_1$.

Theorem. If f is an analytic function in an open set $\Omega \subset \mathbb{C}$ with continuous derivative and $\gamma_0 \sim \gamma_1$ are homotopic closed paths in Ω then $\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$.

Proof. Assume that the function Γ is two times continuously differentiable. Let $g(s) = \int_{\Gamma(s,t)} f(z)\,\mathrm{d}z = \int_0^1 f(\Gamma(s,t))\,\partial_t\Gamma(s,t)\,\mathrm{d}t$. Then, by the chain rule,

$$\partial_s\left(f(\Gamma(t,s))\,\partial_t\Gamma(t,s)\right) = \partial_t\left(f(\Gamma(t,s))\,\partial_s\Gamma(t,s)\right)$$

where ∂_t and ∂_s denote the partial derivatives. Therefore

$$g'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^1 f(\Gamma(s,t))\,\partial_t\Gamma(s,t)\,\mathrm{d}t = \int_0^1 \partial_s\left(f(\Gamma(s,t))\,\partial_t\Gamma(s,t)\right)\mathrm{d}t$$

$$= \int_0^1 \partial_t\left(f(\Gamma(s,t))\,\partial_s\Gamma(s,t)\right)\mathrm{d}t = f(\Gamma(s,1))\,\partial_s\Gamma(s,1) - f(\Gamma(s,0))\,\partial_s\Gamma(s,0) = 0$$

because the paths $\Gamma(s,t)$ are closed. Thus g is a constant function.

The proof without assuming differentiability is obtained by approximation arguments.

CAUCHY'S THEOREM

We have proved the following result: If γ_0 and γ_1 are closed paths which are homotopic in Ω then $\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$ for every function f analytic in Ω. In particular, this implies

Cauchy's theorem. If f is analytic on an open set Ω with continuous derivative f' then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed contour γ which is homotopic in Ω to one point.

An open set $\Omega \in \mathbb{C}$ is said to be *simply connected* if every closed contour in Ω is homotopic to a point. In particular, a convex domain Ω are simply connected. Indeed, let $\gamma : [0,1] \to \Omega$ be a closed path in a convex domain Ω and z_0 be an arbitrary point of Ω. Then $\Gamma(t,s) = (1-s)\gamma_0(t) + sz_0$ is a homotopy in Ω of γ to the constant path $\gamma(t) \equiv z_0$.

Example. An annulus is connected but not simply connected not simply connected. The union of two open discs is simply connected but not connected.

The previous corollary immediately implies the following result.

Corollary. If f is analytic on an open simply connected set Ω and f' is continuous then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed contour γ in Ω.

WINDING NUMBER

Definition. Let γ be a piecewise smooth closed curve in $\mathbb{C}$ and $a \in \mathbb{C} \setminus a$. Then the *index* of γ with respect to a, or the *winding number* of γ about a is defined to be $n(\gamma;a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z$.

Intuitive meaning of winding number: if a pole is planted in the plane at a point a and a closed curve is drawn in the plane not meeting a, if the curve were made of well-lubricated rubber and were to be contracted as small as possible, if it contracts to a point other than a its winding number is 0. If it contracts around the pole planted at a, its winding number is the number of times it circles the pole (signed by orientation).

Lemma. The index $n(\gamma; a)$ is an integer for all $a \in \mathbb{C}$.

Proof. Let γ be parameterized by $[0, 1]$. Define $g(t) = \int_0^t \frac{\dot\gamma(s)}{\gamma(s)-a}\, \mathrm{d}s$. A direct calculation, using the chain rule and the fundamental theorem of calculus, shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((\gamma(t) - a)\, e^{-g(t)}\right) = 0\,.$$

Therefore $(\gamma(0) - a) = (\gamma(t) - a)\, e^{-g(t)}$ for all $t \in [0, 1]$. Taking $t = 1$, we obtain

$$(\gamma(0) - a) = (\gamma(1) - a)\, e^{-g(1)} = (\gamma(0) - a)\, e^{-g(1)} = (\gamma(0) - a)\ e^{2\pi i\, n(\gamma;a)}$$

which implies that $e^{2\pi i\, n(\gamma;a)} = 1$, that is, $n(\gamma; a)$ is an integer.

Remark. The function $n(\gamma; a)$ is obviously continuous with respect to a and, being integer, it is constant on connected components of $\mathbb{C} \setminus \gamma$. Clearly also, it tends to 0 as $a \to \infty$, so it is identically 0 on the unbounded component.

Theorem (Cauchy's Integral Formula). Let f be analytic in an open set $\Omega \in \mathbb{C}$, and let $\gamma \subset \Omega$ be a contour homotopic to a point in Ω. Then

$$n(\gamma; a)\, f(a) \;=\; \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, \mathrm{d}z$$

for each $a \in \Omega$ such that $a \notin \gamma$.

Proof. Let

$$g(z) \;=\; \begin{cases} \frac{f(z)-f(a)}{z-a}\,, & z \neq a\,, \\ f'(z)\,, & z = a\,. \end{cases}$$

It is clear from Taylor's expansion that g is an analytic function in Ω. By Cauchy's theorem, $\int_\gamma g(z)\, \mathrm{d}z = 0$. This implies the theorem.

ZEROS OF ANALYTIC FUNCTIONS

Zero counting theorem. Let γ be a contour homotopic to a point in Ω, and let f be an analytic function in Ω with zeros $a_1, a_2, \ldots, a_p$ of multiplicities $m_1, m_2, \ldots, m_p$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z \;=\; \sum_{k=1}^{p} m_k\, n(\gamma, a_k)\,.$$

Remark. The sum in the right hand side is usually referred to as "the number of zeros of the function f". In other words, when counting zeroes, one assumes that a zero of multiplicity m is counted m times.

Proof of the zero counting theorem. We have

$$f(z) = (z-a_1)^{m_1}(z-a_2)^{m_2}\ldots(z-a_p)^{m_p}\,g(z)$$

where g is an analytic function which does not vanish on Ω. Then

$$\frac{f'(z)}{f(z)} \;=\; \frac{m_1}{z-a_1} + \frac{m_2}{z-a_1} + \cdots + \frac{m_p}{z-a_p} + \frac{g'(z)}{g(z)}$$

and the theorem follows from Cauchy's integral formula.

Rouché Theorem. Let Ω be a simply connected set, and let γ be a simple closed contour in Ω. If f and g are analytic on Ω and $|f(z)| > |g(z)|$ for all $z \in \gamma$ then f and $f+g$ have the same number of zeroes in the domain bounded by γ.
Proof. The inequality $|f| > |g|$ implies that $f \neq 0$ on γ and that the quotient $h = \frac{f+g}{f}$ does not take real nonpositive values on γ. It follows that the principal branch of logarithm $\operatorname{Ln} h$ is analytic on a neighbourhood of γ. Since $(\operatorname{Ln} h)' = \frac{h'}{h}$, the fundamental theorem of calculus implies that $\int_\gamma \frac{h'}{h}\,\mathrm{d}z = 0$. Substituting

$$\frac{h'}{h} \;=\; \frac{(f+g)'f - (f+g)f'}{f^2}\,\frac{f}{f+g} \;=\; \frac{(f+g)'}{f+g} - \frac{f'}{f},$$

we see that

$$\int_\gamma \frac{(f+g)'}{f+g}\,\mathrm{d}z = \int_\gamma \frac{f'}{f}\,\mathrm{d}z\,.$$

Now the required result follows from the zero counting theorem.

Remark. If $a_1, a_2, \ldots, a_p$ are the roots of the equation $f(z) = \alpha$ and $m_1, m_2, \ldots, m_p$ are their multiplicities then, applying the previous theorem to the function $f(z)-\alpha$, we obtain

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}\,\mathrm{d}z \;=\; \sum_{k=1}^{p} m_k\, n(\gamma, a_n)\,.$$

Theorem. Let f be analytic in Ω. If $z = z_0$ is a root of the equation $f(z) = \zeta_0$ with finite multiplicity $m \geqslant 1$, then there exists $\varepsilon, \delta > 0$ such that for all $\zeta \in \mathcal{D}(\zeta_0, \delta)$, the equation $f(z)=\zeta$ has precisely m roots in $\mathcal{D}(z_0,\varepsilon)$ and all the roots are simple.

Proof. That the number of roots is m comes from the continuity of the integral with respect to ζ, and the fact that it is integer valued. We can insure that the roots are simple by taking ε small enough to avoid a root of f'.

Corollary. A nonconstant analytic function maps an open set into an open set.
Proof. Let Ω be an open set and $f(\Omega)$ be its image. Let ζ_0 be an arbitrary point of $f(\Omega)$ and z_0 be a point in Ω such that $f(z_0)=\zeta_0$. By the above theorem, there is $\delta>0$ such that the equation $f(z)=\zeta$ has a root for each $\zeta\in\mathcal{D}(\zeta_0,\delta)$. This implies that $\mathcal{D}(\zeta_0,\delta)\subset f(\Omega)$.

PRIMITIVES AND DERIVATIVES OF ANALYTIC FUNCTIONS

Lemma. If a continuous function f has a primitive on an open set Ω then f is analytic on Ω with continuous derivative f'.
Proof. The primitive F is an analytic function with continuous derivative $F'=f$. Therefore, by a theorem of Week 8, $F(z)=\sum_{n=0}^{\infty}a_n\,(z-z_0)^n$ on every disc $\mathcal{D}\subset\Omega$, where z_0 is the centre of $\mathcal{D}$. Since we can differentiate power series term by term (see Week 4), the functions f' is also given by a power series on $\mathcal{D}$. This implies that f' is analytic and, consequently, is continuous (see Week 3).

Morera's theorem. Let f be a continuous function on an open simply connected set Ω. If $\int_\gamma f(w)\,\mathrm{d}w=0$ for any triangular contour γ in Ω then f has a primitive on Ω.
Proof. Consider a disc $\mathcal{D}\subset\Omega$ be a disc centred at a, and let $\gamma_{z_1,z_2}(t)=z_1+t(z_2-z_1)$ be the line segment joining the points $z_1,z_2\in\mathcal{D}$. Denote $F_{\mathcal{D}}(z)=\int_{\gamma_{a,z}}f(w)\,\mathrm{d}w$. If $z,z_0\in\Omega$ then the line segments γ_{a,z_0}, $\gamma_{z_0,z}$, $\gamma_{z,a}$ form a triangular contour $\gamma\subset\Omega$ and, by our assumption,

$$\int_{\gamma_{a,z_0}}f(w)\,\mathrm{d}w+\int_{\gamma_{z_0,z}}f(w)\,\mathrm{d}w\;=\;-\int_{\gamma_{z,a}}f(w)\,\mathrm{d}w\;=\;\int_{\gamma_{a,z}}f(w)\,\mathrm{d}w\,.$$

It follows that

$$\frac{F_{\mathcal{D}}(z)-F_{\mathcal{D}}(z_0)}{z-z_0}=\frac{1}{z-z_0}\int_{\gamma_{z_0,z}}f(w)\,\mathrm{d}w=\int_0^1 f(z_0+(1-t)\,(z-z_0))\,\mathrm{d}t\,.$$

Since f is continuous, letting $z\to z_0$ we see that $F_{\mathcal{D}}$ is analytic on $\mathcal{D}$ and $F'_{\mathcal{D}}=f$. Thus we see that f has a primitive $F_{\mathcal{D}}$ on any open disc $\mathcal{D}\in\Omega$ and, consequently, f' is continuous.

Now let us fixed an arbitrary point z_k in each connected component Ω_k of the set Ω and define $F(z)=\int_\gamma f(w)\,\mathrm{d}w$ for all other $z\in\Omega_k$, where γ is an arbitrary path joining z_k and z. Note that $f(z)$ does not depend on the choice of γ because, by Cauchy's theorem, the integral of f over any closed contour in Ω is equal to zero (see Week 9). Moreover, $F(z)-F(a)=\int_{\gamma'_{a,z}}f(w)\,\mathrm{d}w$ for all $a,z\in\Omega_k$ where $\gamma'_{a,z}$ is an arbitrary path in Ω_k joining a and z. On the other hand, if z lie in a disc $\mathcal{D}$ centred at a then $\int_{\gamma'_{a,z}}f(w)\,\mathrm{d}w=F_{\mathcal{D}}(z)$. This implies that F is analytic on Ω and $F'(z)=f(z)$ for all $z\in\Omega$.

Goursat's theorem. Every analytic function f on an open simply connected set Ω has a primitive.

Proof. By Morera's theorem, it is sufficient to show that the integral of f over the anticlockwise oriented boundary γ_0 of an arbitrary triangle $\Delta_0 \subset \Omega$ is equal to zero.

Let us fix such a triangle Δ_0 and join the middle points of its edges by line segments. Then we obtain a partition of Δ_0 into the union of four triangles $\Delta'_{0,k}$ with anticlockwise oriented boundaries $\gamma'_{0,k}$, where $k = 1, 2, 3, 4$. Since the diameter of a triangle is equal to the length of its longest edge, we have $\operatorname{diam} \Delta'_{0,k} \leqslant \frac{1}{2} \operatorname{diam} \Delta_0$ for all k. Also, $|\gamma'_{0,k}| \leqslant \frac{1}{2} |\gamma_0|$ for each k because the length of each edge of $\Delta'_{0,k}$ is half of the length of the parallel edge of Δ_0. Finally,

$$\int_{\gamma_0} f(z)\, \mathrm{d}z \;=\; \sum_{k=1}^{4} \int_{\gamma_{0,k}} f(z)\, \mathrm{d}z$$

because the integral over each added internal edge appears in the above sum twice with opposite directions. Since

$$\left| \int_{\gamma_0} f(z)\, \mathrm{d}z \right| \;\leqslant\; \sum_{k=1}^{4} \left| \int_{\gamma_{0,k}} f(z)\, \mathrm{d}z \right|,$$

we have $\left| \int_{\gamma_{0,k}} f(z)\, \mathrm{d}z \right| \geqslant \frac{1}{4} \left| \int_{\gamma_0} f(z)\, \mathrm{d}z \right|$ for at least one of the smaller triangles $\Delta'_{0,k}$. Let us denote this triangle by Δ_1, and let γ_1 be its anticlockwise oriented boundary.

Now, applying the same procedure to the triangle Δ_1, we find a triangle $\Delta_2 \subset \Delta_1$ with boundary γ_2 such that

$$\operatorname{diam} \Delta_2 \;\leqslant\; \frac{1}{2} \operatorname{diam} \Delta_1 \;\leqslant\; \left(\frac{1}{2}\right)^2 \operatorname{diam} \Delta_0 , \qquad |\gamma_2| \;\leqslant\; \frac{1}{2} |\gamma_1| \;\leqslant\; \left(\frac{1}{2}\right)^2 |\gamma_0|$$

and

$$\left| \int_{\gamma_2} f(z)\, \mathrm{d}z \right| \;\geqslant\; \frac{1}{4} \left| \int_{\gamma_1} f(z)\, \mathrm{d}z \right| \;\geqslant\; \left(\frac{1}{4}\right)^2 \left| \int_{\gamma_0} f(z)\, \mathrm{d}z \right|.$$

Iterating, we obtain a family of embedded triangles Δ_n with boundaries γ_n such that $\Delta_{n+1} \subset \Delta_n$, $\operatorname{diam} \Delta_n \leqslant 2^{-n} \operatorname{diam} \Delta_0$, $|\gamma_n| \leqslant 2^{-n} |\gamma_0|$ and

$$4^n \left| \int_{\gamma_n} f(z)\, \mathrm{d}z \right| \;\geqslant\; \left| \int_{\gamma_0} f(z)\, \mathrm{d}z \right|.$$

Denote by z_0 their intersection (it exists and is unique, since $\operatorname{diam} \Delta_n \to 0$).

Since f is differentiable, for every $\varepsilon > 0$ we have

$$|f(z) - f(z_0) - (z - z_0)\, f'(z_0)| \;\leqslant\; \varepsilon\, |z - z_0|$$

for all z lying in a sufficiently small disc about z_0. This implies that the above inequality holds for all $z \in \Delta_n$ provided that n is large enough.

Since polynomials have primitives, $\int_{\gamma_n} (f(z_0) + (z - z_0)\, f'(z_0))\ \mathrm{d}z = 0$ for all closed contours γ_n. Now, applying the basic estimate and the above inequalities, we see that

$$\begin{aligned}\left|\int_{\gamma_0} f(z)\,\mathrm{d}z\right| \;&\leqslant\; 4^n \left|\int_{\gamma_n} f(z)\,\mathrm{d}z\right| \;=\; 4^n \left|\int_{\gamma_n} (f(z) - f(z_0) - (z - z_0)\, f'(z_0))\,\mathrm{d}z\right| \\ &\leqslant\; 4^n\,|\gamma_n| \sup_{z\in\Delta_n} |f(z) - f(z_0) - (z - z_0)\, f'(z_0)| \;\leqslant\; \varepsilon\, 4^n\,|\gamma_n| \sup_{z\in\Delta_n} |z - z_0| \\ &\leqslant\; \varepsilon\, 4^n\,|\gamma_n|\,\mathrm{diam}\,\Delta_n \;\leqslant\; \varepsilon\,|\gamma_0|\,\mathrm{diam}\,\Delta_0\end{aligned}$$

for all sufficiently large n. Since ε can be chosen arbitrarily small, this implies that $\int_{\gamma_0} f(z)\,\mathrm{d}z = 0$.

PRIMITIVES AND DERIVATIVES: A SUMMARY

Recall that in the last lecture we have proved that

(1) if a continuous function has a primitive then it is analytic with continuous derivative;

(2) if the integral of a continuous (or an analytic) function f over any triangular contour in a simply connected domain Ω is equal to zero then f has a primitive (Morera's theorem);

(3) if Ω is simply connected and f is analytic on Ω then the integral of f over any triangular contour in Ω is equal to zero and, consequently, f has a primitive (Goursat's theorem).

Putting (1)-(3) together, we obtain the folowing corollaries

Corollary. If f is analytic then f' is continuous.

Proof. Since a disc is simply connected, (2) and (3), f has a primitive on every disc. Now (1) implies that f' is continuous.

Corollary. A continuous function f on a simply connected domain is analytic if and only if $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed contour γ in Ω.

Proof. If f is analytic then, by (1), it has a primitive and, consequently, $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all closed contours γ in Ω. On the other hand, if $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all closed contours then, by (2), f has a primitive and is therefore analytic.

Remark. Another necessary and sufficient condition for a function to be analytic in an open set Ω is that its Taylor's series at any point $z_0 \in \Omega$ is absolutely convergent in an open disc about z_0.

Remark. Morera's and Goursat's theorem may not be true if Ω is not simply connected. For instance, the function z^{-1} is analytic on an open annulus Ω about the origin and its integrals over all triangular contours in Ω are equal to zero. However, it does not have a primitive on Ω.

SINGULARITIES OF ANALYTIC FUNCTIONS

We shall say that $f(z)$ has an isolated singularity at the point $z = z_0$ if f is analytic in a punctured disc $\mathcal{D}(z_0, \varepsilon) \setminus z_0$ but is not analytic at z_0.

Example. If f is an analytic function then $(z - z_0)^{-1} f(z)$ has an isolated singularity at $z = z_0$. The Cauchy's integral formula shows that, in this case, $\int_\gamma f(z)\,\mathrm{d}z$ may not be equal to zero if γ is a contour going around z_0 (it is not surprising as the sets $\mathcal{D}(z_0, \varepsilon) \setminus z_0$ are not simply connected for all $\varepsilon > 0$).

Assume that f has an isolated singularity at $z = z_0$. Then there are three possibilities:

(a) the limit $\lim_{z\to z_0} |f(z)|$ exists as a finite real number,
(b) $\lim_{z\to z_0} |f(z)| = \infty$,
(c) the limit $\lim_{z\to z_0} |f(z)|$ does not exist as a finite number or ∞.

The first case is that of a *removable singularity*. The second is called *a pole*, the third is *an essential singularity*.

If f has a removable singularity then one can extend f to z_0 in such a way that the new function is analytic in z_0. Indeed, let $g(z) = (z - z_0)f(z)$. The basic estimate for contour integrals implies that $\int_\gamma g(z)\,\mathrm{d}z \to 0$ as γ shrinks to z_0. Since any closed contour in $\mathcal{D}(z_0, r)$ going around z_0 is homotopic to a closed contour lying in an arbitrarily small disc about z_0, the Morera's theorem implies that $g(z)$ is analytic in some disc $\mathcal{D}(z_0, \varepsilon)$. Therefore $g(z) = (z - z_0)\,h(z)$ with an analytic function $h(z)$. Clearly, h is the required extension of f. Note that the same arguments work if f is bounded in a disc about z_0. Therefore f has a removable singularity if and only if it is bounded near z_0.

If f has a pole at z_0 then the function $1/f(z)$ has a removable singularity at z_0. By the above, it can be extended to an analytic function $g(z)$ on a disc about z_0. Clearly, $g(z_0) = \lim_{z\to z_0}(1/f(z)) = 0$, that is, z_0 is a root of g. If m is the multiplicity of this root then $1/f(z) = (z - z_0)^m g_0(z)$ where $g_0(z)$ is an analytic function such that $g_0(z_0) \neq 0$. Thus $f(z) = (z - z_0)^{-m}h(z)$, where $h(z) = 1/g_0(z)$ is analytic near z_0.

Theorem (Casorati–Weierstrass). If f has an essential singularity at z_0, then the image under f of any punctured disk around z_0 is dense in $\mathbb{C}$ (in other words, for every open disc $\mathcal{D}$ there is a point z near z_0 such that $f(z) \in \mathcal{D}$).

Proof. For the sake of simplicity, let us assume that $z_0 = 0$ (otherwise one can consider the function $f_0(z) = f(z - z_0)$ instead of f). If the conclusion is false, there is a punctured disk around 0 in which $f(z)$ stays a fixed positive distance ε away from some number $c \in \mathbb{C}$. Consider the function $g(z) = (f(z) - c)/z$. It tends to ∞ as $z \to 0$, so it has a pole at 0. Therefore $z^m[f(z) - c] \to 0$ as $z \to 0$ for a sufficiently large m. This implies that $z^m f(z) \to 0$ and, consequently, $z^m f(z)$ has a removable singularity. By the above, in this case $f(z) = z^k h(z)$ with analytic $h(z)$ and $k \in \mathbb{Z}$, which is not possible if f has an essential singularity.

Example. The function $f(z) = e^{(z-z_0)^{-1}}$ has an essential singularity at $z = z_0$. Indeed, if $z - z_0 \in \mathbb{R}$ and $z - z_0 > 0$ then $f(z) \to \infty$ as $z \to z_0$, if $z - z_0 \in \mathbb{R}$ and $z - z_0 < 0$ then $f(z) \to 0$ as $z \to z_0$.

LAURENT SERIES

Definition. Let f be an analytic function in the complement of a disc about the origin. We shall say that f is bounded at infinity if $f(1/z)$ has a removable singularity at $z = 0$.

Let f be analytic in an annulus $R_1 < |z| < R_2$ and γ_2 be the anticlockwise oriented circle about the origin of radius r_2. Consider the function

$$f_2(z) \;=\; \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w - z}\,\mathrm{d}w$$

where $|z| < r_2 < R_2$. Since we can differentiate under the integral sign, f_2 satisfies the Cauchy–Riemann equations outside $\operatorname{tr}\gamma_2$ and, therefore, is analytic in the

complement to the set $\{|z| = r_2\}$. Thus we have $f_2(z) = \sum_{n=0}^{\infty} c_n z^n$ with some $c_n \in \mathbb{C}$ for all z with $|z| < r_2$.
Let γ_1 be the anticlockwise oriented circle about the origin of radius r_1 and

$$f_1(z) \;=\; -\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w-z}\,\mathrm{d}w$$

where $R_1 < r_1 < |z|$. This function is analytic in the set $\{|z| \neq r_1\}$ and is bounded at infinity. Therefore the function $f(1/z)$ is analytic in the disc $\{|z| < r_1\}$ and has a removable singularity at $z = 0$. Expanding $f(1/z)$ into Taylor's series in the disc $\mathcal{D}(0, r_1)$, we see that $f_1(z) = \sum_{n=-1}^{\infty} c_n z^n$ for all z with $|z| > r_1$.
Finally, Cauchy's integral formula implies that $f_1(z) + f_2(z) = f(z)$ whenever $r_1 < |z| < r_2$. Indeed, $f_1(z) + f_2(z)$ is the sum of integrals over the anticlockwise oriented paths γ_2 and clockwise oriented path γ_1. Let us join these paths by a path γ_+, whose trace is a line segment not passing though z, and denote by γ_- the reverse path. Then $\int_{\gamma_+} + \int_{\gamma_-} = 0$ and $\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_+} + \int_{\gamma_-}$ is the integral over a closed contour running about z in the annulus $R_1 < |z| < R_2$ whose winding number is equal to 1.

Thus we have proved the following theorem.

Theorem (Laurent expansion). Let $0 \leqslant R_1 < R_2 \leqslant \infty$ and $\mathcal{D}(0, R_1, R_2)$ be the open annulus $R_1 < |z| < R_2$. If f is analytic in $\mathcal{D}(0, R_1, R_2)$ then there exist complex numbers c_n such that $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ for all $z \in \mathcal{D}(0, R_1, R_2)$, where the series is absolutely and uniformly convergent on any annulus $\mathcal{D}(0, r_1, r_2)$ with $R_1 < r_1 < r_2 < R_2$.

Since the functions z^k have primitives for all $k \neq -1$, their integral over any circle γ is zero. Integrating the Laurent series term by term, we obtain

$$c_m \;=\; \frac{1}{2\pi i}\int_\gamma c_m\, z^{-1}\,\mathrm{d}z = \frac{1}{2\pi i}\int z^{-m-1}\left(\sum_{n=-\infty}^{\infty} c_n z^n\right)\mathrm{d}z \;=\; \frac{1}{2\pi i}\int \frac{f(z)}{z^{m+1}}\,\mathrm{d}z\,,$$

where γ is an arbitrary anticlockwise oriented circle in $\mathcal{D}(0, R_1, R_2)$. Thus, the coefficients c_n are uniquely defined by f. Therefore the Laurent expansion is unique.

MEROMORPHIC FUNCTIONS

If f has an isolated singularity at $z = z_0$ then, applying the Laurent expansion theorem to $f(z - z_0)$, we see that

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} c_n\,(z-z_0)^n$$

for all z lying in a punctured open disc $\mathcal{D}(z_0, R) \setminus \{z_0\}$.

If all coefficients with negative indices are equal to zero then f has a removable singularity at z_0. If there are infinitely many nonzero coefficients with negative indices then it is an essential singularity. Finally, if

$$f(z) \;=\; \sum_{n=m}^{\infty} c_n\,(z-z_0)^n\,,$$

where m is a negative integer and $c_m \neq 0$ then f has a pole at z_0. The number m is called the order of the pole.

Definition. More generally, if z_0 is not an essential singularity of f then

$$f(z) \;=\; \sum_{n=m}^{\infty} c_n \, (z - z_0)^n \,,$$

where m is an integer number (positive or negative) and $c_m \neq 0$. We shall denote this number m by $\mathrm{ord}\,(f, z_0)$.

If $\mathrm{ord}\,(f, z_0) > 0$ then it is the multiplicity of the zero of f at z_0 (see Week 9); if $\mathrm{ord}\,(f, z_0) < 0$ then it is the order of the corresponding pole. Clearly,

$$f(z) \;=\; (z - z_0)^m g(z)$$

where $m = \mathrm{ord}\,(f, z_0)$ and g is an analytic function in a disc about z_0 such that $g(z_0) \neq 0$.

Definition. A complex function on an open set Ω is called *meromorphic* if it is analytic in Ω except for a set of poles. Note that the pole set of a meromorphic function is discrete (but may be infinite).

RESIDUE THEOREM AND ARGUMENT PRINCIPLE

Definition. If f has an isolated singularity at z_0 then the coefficient c_{-1} is called the *residue* of f at the point z_0 and is denoted $\operatorname{Res}(f, z_0)$.

Let γ_0 be a sufficiently small closed contour about z_0 (such that f has only one singularity inside γ). Integrating its Laurent series term by term, we see that $\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_0} c_{-1}(z-z_0)^{-1}\,\mathrm{d}z$ (since the functions $(z-z_0)^n$ with $n \neq -1$ have primitives, their integrals are equal to zero). Now Cauchy's integral formula implies that

$$n(\gamma, z_0)\, c_{-1} \;=\; \frac{1}{2\pi i}\int_{\gamma_0} f(z)\,\mathrm{d}z\,.$$

Theorem (residue theorem). Let f be a meromorphic function in Ω with poles at the points $a_1, a_2 \ldots$ Then for every closed contour γ, which homotopic to a point in Ω and does not pas through a_k, we have

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\,\mathrm{d}z \;=\; \sum_k n(\gamma, a_k)\operatorname{Res}(f, a_k)$$

Proof. If a point a lies outside the domain bounded by γ then the winding number $n(\gamma, a_k)$ is equal to zero. Thus the summation in the above formula is taken only over the poles a_k which belong to this bounded domain. Since all poles are separated from each other, there are only finitely many such a_k (even if the domain Ω is unbounded and the number of poles in Ω is infinite). Let us enumerate them $a_1, a_2, \ldots, a_p$.

The easiest way to prove the residue theorem is by induction in p.

If $p = 1$ then the result follows from the above formula for $\int_{\gamma_0} f(z)\,\mathrm{d}z$, since any homotopic to a point contour in Ω can be continuously transformed into an arbitrarily small circle.

Assume that the formula holds for functions with $p-1$ poles. Let f has poles at some points $a_1, \ldots, a_{p-1}$ and, in addition, at the point a_p. Consider the Laurent expansion $f(z) = \sum_{n=m_p}^{\infty} c_n(z-a_p)^n$ in an annulus about a_p (here $m_p < 0$ is the order of the pole at a_p). Define $g(z) = \sum_{n=m_p}^{-1} c_n(z-a_p)^n$. Then $f-g$ is a meromorphic function in Ω, which has poles only at $a_1, \ldots, a_{p-1}$. By the induction assumption, the formula holds for g and $f-g$. Adding up these two equalities, we obtain the required formula for f.

Theorem (argument principle). Let f be a meromorphic function in Ω with poles at the points $a_1, a_2 \ldots$ Then for every closed contour γ, which homotopic to a point in Ω and does not pas through a_k, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z \;=\; \sum_k n(\gamma,a_k)\,\mathrm{ord}\,(f,a_k)\,,$$

where the summation is taken over all the poles and zeros a_k of the function f lying in the domain bounded by γ.

Proof. We have $f(z) = (z-a_1)^{m_1} f_1(z)$, where $m_1 = \mathrm{ord}\,(f,a_1)$ and g_1 is a meromorphic in Ω function such that f_1 is analytic near a_1 and $g(a_1) \neq 0$. Similarly, $f_1(z) = (z-a_2)^{m_2} f_2(z)$, where $m_2 = \mathrm{ord}\,(f,a_2)$ and f_2 is a meromorphic in Ω function such that f_2 is analytic near a_1 and a_2, $f_1(a_1) \neq 0$ and $f_2(a_2) \neq 0$. Applying this formula p times, we see that

$$f(z) \;=\; (z-a_1)^{m_1}(z-a_2)^{m_2}\ldots(z-a_p)^{m_p}\,g(z)$$

where $m_k = \mathrm{ord}\,(f,a_k)$ and g is a holomorphic function in Ω which does not have zeros and poles. The direct calculation shows that

$$\frac{f'(z)}{f(z)} \;=\; \frac{m_1}{z-a_1} + \frac{m_2}{z-a_2} + \cdots + \frac{m_p}{z-a_p} + \frac{g'(z)}{g(z)}\,.$$

Since the function $\frac{g'}{g}$ is analytic and, consequently, $\int_\gamma \frac{g'}{g} = 0$, now the required result is obtained by applying the residue theorem to the sum of the first terms in the above formula.

Remark. If $\gamma(t) : [a,b] \to \mathbb{C}$ then by definition,

$$\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\,\dot\gamma(t)\,\mathrm{d}t.$$

Applying the chain rule, we obtain

$$\frac{f'(\gamma(t))}{f(\gamma(t))}\,\dot\gamma(t) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\,(\ln f(\gamma(t)) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(\ln|f(\gamma(t))| + i\arg(f(\gamma(t))\Big)\,.$$

The Argument Principle implies that the integral $\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$ is an imaginary number. Thus we have $\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\ln|f(\gamma(t))|\,\mathrm{d}t = 0$ and

$$\begin{aligned}\sum_k n(\gamma,a_k)\,\mathrm{ord}\,(f,a_k) \;&=\; \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z \;=\; \frac{1}{2\pi i}\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\Big(i\arg(f(\gamma(t))\Big)\,\mathrm{d}t\\ &=\; \frac{1}{2\pi}\left(\arg(f(\gamma(b)) - \arg(f(\gamma(a)))\right)\,.\end{aligned}$$

The above formula tells us how the argument changes as we traverse the path γ. This justifies the name of the theorem.

APPLICATIONS

Let $P(z)$ be a polynomial which does not have real roots, and let $Q(z)$ be an entire function on $\mathbb{C}$ such that $|zQ(z)|/|P(z)| \to 0$ as $|z| \to \infty$ with $\operatorname{Re} z \geqslant 0$ (the case $\operatorname{Re} z \leqslant 0$ is treated in a similar manner). If $\tilde{\gamma}_R$ is the half-circle about the origin running from R to $-R$ in $\mathbb{C}$ then, in view of the basic estimate, the last condition implies that $\int_{\tilde{\gamma}_R} Q(z)/P(z)\,\mathrm{d}z \to 0$ as $R \to \infty$ (because $|\tilde{\gamma}_R| = \pi R$).

$$\int_{-\infty}^{\infty} \frac{Q(t)}{P(t)}\,\mathrm{d}t \;=\; \lim_{R\to\infty} \int_{-R}^{R} \frac{Q(t)}{P(t)}\,\mathrm{d}t \;=\; \lim_{R\to\infty} \int_{\gamma_R} \frac{Q(z)}{P(z)}\,\mathrm{d}z\,,$$

where γ_R is the contour obtained from $\tilde{\gamma}_R$ by joining its end points by the line segment $[-R, R]$. If R_0 is large enough then all poles of $Q(z)/P(z)$ lie inside γ_R for all $R \geqslant R_0$ and then, by Cauchy's formula, the integral $\int_{\gamma_R} \frac{Q(z)}{P(z)}\,\mathrm{d}z$ is independent of $R \geqslant R_0$ and coincides with $2\pi i \sum_k \operatorname{Res}(Q/P, a_k)$ where a_k are the poles of Q/P inside γ_R. These residues can be easily found. Indeed, a_k are the roots of P. If $Q(a_k) \neq 0$ then $\operatorname{Res}(Q/P, a_k) \neq 0$ if and only if P has a simple root at a_k (that is, a root of multiplicity one), and

$$\operatorname{Res}(Q/P, a_k) \;=\; Q(a_k) \lim_{z\to a_k} \frac{z - a_k}{P(z)} \;=\; \frac{Q(a_k)}{P'(a_k)}\,.$$

In this case $\int_{-\infty}^{\infty} \frac{Q(t)}{P(t)}\,\mathrm{d}t = 2\pi i \sum_n \frac{Q(a_n)}{P'(a_n)}$, where a_n are the simple roots of P.

This amazingly simple formula has proved to be very useful. It can be extended in various ways (think, for instance, of a more general meromorphic function P or of choosing a different auxiliary contour $\tilde{\gamma}_R$). If $Q(a_k) = 0$ then one has to consider Taylor's expansion of Q at a_k and multiple roots of P at a_k, generating nonzero $\operatorname{Res}(Q/P, a_k)$.

One has to be careful, when applying the above arguments, to make sure that $|zQ(z)|/|P(z)| \to 0$ as $|z| \to \infty$. To illustrate possible problems, let us consider the following example.

Example. Let us evaluate the integral $\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,\mathrm{d}x$, where a is a real constant. By the above, it coincides with the integral $\int \gamma \frac{e^{iz}}{z^2+1}\,\mathrm{d}z$ over a contour γ composed of the line segment $[-R, R]$ and a half-circle $\tilde{\gamma}_R$ around the origin in $\mathbb{C}$, going from R to $-R$, where R is large enough. The question is: shall we take the half-circle $\tilde{\gamma}_R$ in the upper or lower half-plane? Formal calculations via the residue theorem give different results.

The correct answer is obtained if we choose the contour in the upper half-plane $\operatorname{Im} z = y > 0$ for $a > 0$ (because in this case $|e^{ia(x+iy)}| = e^{-ay}$ as $|x + iy| \to \infty$) and if we choose the contour in the lower half-plane $\operatorname{Im} z = y < 0$ for $a < 0$ (because in this case $|e^{ia(x+iy)}| = e^{-ay}$ as $|x + iy| \to \infty$). Since $z^2 + 1 = (z + i)(z - i)$, the answer for $a > 0$ is

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,\mathrm{d}x = \int_{\gamma_R} \frac{e^{iaz}}{z^2+1}\,\mathrm{d}z = 2\pi i \operatorname{Res}\left(\frac{e^{iax}}{(z+i)(z-i)}\,, i\right)$$

$$= \; 2\pi i \left(\frac{e^{iax}}{z+i}\right)\Bigg|_{z=i} \;=\; \pi\, e^{-a}\,.$$