CM 321A REAL ANALYSIS II

Yu. Safarov

1. Metrics and norms

Definition 1.1. Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}$ is called a *metric* on X if it satisfies

(1) $\rho(x,y) > 0$ if $x \neq y$ and $\rho(x,x) = 0$,
(2) $\rho(x,y) = \rho(y,x)$,
(3) $\rho(x,z) \leqslant \rho(x,y) + \rho(y,z)$,

where x, y and z are arbitrary elements of X.

Example 1.2. $\mathbb{R}$ or $\mathbb{C}$ with the usual (Euclidean) metric $\rho(x,y) = |x-y|$.

Example 1.3. $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean metric

$$\rho(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2},$$

where x_i and y_i are coordinates of the points x and y respectively.

Example 1.4. $\mathbb{R}^n$ with the metric

$$\rho(x,y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|\,.$$

Example 1.5. $C[a,b]$, the set of all continuous (real or complex-valued) functions on $[a,b]$, with the metric

$$\rho(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|\,. \tag{1.1}$$

Example 1.6. $B(S)$, the set of all bounded (real or complex-valued) functions on a set S, with the metric

$$\rho(f,g) = \sup_{x \in S} |f(x) - g(x)|\,. \tag{1.2}$$

Note that (1.2) turns into (1.1) if $S = [a,b]$. In other words, if $S = [a,b]$ then (1.1) and (1.2) define the same metric. However, the space $C[a,b]$ is strictly smaller than $B[a,b]$ (every continuous function on $[a,b]$ is bounded but there are bounded functions which are not continuous).

Example 1.7 (discrete metric). For any set X, define the metric ρ by

$$\begin{cases} \rho(x,y) = 1\,, & \text{if } x \neq y, \\ \rho(x,x) = 0\,. \end{cases}$$

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Definition 1.8. We call the pair (X,ρ) a *metric space* if X is a non-empty set and ρ is a metric on X.

Definition 1.9. If (X,ρ) is a metric space and $A\subset X$ then ρ is also a metric on A. The metric space (A,ρ) is called a subspace of (X,ρ).

Example 1.10. A continuous function on a bounded closed interval is always bounded. Therefore $C[a,b]$ is a subspace of $B[a,b]$ whenever $-\infty<a<b<+\infty$.

If the X is a linear space, it is often possible to express the metric ρ in terms of a function of one variable that can be thought of as the length of each element (i.e., its distance from 0).

Definition 1.11. Let X be a vector space over $\mathbb{R}$ (or over $\mathbb{C}$). A function $\|\cdot\|:X\to\mathbb{R}$ is called a *norm* on X if it satisfies

(1) $\|x\|>0$ if $x\neq 0$ and $\|0\|=0$,
(2) $\|\lambda x\|=|\lambda|\,\|x\|$ for all $x\in X$ and $\lambda\in\mathbb{R}$ (or $\lambda\in\mathbb{C}$),
(3) $\|x+y\|\leqslant\|x\|+\|y\|$ for all $x,y\in X$.

A linear space with a norm is called a *normed* space.

Given a norm on X, the function $\rho(x,y)=\|x-y\|$ is a metric on X (one can easily check that each property of the norm $\|\cdot\|$ implies the corresponding property of the metric $\rho(x,y)=\|x-y\|$). However, not every metric arises in this way; one can have a metric d on a vector space X such that $d(x,0)$ does not have the properties of a norm.

Example 1.12. In Examples 1.2, 1.3 and 1.4 the metrics are generated by the norms $\|x\|=|x|$, $\|x\|=(\sum_{i=1}^n|x_i|^2)^{1/2}$ and $\|x\|=\sum_{i=1}^n|x_i|$ respectively.

Example 1.13. The metrics (1.1) and (1.2) are generated by the norms

$$\|f\| = \sup_{x\in[a,b]}|f(x)| \qquad\text{and}\qquad \|f\| = \sup_{x\in S}|f(x)|\,.$$

2. Convergence

Definition 2.1. A sequence x_n of elements of a metric space (X,ρ) is said to *converge* to $x\in X$ if for any $\varepsilon>0$ there exists an integer n_ε such that $\rho(x_n,x)<\varepsilon$ for all $n>n_\varepsilon$.

Lemma 2.2 (another definition of convergence). *$x_n\to x$ in (X,ρ) if and only if $\rho(x_n,x)\to 0$ in $\mathbb{R}^1$.*

Proof. By definition, the sequence of non-negative numbers $\rho(x_n,x)$ converges to zero if and only if for any $\varepsilon>0$ there exists an integer n_ε such that $\rho(x_n,x)\leqslant\varepsilon$ for all $n>n_\varepsilon$. □

Lemma 2.3. *If r_n are non-negative numbers such that $r_n\to 0$ in $\mathbb{R}^1$ and $\rho(x_n,x)\leqslant r_n$ for all (sufficiently large) n then $x_n\to x$ in (X,ρ).*

Proof. By definition, the sequence of real numbers r_n converges to zero if and only if for any $\varepsilon>0$ there exists an integer n_ε such that $r_n\leqslant\varepsilon$ for all $n>n_\varepsilon$. Then $\rho(x_n,x)\leqslant r_n\leqslant\varepsilon$ for all $n>n_\varepsilon$ which means that $\rho(x_n,x)\to 0$. □

Definition 2.4. We say that the metrics ρ_1 and ρ_2 defined on the same set X are equivalent if

$$\alpha_n \to \alpha \text{ in } (X, \rho_1) \quad \text{if and only if} \quad \alpha_n \to \alpha \text{ in } (X, \rho_2)\,.$$

One can easily check the metric introduced in Example 1.4 is equivalent to the Euclidean metric (Example 1.3), whereas the discrete metric (Example 1.7) on $X = \mathbb{R}^2$ is not.

Definition 2.5. Convergence in the metric space $B(S)$ (or any subspace of $B(S)$) is called *uniform convergence* on S.

Lemma 2.6 (another definition of uniform convergence). *Let f_n be a sequence in $B(S)$ and $f \in B(S)$. Then $f_n \to f$ in $B(S)$ if and only if for any $\varepsilon > 0$ there exists an integer n_ε such that $|f(x) - f_n(x)| \leqslant \varepsilon$ for all $n > n_\varepsilon$ and all $x \in S$.*

Proof. $f_n \to f$ in $B(S)$ if and only if $\rho(f, f_n) = \sup_{x\in S} |f(x) - f_n(x)| \to 0$ (Lemma 2.2). By definition, the sequence of real numbers $\sup_{x\in S} |f(x) - f_n(x)|$ converges to zero if and only if for any $\varepsilon > 0$ there exists an integer n_ε such that

$$\sup_{x\in S} |f(x) - f_n(x)| \leqslant \varepsilon\,, \qquad \forall n > n_\varepsilon\,.$$

Therefore we only have to prove that $\sup_{x\in S} |f(x) - f_n(x)| \leqslant \varepsilon$ if and only if $|f(x) - f_n(x)| \leqslant \varepsilon$ for all $x \in S$.

If $\sup_{x\in S} |f(x) - f_n(x)| \leqslant \varepsilon$ then, obviously,

$$|f(x) - f_n(x)| \leqslant \sup_{x\in S} |f(x) - f_n(x)| \leqslant \varepsilon\,, \qquad \text{for all } x \in S.$$

Assume now that $|f(x) - f_n(x)| \leqslant \varepsilon$ for all $x \in S$. By definition, the supremum coincides with the least upper bound. Therefore for any $\delta > 0$ there exists a point $x_\delta \in S$ such that

$$\sup_{x\in S} |f(x) - f_n(x)| \leqslant \delta + |f(x_\delta) - f_n(x_\delta)| \leqslant \delta + \varepsilon.$$

Since δ can be chosen arbitrarily small, this implies that $\sup_{x\in S} |f(x) - f_n(x)| \leqslant \varepsilon$. □

Remark 2.7. A sequence of functions $f_n \in B(S)$ converges to $f \in B(S)$ *pointwise* if for any $x \in S$ and $\varepsilon > 0$ there exists an integer $n_{\varepsilon,x}$ such that $|f(x) - f_n(x)| \leqslant \varepsilon$ for all $n > n_{\varepsilon,x}$. In this definition the integer $n_{\varepsilon,x}$ may depend on x. If for any ε the set $\{n_{\varepsilon,x}\}_{x\in S}$ is bounded from above, that is $n_{\varepsilon,x} \leqslant n_\varepsilon$ for all $x \in S$, then $f_n \to f$ uniformly.

3. Open Sets and Closed Sets

1. Open and closed balls. Let (X, ρ) be a metric space and r be a strictly positive number.

Definition 3.1. If $\alpha \in X$ then the set

$$B_r(\alpha) \;=\; \{x \in X : \rho(x,\alpha) < r\}$$

is called the *open ball* and the set

$$B_r[\alpha] \;=\; \{x \in X : \rho(x,\alpha) \leqslant r\}$$

is called the *closed ball* centre α radius r.

If there is a need to emphasize the metric, we write $B_r^\rho(\alpha)$ and $B_r^\rho[\alpha]$. Clearly,

$$\alpha \in B_{r-\varepsilon}(\alpha) \subset B_r(\alpha) \subset B_r[\alpha] \subset B_{r+\varepsilon}(\alpha)\,, \qquad \forall r > \varepsilon > 0\,. \tag{3.1}$$

Definition 3.2. A set $A \subset X$ is said to be a *neighbourhood* of $\alpha \in X$ if A contains an open ball $B_r(\alpha)$.

In view of (3.1) the balls $B_r(\alpha)$ and $B_r[\alpha]$ are neighbourhoods of the point α. Now we can rephrase Definition 2.1 as follows.

Definition 2.1'. A sequence x_n in a metric space converges to x if for any ball $B_r(x)$ (or any neighbourhood A of x) there exists an integer n' such that for all $n > n'$ we have $x_n \in B_r(x)$ (or $x_n \in A$).

Theorem 3.3. *Two metrics ρ and σ on the same set X are equivalent if and only if every open ball $B_r^\rho(x)$ contains an open ball $B_s^\sigma(x)$ and every open ball $B_s^\sigma(x)$ contains an open ball $B_r^\rho(x)$.*

Proof. If every open ball $B_r^\rho(x)$ contains an open ball $B_s^\sigma(x)$ and $x_n \xrightarrow{\sigma} x$ then for any $r > 0$ we can choose $s > 0$ and n_s such that

$$x_n \in B_s^\sigma(x) \subset B_r^\rho(x)\,, \qquad \forall n > n_s\,.$$

This implies that $x_n \xrightarrow{\rho} x$. Therefore if every ball with respect to one metric contains a ball with respect to another metric then convergence in one metric implies convergence in another metric, that is, the metrics are equivalent.

Assume now that the metrics are equivalent but there exists a ball $B_r^\rho(x)$ which does not contain $B_s^\sigma(x)$ for all positive s. Let us choose a sequence $s_n \to 0$ and let $x_n \in B_{s_n}^\sigma(x)$ and $x_n \notin B_r^\rho(x)$. Then $x_n \xrightarrow{\sigma} x$. However, the sequence x_n does not converge to x in the metric ρ because $x_n \notin B_r^\rho(x)$ for all n. Therefore the metrics ρ and σ are not equivalent, and we obtain a contradiction. $\square$

By Theorem 3.3, if A is a neighbourhood of x in (X,ρ) then it is a neighbourhood of x with respect to every metric which is equivalent to ρ.

2. Open sets.

Definition 3.4. A set is *open* if if it contains a ball about each of its points. (Equivalently, a set is open it contains a neighbourhood of each of its points.)

Lemma 3.5. *An open ball in a metric space (X,ρ) is open.*

Proof. If $x \in B_r(\alpha)$ then $\rho(x,\alpha) = r - \varepsilon$ where $\varepsilon > 0$. If $y \in B_\varepsilon(x)$ then $\rho(x,y) < \varepsilon$ and, by the triangle inequality,

$$\rho(y,\alpha) \leqslant \rho(x,\alpha) + \rho(y,x) < r - \varepsilon + \varepsilon = r\,.$$

This implies that $y \in B_r(\alpha)$ for all $y \in B_\varepsilon(x)$, that is, $B_\varepsilon(x) \subset B_r(\alpha)$. $\square$

Theorem 3.6. *If (X, ρ) is a metric space then*

(1) *the whole space X and the empty set $\varnothing$ are both open,*
(2) *the union of any collection of open subsets of X is open,*
(3) *the intersection of any finite collection of open subsets of X is open.*

Proof.
(1) The whole space is open because it contains all open balls and the empty set is open because it does not contain any points.

(2) If x belongs to the union of open sets A_ν then x belongs to at least one of the sets A_ν. Since this set is open, it also contains an open ball about x. This ball lies in the union of A_ν, so the union is an open set.

(3) If $A_1, A_2, \dots, A_k$ are open sets and $x \in \cap_{n=1}^k A_n$ then $x \in A_n$ for every $n = 1, \dots, k$. Since A_n are open, for each n there exists r_n such that $B_{r_n}(x) \subset A_n$. Let $r = \min\{r_1, r_2, \dots, r_k\}$. Then, in view of (3.1), $B_r(x) \subset B_{r_n}(x) \subset A_n$ for all $n = 1, \dots, k$, so $B_r(x) \subset \left(\cap_{n=1}^k A_n\right)$. □

An infinite intersection of open sets is not necessarily open.

Example 3.7. Let A_n be the open intervals $(-1/n, 1/n)$ in $\mathbb{R}$. Then A_n are open sets but the intersection $\cap_{n=1}^\infty A_n = \{0\}$ is not open.

Lemma 3.8. *A set is open if and only if it coincides with the union of a collection of open balls.*

Proof. According to Theorem 3.6 the union of any collection of open balls is open. On the other hand, if A is open then for every point $x \in A$ there exists a ball $B(x)$ about x lying in A. We have $A = \cup_{x \in A} B(x)$. Indeed, the union $\cup_{x \in A} B(x)$ is a subset of A because every ball $B(x)$ is a subset of A, and the union contains every point $x \in A$ because $x \in B(x)$. □

Definition 3.9. A point $x \in A$ is said to be an *interior* point of the set A if there exists an open ball $B_r(x)$ lying in A. The *interior* of a set A is the union of all open sets contained in A, that is, the maximal open set contained in A. The interior of A is denoted by $int(A)$.

Clearly, the interior of A coincides with set of interior points of A. Indeed, if x is an interior point then there exists an open ball $B_r(x)$ lying in A. This ball is an open set lying in A and therefore is a subset of the maximal open set $int(A) \subset A$. Conversely, if $x \in int(A)$ then (since $int(A)$ is open) there exists a ball $B_r(x) \subset int(A) \subset A$, so x is an interior point of A.

3. Closed sets.

Definition 3.10. A point $x \in X$ is called a *limit* point of a set A if every ball about x contains a point of A distinct from x. Other terms for "limit point" are point of accumulation or cluster point. The set of limit points of A is denoted A'.

Lemma 3.11. *A point x is a limit point of a set A if and only if there is a sequence x_n of elements of A distinct from x which converges to x.*

Proof. If $x_n \to x$ then every ball about x contains a point x_n (see Definition 2.1'). If every ball about x contains a point of A distinct from x then there exists a sequence of points $x_n \in A$ distinct from x and lying in the balls $B_{1/n}(x)$. Obviously, this sequence converges to x. □

Definition 3.12. A set is *closed* if it contains all its limit points.

Lemma 3.13 (another definition of closed sets). *A set A is closed if and only if the limit of any convergent sequence of elements of A lies in A.*

Proof. If a sequence of elements of A has a limit then either this limit coincides with one of the elements of the sequence (and then it lies in A) or it is a limit point of A. Therefore a closed set A contains the limits of all convergent sequences $\{x_n\} \subset A$.

Conversely, every limit point is a limit of some sequence $\{x_n\} \subset A$. Therefore A contains all its limit points, provided that the limit of any convergent sequence of elements of A lies in A. □

Lemma 3.14. *A closed ball in a metric space (X, ρ) is closed.*

Proof. Let x_n be a convergent subsequence lying in the closed ball $B_r[\alpha]$ and x be its limit. Then, by the triangle inequality,

$$\rho(x,\alpha) \leqslant \rho(x_n,\alpha) + \rho(x,x_n) \leqslant r + \rho(x,x_n)\,, \qquad \forall n\,.$$

Since $x_n \to x$, we can make $\rho(x,x_n)$ arbitrarily small by choosing large n. This implies that $\rho(x,\alpha) \leqslant r$, that is, $x \in B_r[\alpha]$. □

Definition 3.15. If $A \subset X$ then $\mathcal{C}(A)$ denotes the complement of the set A in X, that is, the set of all points $x \in X$ which do not belong to A.

Theorem 3.16. *If A is open then $\mathcal{C}(A)$ is closed. If A is closed then $\mathcal{C}(A)$ is open.*

Proof. If A is open then for every point of A there exists a ball about this point lying in A. Clearly, such a ball does not contain any points from $\mathcal{C}(A)$. This means that every point of A is not a limit point of $\mathcal{C}(A)$, that is, $\mathcal{C}(A)$ contains all its limit points.

If A is closed then it contains all its limit points, so any point $x \in \mathcal{C}(A)$ is not a limit point of A. This means that there exists a ball $B_r(x)$ which lies in $\mathcal{C}(A)$, that is, $\mathcal{C}(A)$ is open. □

Theorem 3.17. *In a metric space (X, ρ)*

1. *the whole space X and the empty set $\varnothing$ are both closed,*
2. *the intersection of any collection of closed sets is closed,*
3. *the union of any finite collection of closed sets is closed.*

Proof. The theorem follows from Theorems 3.6, Theorem 3.16 and the following elementary results.

(1) $\mathcal{C}(X) = \varnothing$ and $\mathcal{C}(\varnothing) = X$ (obvious).

(2) The complement of the intersection of sets A_ν coincides with the union of the complements $\mathcal{C}(A_\nu)$. Indeed, $x \in \mathcal{C}(\bigcap_\nu A_\nu) \Leftrightarrow x \notin \bigcap_\nu A_\nu \Leftrightarrow \{x \notin A_\nu \text{ for some } \nu\}$ $\Leftrightarrow \{x \in \mathcal{C}(A_\nu) \text{ for some } \nu\} \Leftrightarrow x \in \bigcup_\nu \mathcal{C}(A_\nu)$.

(3) The complement of the union of sets A_j coincides with the intersection of the complements $\mathcal{C}(A_j)$. Indeed, $x \in \mathcal{C}(\bigcup_\nu A_\nu) \Leftrightarrow x \notin \bigcup_\nu A_\nu \Leftrightarrow \{x \notin A_\nu \text{ for all } \nu\}$ $\Leftrightarrow \{x \in \mathcal{C}(A_\nu) \text{ for all } \nu\} \Leftrightarrow x \in \bigcap_\nu \mathcal{C}(A_\nu)$. □

Definition 3.18. The *closure* of a set A is the intersection of all closed sets containing A, that is, the minimal closed set containing A. The closure is denoted by $cl(A)$ or $\bar{A}$.

Theorem 3.19. $\bar{A} = A \cup A'$.

Proof. Let $x \in \mathcal{C}(A \cup A')$ be an arbitrary point. Then $x \notin A'$. Therefore there exists a ball $B_r(x)$ which does not contain elements of A distinct from x (Definition 3.10). Since $x \notin A$, this implies that $B_r(x) \subset \mathcal{C}(A)$. By Lemma 3.5, for every point $y \in B_r(x)$ there exists a ball $B_\varepsilon(y) \subset B_r(x) \subset \mathcal{C}(A)$, that is, a point $y \in B_r(x)$ cannot be a limit point of A. Therefore $B_r(x) \subset \mathcal{C}(A \cup A')$. Thus, for every point $x \in \mathcal{C}(A \cup A')$ we can find a ball $B_r(x) \subset \mathcal{C}(A \cup A')$. This means that the set $\mathcal{C}(A \cup A')$ is open. Now, by Theorem 3.16, the set $A \cup A'$ is closed.

It remains to prove that $A \cup A'$ is the minimal closed set which contains A. Let $A \subset K \subset A \cup A'$ and $K \neq A \cup A'$. Then K does not contain at least one point of A'. By Lemma 3.11, there exists a sequence of elements $x_n \in A \subset K$ which converges to this point. Therefore, in view of Lemma 3.13, the set K is not closed. □

Corollary 3.20. *$x \in \bar{A}$ if and only if there exists a sequence $\{x_n\} \subset A$ which converges to x.*

Proof. If $x \in \bar{A}$ then either $x \in A$ or $x \in A'$ (Theorem 3.19). In the first case the sequence $\{x, x, \dots\} \subset A$ converges to x, in the second case a sequence $\{x_n\} \to x$ exists according to Lemma 3.11.

Conversely, assume that there exists a sequence $\{x_n\} \subset A$ which converges to x. If $x_n = x$ for some n then $x \in A$. If x_n are distinct from x then, by Lemma 3.11, $x \in A'$. Therefore $x \in \bar{A} = A \cup A'$. □

Example 3.21. Let (X, ρ) be a metric space with the discrete metric

$$\rho(x,y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then

$$B_r[a] = \begin{cases} a, & \text{if } r < 1, \\ X, & \text{if } r \geqslant 1. \end{cases} \qquad B_r(a) = \begin{cases} a, & \text{if } r \leqslant 1, \\ X, & \text{if } r > 1, \end{cases}$$

Since the open ball is open, this implies that any point is an open set. Since any set coincides with the union of its elements, Theorem 3.6 implies that any subset of X is open. Therefore, by Theorem 3.16, any subset of X is closed.

The closure of the open ball $B_r(a)$ does not necessarily coincide with the closed ball $B_r[a]$. In particular, in the Example 3.21 $\overline{B}_1(a) = B_1(a) = a$ (since $B_1(a)$ is closed) but $B_1[a] = X$.

Theorem 3.22. *In a normed linear space $\overline{B_r(\alpha)} = B_r[\alpha]$.*

Proof. Let x be a limit point of $B_r(\alpha)$. Then, by Lemma 3.11, there exists a sequence of elements $x_n \in B_r(\alpha)$ which converges to x. By the triangle inequality

$$\|\alpha - x\| \;\leqslant\; \|\alpha - x_n\| + \|x_n - x\| \;\leqslant\; r + \|x_n - x\|$$

for all n. Since $\|x_n - x\|$ can be made arbitrarily small by choosing large n, we see that $\|\alpha - x\| \leqslant r$. This implies that $x \in B_r[\alpha]$. Therefore, in view of (3.1),

$$\overline{B_r(\alpha)} = B_r(\alpha) \cup \big(B_r(\alpha)\big)' \subset B_r[\alpha]\,.$$

It remains to prove that

$$B_r[\alpha] \;\subset\; \overline{B_r(\alpha)} \;=\; B_r(\alpha) \cup \big(B_r(\alpha)\big)' . \tag{3.2}$$

Assume that $x \in B_r[\alpha]$, that is, $\rho(\alpha, x) \leqslant r$. If $\rho(\alpha, x) < r$ then $x \in B_r(\alpha) \subset \overline{B_r(\alpha)}$. If $\rho(\alpha, x) = r$ then we define $x_n = x + n^{-1}(\alpha - x)$, $n = 1, 2, \ldots$ We have

$$\rho(x_n, \alpha) = \|x_n - \alpha\| = \|x - \alpha + n^{-1}(\alpha - x)\|$$
$$= (1 - n^{-1})\|x - \alpha\| = (1 - n^{-1})\, r < r \tag{3.3}$$

and

$$\rho(x_n, x) = \|x_n - x\| = \|n^{-1}(\alpha - x)\| = n^{-1}\|x - \alpha\| = n^{-1}\, r \to 0\,, \qquad n \to \infty\,. \tag{3.4}$$

The inequality (3.3) implies that $x_n \in B_r(\alpha)$ for all n, and (3.4) implies that $x_n \to x$. Therefore x is a limit point of the set $B_r(\alpha)$, so $x \in \big(B_r(\alpha)\big)' \subset \overline{B_r(\alpha)}$ and (3.2) follows. □

4. Continuity

Definition 4.1. Let (X, ρ) and (Y, d) be metric spaces. A map $T : X \to Y$ is said to be *continuous at* $\alpha \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(Tx, T\alpha) < \varepsilon$ whenever $\rho(x, \alpha) < \delta$. The map T is said to be *continuous* if it is continuous at every point $\alpha \in X$.

If T is a function on $\mathbb{R}^n$, that is, a map from $\mathbb{R}^n$ to $\mathbb{R}$ or to $\mathbb{C}$, then the above definition coincides with the usual definition of continuity.

Theorem 4.2 (sequential characterisation of continuity). *The map $T : (X, \rho) \to (Y, d)$ is continuous at $\alpha \in X$ if and only if for every sequence x_n converging to α in (X, ρ), the sequence Tx_n converges to $T\alpha$ in (Y, d).*

Proof. Assume that T is continuous and that $x_n \to \alpha$ in (X, ρ). In order to prove that $Tx_n \to T\alpha$ in (Y, d) we have to show that for every $\varepsilon > 0$ there exists n_ε such that

$$d(Tx_n, T\alpha) \leqslant \varepsilon\,, \qquad \forall n > n_\varepsilon\,. \tag{4.1}$$

Since T is continuous, given $\varepsilon > 0$ we can find $\delta > 0$ such that $\rho(x_n, \alpha) \leqslant \delta$ implies $d(Tx_n, T\alpha) \leqslant \varepsilon$. Since the sequence x_n converges to α in (X, ρ), for this δ there exists n_δ such that $\rho(x_n, \alpha) \leqslant \delta$ for all $n > n_\delta$. Obviously, (4.1) holds true for $n_\varepsilon = n_\delta$.

Now assume that for every sequence x_n converging to α in (X, ρ), the sequence Tx_n converges to $T\alpha$ in (Y, d). If T is not continuous then there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ we can find $x \in X$ for which $\rho(x, \alpha) < \delta$ and $d(Tx, T\alpha) > \varepsilon_0$. Let $x_n \in X$ be such that $\rho(x_n, \alpha) < 1/n$ and $d(Tx_n, T\alpha) > \varepsilon_0$. Then $x_n \to \alpha$ in (X, ρ) but Tx_n does not converge to $T\alpha$ in (Y, d), and we obtain a contradiction. □

Theorem 4.3. *If the map $T_1 : (X, \rho) \to (Y, d)$ is continuous at α and the map $T_2 : (Y, d) \to (Z, \sigma)$ is continuous at $T_1\alpha$ then $T_2T_1 : (X, \rho) \to (Z, \sigma)$ is continuous at α.*

Proof. Let x_n be an arbitrary sequence converging to α in (X, ρ). Then, since T_1 is continuous, $T_1x_n \to T_1\alpha$ in (Y, d) and, since T_2 is continuous, $T_2T_1x_n = T_2\,(T_1x_n) \to T_2\,(T_1\alpha)$ in (Z, σ). In view of Theorem 4.2 this means that T_2T_1 is continuous. □

Lemma 4.4. *If (X, ρ) is a metric space and $x_0 \in X$ is a fixed element then $T : x \to \rho(x, x_0)$ is a continuous map from (X, ρ) to $\mathbb{R}$.*

Proof. By the triangle inequality

$$|Tx_n - T\alpha| \;=\; |\rho(x_n, x_0) - \rho(\alpha, x_0)| \;\leqslant\; \rho(x_n, \alpha)\,.$$

This implies that $Tx_n \to T\alpha$ whenever $x_n \to \alpha$. □

Definition 4.5 (direct products). Given metric spaces (X, ρ) and (Y, d), a metric σ can be defined on the direct product $X \times Y$ by

$$\sigma\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{\big(\rho(x_1, x_2)\big)^2 + \big(d(y_1, y_2)\big)^2}\,.$$

The metric space $(X \times Y, \sigma)$ is said to be the direct product of the metric spaces (X, ρ) and (Y, d).

Using the above definition for $X = Y = \mathbb{R}$ makes $\mathbb{R} \times \mathbb{R}$ the same as $\mathbb{R}^2$ with the Euclidean metric. There are many other metrics that may be defined on $X \times Y$; in particular,

$$\sigma\left((x_1, y_1), (x_2, y_2)\right) = \rho(x_1, x_2) + d(y_1, y_2)$$

is also a metric.

Note that continuity of a function of two variables with this metric is not the the same as continuity in each variable separately.

Example 4.6. The real-valued function

$$f(x, y) = \begin{cases} (x\,y)/(x^2 + y^2), & \text{if } x^2 + y^2 \neq 0, \\ 0, & \text{if } x = y = 0\,. \end{cases}$$

defined on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is continuous at the origin in each variable separately but is not continuous as a function of two variables.

Definition 4.7. Given a map $T : X \to Y$ and a subset $A \subset Y$, the set $\{x \in X : Tx \in A\}$ is denoted $T^{-1}(A)$ and called the *inverse image* of A.

Note that $T^{-1}(A)$ is a well-defined set irrespective of whether T has any inverse. Now Definition 4.1 can be rephrased as follows.

Definition 4.1′. Let (X, ρ) and (Y, d) be metric spaces. A map $T : X \to Y$ is said to be *continuous at* $\alpha \in X$ if for any open ball $B_\varepsilon(T\alpha)$ about $T\alpha$ there exists a ball $B_\delta(\alpha)$ about α such that $B_\delta(\alpha) \subset T^{-1}\left(B_\varepsilon(T\alpha)\right)$.

Theorem 4.8 (inverse image characterization of continuity). *Let (X, ρ) and (Y, σ) be metric space and $T : X \to Y$ be a map from X to Y. Then the following statements are equivalent:*

1. *T is continuous,*
2. *the inverse image of every open subset of Y is an open subset of X,*
3. *the inverse image of every closed subset of Y is a closed subset of X.*

Proof. The inverse image of the complement of a set A coincides with the complement of the inverse image $T^{-1}(A)$. Therefore Theorem 3.16 implies that (2) is equivalent to (3). Let us prove that (2) is equivalent to (1).

Assume first that T is continuous. Let A be an open subset of Y and $x \in T^{-1}(A) \subset X$. Since A is open, there exists a ball $B_\varepsilon(Tx)$ about the point Tx such that $B_\varepsilon(Tx) \subset A$. Since T is continuous, there exists a ball $B_\delta(x)$ about x such that $B_\delta(x) \subset T^{-1}(B_\varepsilon(Tx)) \subset T^{-1}(A)$ (Definition 4.1′). Therefore for every point $x \in T^{-1}(A)$ there exists a ball $B_\delta(x)$ lying in $T^{-1}(A)$, which means that $T^{-1}(A)$ is open.

Assume now that the inverse image of any open set is open. Let $x \in X$ and $B_\varepsilon(Tx)$ is a ball about $Tx \in Y$. The inverse image $T^{-1}(B_\varepsilon(Tx))$ is an open set which contains the point x. Therefore there exists a ball $B_\delta(x)$ about x such that $B_\delta(x) \subset T^{-1}(B_\varepsilon(Tx))$. This implies that T is continuous (Definition 4.1′). □

Let X and Y be linear spaces and $T : X \to Y$. Recall that T is called a linear map if $T(x+y) = Tx+Ty$ and $T(\lambda x) = \lambda Tx$ for all $x, y \in X$ and $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$). In particular, if T is linear then $T0 = 0$

Definition 4.9. Let X , Y be normed linear spaces and $\|\cdot\|_X$, $\|\cdot\|_Y$ be the corresponding norms. A linear map $T : X \to Y$ is said to be *bounded* if there exists a positive constant C such that $\|Tx\|_Y \leqslant C\|x\|_X$ for all $x \in X$.

Theorem 4.10. *Let X and Y be normed linear spaces and $T : X \to Y$ be a linear map. Then T is continuous if and only if it is bounded.*

Proof. Let $\rho(x_1, x_2) = \|x_1 - x_2\|_X$ and $d(y_1, y_2) = \|y_1 - y_2\|_Y$ be the metrics on X and Y generated by the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$.

Assume first that T is bounded, that is, $\|Tx\|_Y \leqslant C\|x\|_X$. Let $\alpha \in X$ and $\varepsilon > 0$. Take $\delta = C^{-1}\varepsilon$. Then, for all $x \in X$ satisfying $\rho(x, \alpha) = \|x - \alpha\|_X \leqslant \delta$, we have

$$d(Tx, T\alpha) = \|Tx - T\alpha\|_Y = \|T(x - \alpha)\|_Y \leqslant C\|x - \alpha\|_X \leqslant \varepsilon.$$

This implies that T is continuous.

Assume now that T is continuous. Then T is continuous at 0, and therefore there exists $\delta > 0$ such that

$$d(0, Tx_0) = \|Tx_0\|_Y \leqslant 1 \quad \text{whenever} \quad \rho(0, x_0) = \|x_0\|_X \leqslant \delta\,. \tag{4.2}$$

If $x \in X$, let us denote $c = \delta\|x\|_X^{-1}$ and $x_0 = c\,x$. Then $\|x_0\|_X = \delta$. Since T is a linear map, (4.2) implies

$$\|Tx\|_Y = c^{-1}\|Tx_0\|_Y \leqslant c^{-1} = \delta^{-1}\|x\|_X\,,$$

which means that T is bounded. □

5. Completeness

1. Cauchy sequences.

Definition 5.1. A sequence x_n of elements of a metric space (X, ρ) is called a *Cauchy* sequence if, given any $\varepsilon > 0$, there exists n_ε such that $\rho(x_n, x_m) < \varepsilon$ for all $n, m > n_\varepsilon$.

Lemma 5.2. *Every convergent sequence is a Cauchy sequence.*

Proof. If $x_n \to x$ then for any $\varepsilon > 0$ there exists n_ε such that $\rho(x_n, x) \leqslant \varepsilon/2$ for all $n > n_\varepsilon$. Applying the triangle inequality we obtain

$$\rho(x_n, x_m) \leqslant \rho(x_n, x) + \rho(x_m, x) \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $n, m > n_\varepsilon$. This implies that x_n is a Cauchy sequence. □

Lemma 5.3. *If a Cauchy sequence has a convergent subsequence then it is convergent with the same limit.*

Proof. Let x_n be a Cauchy sequence and $x_{n_k} \to x$ be a convergent subsequence. Then for any $\varepsilon > 0$ there exists n_ε such that $\rho(x_n, x_{n_k}) \leqslant \varepsilon/2$ and $\rho(x_{n_k}, x) \leqslant \varepsilon/2$ for all $n, n_k > n_\varepsilon$. Applying the triangle inequality we obtain

$$\rho(x_n, x) \leqslant \rho(x_{n_k}, x) + \rho(x_n, x_{n_k}) \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon\,, \qquad \forall n > n_\varepsilon\,.$$

Since ε is an arbitrary positive number, this implies that $x_n \to x$. □

2. Complete metric spaces.

Definition 5.4. A metric space (X, ρ) is said to be *complete* if any Cauchy sequence $\{x_n\} \subset X$ converges to a limit $x \in X$.

There are incomplete metric spaces. If a metric space (X, ρ) is not complete then it has Cauchy sequences which do not converge. This means, in a sense, that there are gaps (or missing elements) in X. Every incomplete metric space can be made complete by adding new elements, which can be thought of as the missing limits of non-convergent Cauchy sequences. More precisely, we have the following theorem.

Theorem 5.5. *Let (X, ρ) be an arbitrary metric space. Then there exists a complete metric space $(\widetilde{X}, \widetilde{\rho})$ such that*

1. *$X \subset \widetilde{X}$ and $\widetilde{\rho}(x, y) = \rho(x, y)$ whenever $x, y \in X$;*
2. *for every $\widetilde{x} \in \widetilde{X}$ there exists a sequence of elements $x_n \in X$ such that $x_n \to \widetilde{x}$ as $n \to \infty$ in the space $(\widetilde{X}, \widetilde{\rho})$.*

Proof. See Appendix.

The metric space $(\widetilde{X}, \widetilde{\rho})$ is said to be the *completion* of (X, ρ). If (X, ρ) is already complete then necessarily $X = \widetilde{X}$ and $\rho = \widetilde{\rho}$.

Theorem 5.6. *Let (A, ρ) be a subspace of a complete metric space (X, ρ) and $\bar{A}$ be the closure of A in (X, ρ). Then $(\bar{A}, \rho)$ is the completion of (A, ρ).*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\bar{A}$. Since (X, ρ) is complete and $\bar{A} \subset X$, this sequence converges to some element $x \in X$. Since $\bar{A}$ is closed, by Lemma 3.13, we have $x \in \bar{A}$. Therefore the space $(\bar{A}, \rho)$ is complete. Now the theorem follows from Corollary 3.20. □

Example 5.7. Let X be the set of rational numbers with the standard metric $\rho(x,y) = |x-y|$. This metric space is not complete because any sequence x_n which converges to an irrational number is a Cauchy sequence but does not have a limit in X. The completion of this space is the set of all real numbers $\mathbb{R}$ with the same metric $\rho(x,y) = |x-y|$. Any irrational number can be written as an infinite decimal fraction $0.a_1a_2\ldots$ or, in other words, can be identified with the Cauchy sequence 0, $0.a_1$, $0.a_1a_2$, ... of rational numbers which does not converge to a rational limit.

The space of real numbers $\mathbb{R}$ is <u>defined</u> as the completion of the space of rational numbers and therefore, by definition, is complete.

Example 5.8. Since $\mathbb{R}$ is complete, the space of complex numbers $\mathbb{C}$ with the standard metric $\rho(x,y) = |x-y|$ is also complete. Indeed, if $\{c_n\}$ is a sequence of complex numbers and $c_n = a_n + ib_n$, where $a_n = \operatorname{Re} c_n$ and $b_n = \operatorname{Im} c_n$, then

> $\{c_n\}$ is a Cauchy sequnce if and only if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequnces of real numbers;
> the sequence $\{c_n\}$ converges if and only if the sequences $\{a_n\}$ and $\{b_n\}$ converge.

Theorem 5.9. *$B(S)$ is complete.*

Proof. Let $f_1, f_2, \ldots$ be a Cauchy sequence in $B(S)$. Then for any $\varepsilon > 0$ there exists n_ε such that

$$\sup_{x\in S} |f_n(x) - f_m(x)| \leqslant \varepsilon/2\,, \qquad \forall n,m > n_\varepsilon\,.$$

This implies that for each fixed $x \in S$ the numbers $f_n(x)$ form a Cauchy sequence of real (or complex, if f_n are complex-valued functions) numbers. Since the space of real (or complex) numbers is complete, this sequence has a limit. Let us denote this limit by $f(x)$. Then $f_n(x) \to f(x)$ for each fixed $x \in S$, that is, for any $\varepsilon > 0$ there exists an integer $n_{\varepsilon,x}$ (which may depend on x) such that

$$|f_n(x) - f(x)| \leqslant \varepsilon/2\,, \qquad \forall n > n_{\varepsilon,x}\,.$$

We have

$$|f_n(x) - f(x)| \leqslant |f_n(x) - f_m(x)| + |f_m(x) - f(x)|$$

If $n,m > n_\varepsilon$ and $m > n_{\varepsilon,x}$ then the right hand side is estimated by ε. Therefore the left hand side is estimated by ε for all $x \in S$ (indeed, given x we can always choose m in the right hand side to be greater than n_ε and $n_{\varepsilon,x}$). This implies that $\sup_{x\in S} |f_n(x) - f(x)| \leqslant \varepsilon$ for all $n > n_\varepsilon$, which means that $f_n \to f$ uniformly.

It remains to prove that f is bounded. Choosing $n > n_\varepsilon$ we obtain

$$\begin{aligned}\sup_{x\in S} |f(x)| \leqslant \sup_{x\in S} |f_n(x) - f_n(x) + f(x)| &\leqslant \sup_{x\in S} \left(|f_n(x)| + |f_n(x) - f(x)|\right) \\ &\leqslant \sup_{x\in S} |f_n(x)| + \sup_{x\in S} |f_n(x) - f(x)| \leqslant \sup_{x\in S} |f_n(x)| + \varepsilon\,.\end{aligned}$$

Since f_n is bounded, this estimate implies that f is also bounded. $\square$

Corollary 5.10. *$C[a,b]$ is complete.*

Proof. Since continuous functions on $[a,b]$ are bounded, Theorem 5.9 implies that any Cauchy sequence of continuous functions f_k uniformly converges to a bounded function f on $[a,b]$, and we only need to prove that the function f is continuous.

In order to prove that we have to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| \leqslant \varepsilon$ whenever $|x - y| \leqslant \delta$. We have

$$|f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$$
$$\leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|\,.$$

Since $f_n \to f$ in $B(S)$, we can choose n such that $|f(x) - f_n(x)| \leqslant \varepsilon/3$ and $|f(y) - f_n(y)| \leqslant \varepsilon/3$. Since the function f_n is continuous, there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| \leqslant \varepsilon/3$ whenever $|x - y| \leqslant \delta$. Therefore

$$|f(x) - f(y)| \leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

whenever $|x - y| \leqslant \delta$. □

3. Series in Banach spaces.

Definition 5.11. A complete normed linear space is called a *Banach space*.

Example 5.12. In view of Theorem 5.9 and Corollary 5.10, $B(S)$ and $C[a,b]$ are Banach spaces.

Let x_n be a sequence of elements of a normed linear space X.

Definition 5.13. The series $\sum_{n=1}^{\infty} x_n$ is said to be *convergent* if the sequence σ_k defined by $\sigma_k = \sum_{n=1}^{k} x_n$ is convergent in X. If $\sigma_k \to x \in X$ as $k \to \infty$ then we write $\sum_{n=1}^{\infty} x_n = x$.

Definition 5.14. The series $\sum_{n=1}^{\infty} x_n$ is said to be *absolutely convergent* if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Theorem 5.15. *In a Banach space every absolutely convergent series is convergent.*

Proof. Let $s_k = \sum_{n=1}^{k} \|x_n\|$. Since the series $\sum_{n=1}^{\infty} \|x_n\|$ is convergent, the sequence of positive numbers $\{s_k\}$ converges and therefore it is a Cauchy sequence. If $m > k$, $\sigma_m = \sum_{n=1}^{m} x_n$ and $\sigma_k = \sum_{n=1}^{k} x_n$ then

$$\rho(\sigma_m, \sigma_k) = \|\sigma_m - \sigma_k\| = \|\sum_{n=k+1}^{m} x_n\| \leqslant \sum_{n=k+1}^{m} \|x_n\| = |s_m - s_k|\,.$$

This implies that $\{\sigma_n\}_{n=1,2,\ldots}$ is a Cauchy sequence in our Banach space, and therefore it converges. □

Corollary 5.16. *Let f_n be bounded functions defined on a set S. If $\sum\limits_{n=1}^{\infty} \sup\limits_{x \in S} |f_n(x)| < \infty$ then there exists a bounded function f on S such that*

$$\sup_{x \in S} |f(x) - \sum_{n=1}^{k} f_n(x)| \underset{k \to \infty}{\to} 0\,.$$

If $S = [a,b]$ is a bounded interval and the functions f_n are continuous on $[a,b]$ then f is also continuous.

Proof. The corollary immediately follows from Theorem 5.15 and the fact that $B(S)$ and $C[a,b]$ are Banach spaces (see Theorem 5.9 and Corollary 5.10). □

4. Contractions.

Definition 5.17. A map T from a metric space (X,ρ) to itself is called a contraction if $\rho(Tx,Ty)\leqslant c\rho(x,y)$ for some $0\leqslant c<1$ and all $x,y\in X$.

Theorem 5.18 (The Contraction Mapping Theorem). *If T is a contraction on a complete metric space then the equation $Tx=x$ has a unique solution x and, for any $x_0\in X$, the sequence $x_n=T^nx_0$ converges to x.*

Proof. Let $n>m$. Then, since $\rho(Tx,Ty)\leqslant c\rho(x,y)$, we have

$$\rho(x_m,x_n)=\rho(T^mx_0,T^nx_0)\leqslant c\rho(T^{m-1}x_0,T^{n-1}x_0)$$
$$\leqslant c^2\rho(T^{m-2}x_0,T^{n-2}x_0)\ \dots\ \leqslant c^m\rho(x_0,T^{n-m}x_0)\,.$$

By the triangle inequality

$$\rho(x_0,T^{n-m}x_0)$$
$$\leqslant\rho(x_0,Tx_0)+\rho(Tx_0,T^2x_0)+\rho(T^2x_0,T^3x_0)+\cdots+\rho(T^{n-m-1}x_0,T^{n-m}x_0)$$
$$\leqslant\rho(x_0,Tx_0)+c\rho(x_0,Tx_0)+c^2\rho(x_0,Tx_0)+\cdots+c^{n-m-1}\rho(x_0,Tx_0)$$
$$\leqslant(1-c)^{-1}\rho(x_0,Tx_0)\,.$$

These two inequalities imply that

$$\rho(x_m,x_n)\ \leqslant\ c^m(1-c)^{-1}\rho(x_0,Tx_0)\,,\qquad\forall x_0\in X,\quad\forall n>m\,.\tag{5.1}$$

Since $c<1$, the expression on the right hand side can be made arbitrarily small by choosing large m. This implies that $\{x_n\}$ is a Cauchy sequence. Since our metric space is complete, $\{x_n\}$ converges to a limit x. In view of Lemma 4.4, we have

$$\rho(x,Tx)=\lim_{n\to\infty}\rho(x_n,Tx)=\lim_{n\to\infty}\rho(T^nx,Tx)$$
$$\leqslant c\lim_{n\to\infty}\rho(T^{n-1}x,x)=c\lim_{n\to\infty}\rho(x_{n-1},x)=0\,.$$

Therefore $\rho(x,Tx)=0$, that is, $Tx=x$. If y is another solution of the equation $Ty=y$ then

$$0=\rho(Tx,Ty)-\rho(x,y)\leqslant c\rho(x,y)-\rho(x,y)=(c-1)\rho(x,y)$$

and, consequently, $\rho(x,y)=0$. This implies that x is the only solution of the equation $Tx=x$. $\square$

Theorem 5.18 allows one to construct an approximate solution to an equation of the form $Tx=x$ by choosing an arbitrary element $x_0\in X$ and evaluating $x_m=T^mx$ for sufficiently large m. This is called the *method of successive approximations*.

Corollary 5.19 (error estimate). *Under conditions of Theorem 5.18 we have*

$$\rho(x_m,x)\ \leqslant\ c^m(1-c)^{-1}\rho(x_0,Tx_0)\,,\qquad\forall x_0\in X\,,\quad\forall m=0,1,2,\dots\tag{5.2}$$

Proof. (5.2) is obtained from (5.1) by passing to the limit as $n\to\infty$ and applying Lemma 4.4. $\square$

Example 5.20. Let f be a real-valued function defined on an interval $[a,b]$ such that $f(x) \in [a,b]$ and

$$|f(x) - f(y)| \leqslant c\,|x-y| \tag{5.3}$$

for all $x \in [a,b]$ and some constant $c < 1$. Then, for any $x_0 \in [a,b]$, the sequence $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2), \ldots$ converges to the only solution of the equation $f(x) = x$.

Remark 5.21. The inequality (5.3) (with some $c > 0$) is called the *Lipschitz condition.* If f is continuously differentiable on $[a,b]$ then, by the mean value theorem, f satisfies the Lipschitz condition with $c = \sup_{x\in[a,b]} |f'(x)|$.

6. Connectedness

There are two commonly used notions of connectedness in metric spaces.

Definition 6.1. A subset A of a metric space is *disconnected* if there exists two open sets U_1 and U_2 such that $U_1 \cap U_2 = \varnothing$, neither $A \cap U_1$ nor $A \cap U_2$ are empty and $A \subseteq U_1 \cup U_2$. Such a pair of sets U_1, U_2 is called a *disconnection* of A. A set is *connected* if no disconnection of it exists.

Definition 6.2. Given two points x_1, x_2 of a metric space (X,ρ), a continuous function $f : [0,1] \to X$ with $f(0) = x_1$ and $f(1) = x_2$ is called a *path from x_1 to x_2*. A subset A of X is said to be *path-connected* if there exists a path with values in A between any two points of A.

Remark 6.3. Note that if there is a path f_{12} from x_1 to x_2 and a path f_{23} from x_2 to x_3 then

$$f_{13}(t) = \begin{cases} f_{12}(2t), & \text{if } 0 \leqslant t \leqslant 1/2, \\ f_{23}(2t-1), & \text{if } 1/2 \leqslant t \leqslant 1, \end{cases}$$

is a path from x_1 to x_3.

A connected set is not necessarily path-connected.

Theorem 6.4. *A path-connected set is connected.*

Proof. Suppose A is path-connected but not connected and let U_1, U_2 be a disconnection of A. Choose $x_1 \in A \cap U_1$ and $x_2 \in A \cap U_2$ and let f be a path in A from x_1 to x_2. Since f is continuous, the inverse images $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open subsets of $[0,1]$. Clearly, these subsets are not empty and

$$f^{-1}(U_1) \cap f^{-1}(U_2) = \varnothing, \qquad f^{-1}(U_1) \cup f^{-1}(U_2) = [0,1],$$

that is, the pair $f^{-1}(U_1), f^{-1}(U_2)$ is a disconnection of $[0,1]$.

Let us assume that the point $\{0\}$ lies in $f^{-1}(U_1)$. Then, since $f^{-1}(U_1)$ is open, there exists a positive number $r < 1$ such that the open ball $B_r(0) = [0,r)$ lies in $f^{-1}(U_1)$. Let r_0 be the maximal radius such that $[0,r_0) \subset f^{-1}(U_1)$. Then the point r_0 does not belong to $f^{-1}(U_1)$. On the other hand, every neighbourhood of r_0 contains points from $f^{-1}(U_1)$, so r_0 does not belong to $f^{-1}(U_2)$ (because $f^{-1}(U_2)$ is open and contains a neighbourhood of each point $x \in f^{-1}(U_2)$). Therefore $r_0 \notin f^{-1}(U_1) \cup f^{-1}(U_2)$, and we obtain a contradiction.

Every normed linear space X is path-connected. Indeed, if $x_1, x_2 \in X$ then $f(t) = (1-t)x_1 + tx_2$ is a path from x_1 to x_2.

Lemma 6.5. *Any interval of* $\mathbb{R}$ *is path-connected.*

Proof. Let A be an interval, $x_1, x_2 \in A$ and $x_1 < x_2$. Then $[x_1, x_2] \subset A$. We define $f(t) = (1-t)x_1 + tx_2$, $0 \leqslant t \leqslant 1$. Clearly, $f(t) \in [x_1, x_2]$ for all t, so $f(t)$ is a path between x_1 and x_2 lying in A. This proves that the interval is path-connected and therefore it is connected. □

Lemma 6.6. *Any ball in a normed linear space (with scalars* $\mathbb{R}$ *or* $\mathbb{C}$*) is path-connected.*

Proof. Let $A = B_r(x)$ or $A = B_r[x]$ and $x_1 \in A$. Then $f(t) = (1-t)x_1 + tx$ is a path from x_1 to x. We have

$$\|f(t) - x\| = \|(1-t)x_1 - (1-t)x\| = (1-t)\|x_1 - x\| \leqslant \|x_1 - x\|$$

for all $t \in [0,1]$, which implies that $f(t)$ takes its values in A. In other words, we have proved that, for every point $x_1 \in A$, there exists a path in A from x_1 to the centre x. In view of Remark 6.3, this implies that every two points of A can be joined by a path. □

Theorem 6.7. *Every open connected set in a normed linear space* X *is path-connected.*

Proof. Let A be open and connected and let $x \in A$ be a fixed point. Define

$$U_1 = \{y \in A : \text{there is a path from } x \text{ to } y\}.$$

In view of Remark 6.3, U_1 is path-connected. Since A is open, for every $y \in U_1$ there exists $r > 0$ such that $B_r(y) \subset A$. Then, by Lemma 6.6, for any $z \in B_r(y)$ there is a path from z to y to x, which means that $B_r(y) \subset U_1$. Therefore U_1 is open.

If U_1 does not coincide with A, consider $U_2 = \mathcal{C}(U_1) \cap A$. Since A is open, for any $y \in U_2$ there exists $r > 0$ such that $B_r(y) \subset A$. Note that $B_r(y) \cap U_1$ is empty for if it contains some point z then, by Lemma 6.6, there exists a path from x to z to y and therefore, by Remark 6.3, from x to y. This would imply that $y \in U_1$, contrary to the definition of U_2. Thus $B_r(y) \subset U_2$. Therefore U_2 is open. Also $A = U_1 \cup U_2$ and so U_2 must be empty, otherwise U_1, U_2 is a disconnection of A. Thus $A = U_1$ is path-connected. □

The above theorem shows that the two notions of connectedness are equivalent for open subsets of normed spaces. A most important instance of this is an open subset in the complex plane (this is used in complex analysis).

Theorem 6.8. *Every connected subset of* $\mathbb{R}$ *is a (possibly degenerate) interval.*

Proof. For any connected subset A of $\mathbb{R}$, let $a = \inf A$ and $b = \sup A$ (we assume that $a = -\infty$ or $b = \infty$ if A is not bounded above or below). We show that A contains every point between a and b. Let x be such a point. Since $a < x < b$, there is some element α of A with $\alpha < x$ (otherwise x would be a lower bound $> a$) and similarly there is $\beta \in A$ with $\beta > x$. Then, if $x \notin A$, we would have a disconnection $(-\infty, x), (x, \infty)$ of A and so $x \in A$. □

Theorem 6.9. *Let (X, ρ) and (Y, σ) be metric spaces and let $f : X \to Y$ be continuous. Then the image of any connected subset A of X under f is a connected subset of Y.*

Proof. Suppose that $f(A) = \{f(x) : x \in A\}$ is disconnected in Y and let V_1, V_2 be a disconnection of $f(A)$. Let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. We will show that U_1, U_2 is a disconnection of A and thus obtain a contradiction.

Clearly $A \subset U_1 \cup U_2$ since for $x \in A$, either $f(x) \in V_1$ or $f(x) \in V_2$, that is, either $x \in U_1$ or $x \in U_2$. The fact that $U_1 \cap A \neq \varnothing$ is clear since it contains x whenever $f(x)$ is in the non-empty set $f(A) \cap V_1$. Similarly $U_2 \cap A \neq \varnothing$. Also, if $x \in U_1 \cap U_2$ then $f(x) \in V_1 \cap V_2$. But, since V_1, V_2 is a disconnection, $\varnothing = V_1 \cap V_2 \not\ni f(x)$. So $U_1 \cap U_2 = \varnothing$. Finally, the inverse image characterization of continuity (Theorem 4.8) shows that U_1 and U_2 are open. □

Note that in view of Theorem 6.8 the above theorem includes the Intermediate Value Theorem of elementary analysis.

Example 6.10. Let X be a metric space with discrete metric (see Example 1.7 and 3.21). Then every subset of X is open. This implies that every set which consists of more than one element can be represented as a union of two open non-overlapping subsets and therefore is disconnected. Such metric spaces are called *totally disconnected*.

7. Compactness

Intervals which are bounded and closed figure prominently in analysis on the real line. The appropriate generalization of their essential properties that are relevant to analysis in more general spaces is compactness. There are two definitions of compactness which can be shown to be equivalent.

Definition 7.1. A subset K of a metric space (X, ρ) is said to be *(sequentially) compact* if any sequence of elements of K has a subsequence which converges to a limit in K.

It is clear from the definition that K is compact in (X, ρ) if and only if it is compact in (X, σ) for any metric σ equivalent to ρ.

The second definition needs a little terminology. If $\hat{S}$ is a family of subsets of X and $K \subseteq \cup_{\hat{S}} S$ then $\hat{S}$ is called a *cover* of K. If each member of $\hat{S}$ is open, it is called an *open cover* of K. If $\hat{S}$ is a cover of K and a subset $\hat{S}_0$ of $\hat{S}$ also covers K then $\hat{S}_0$ is called a *subcover* of $\hat{S}$. A cover (or subcover) is said to be finite if it has a finite number of members.

Definition 7.2. A subset K of a metric space (X, ρ) is said to be *compact* if any open cover of K has a finite subcover.

Theorem 7.3. *A set is compact if and only if it is sequentially compact.*

Proof. See Appendix.

Definition 7.4. A subset K of a metric space (X, ρ) is *bounded* if, for some $x \in X$ and $r > 0$, we have $K \subset B_r(x)$.

Theorem 7.5. *A compact set K is bounded and closed.*

Proof. If K is not bounded then, for every $x \in X$, the family of balls $B_n(x)$, $n = 1, 2, \ldots$, is an open cover of K which does not have a finite subcover.

If K is not closed, it does not contain at least one of its limit points. Consider a sequence of elements of K which converges to this limit point. Every subsequence of this sequence converges to the same limit point. Therefore such a sequence does not have a subsequence which converges to a limit in K. □

Lemma 7.6. *A closed subset of a compact set is compact.*

Proof. Let K be compact, K_0 be a closed subset of K and $\{x_n\}$ be a sequence of elements of K_0. Since $\{x_n\} \subset K$, this sequence has a convergent subsequence. Since K_0 is closed, the limit of this subsequence lies in K_0. Therefore any sequence of elements of K_0 has a subsequence which converges to a limit in K_0 which means that K_0 is compact. □

Lemma 7.7. *If K and L are compact subsets of metric spaces (X, ρ) and (Y, σ) respectively then $K \times L$ as a subset of $X \times Y$ with the metric*

$$d\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{\rho(x_1, x_2)^2 + \sigma(y_1, y_2)^2}\,.$$

is compact.

Proof. Let (x_n, y_n) be an arbitrary sequence in $K \times L$. Since K is compact, there is a subsequence x_{n_k} which converges to a limit $x \in K$ as $k \to \infty$. Since L is compact, the sequence y_{n_k} has a subsequence $y_{n_{k_i}}$ which converges to a limit $y \in L$ as $i \to \infty$. Since $x_{n_k} \to x$ as $k \to \infty$, we also have $x_{n_{k_i}} \to x$ as $i \to \infty$. By definition of convergence, $\rho(x_{n_{k_i}}, x) \to 0$ and $\sigma(y_{n_{k_i}}, y) \to 0$ as $i \to \infty$. This implies that $d\big((x_{n_{k_i}}, y_{n_{k_i}}), (x, y)\big) \to 0$ as $i \to \infty$, that is, $(x_{n_{k_i}}, y_{n_{k_i}}) \to (x, y) \in K \times L$. Therefore any sequence (x_n, y_n) of elements of $K \times L$ has a subsequence which converges to a limit in $K \times L$. □

Theorem 7.8. *A bounded and closed subset of $\mathbb{R}^n$ is compact.*

Proof. Since any bounded subset lies in a closed cube Q^n, in view of Lemma 7.6 it is sufficient to prove that the closed cube is compact. The closed cube Q^n is a direct product of a one dimensional closed cube $\mathbb{Q}^1$ (a closed interval) and a closed cube $Q^{n-1} \subset \mathbb{R}^{n-1}$. If $\mathbb{Q}^1$ and $\mathbb{Q}^{n-1}$ are compact then, by Lemma 7.7, $\mathbb{Q}^n$ is also compact. Therefore it is sufficient to prove that a closed interval is compact (after that the required result is obtained by induction in n).

Let x_n be an arbitrary sequence of numbers lying in a closed interval $[a, b]$. Let us split $[a, b]$ into the union of two intervals of length $\delta/2$, where $\delta = b - a$. At least one of these intervals contains infinitely many elements x_n of our sequence. Let us choose one of these elements and denote it by y_1. Now we split the interval of length $\delta/2$ which contains infinitely many elements x_n into the union of two intervals of length $\delta/4$. Again, at least one of these intervals contains infinitely many elements x_n. We choose one of these elements (distinct from y_1) and denote it by y_2. Repeating this procedure, we obtain a subsequence $\{y_k\}$ of the sequence $\{x_n\}$ such that y_k lie in an interval of length 2^{-k_0} for all $k \geqslant k_0$. Clearly, $\{y_k\}$ is a Cauchy sequence. Since $\mathbb{R}$ is a complete metric space, $\{y_k\}$ converges to a limit. Since a closed interval is a closed set, this limit belongs to $[a, b]$. Thus, any sequence of elements of $[a, b]$ has a subsequence which converges to a limit in $[a, b]$, which means that the closed interval is compact. □

Theorem 7.9. *The image of a compact set by a continuous map is compact.*

Proof. Let K be a compact set and T be a continuous map. Let y_n be an arbitrary sequence of elements of $T(K)$. Then $y_n = Tx_n$ where $x_n \in K$. Since K is compact, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to a limit $x \in K$. Then by Theorem 4.2 the subsequence $y_{n_k} = Tx_{n_k}$ converges to the limit $Tx \in T(K)$. This proves that $T(K)$ is compact. □

Definition 7.10. A metric space (X, ρ) is said to be compact if the set X is compact.

Theorem 7.11. *A compact metric space (X, ρ) is complete.*

Proof. Let $\{x_n\} \subset X$ be an arbitrary Cauchy sequence. Since X is compact, this sequence has a subsequence which converges to a limit in X. By Lemma 5.3, the whole sequence $\{x_n\}$ converges to the same limit. □

Theorem 7.12. *Let (X, ρ), (Y, σ) be compact metric spaces and $T : (X, \rho) \to (Y, \sigma)$ be a continuous bijection. Then the inverse mapping T^{-1} is continuous.*

Proof. Applying Theorem 4.8 to T^{-1}, we see that it is sufficient to prove that the inverse image $(T^{-1})^{-1}(B) = T(B) \subset Y$ is closed whenever the set $B \subset X$ is closed.

If B is closed then, by Lemma 7.6. it is compact. By Theorem 7.9 $T(B)$ is also compact and therefore is closed (Theorem 7.5). □

Example 7.13. Let X be the space of continuously differentiable functions on a closed interval $[a, b]$ and ρ, σ be the metrics on X defined as follows:

$$\rho(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)| + \sup_{x\in[a,b]} |f'(x) - g'(x)|,$$
$$\sigma(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

The identical map $(X, \rho) \to (X, \sigma)$ is a bijection and is continuous because $f_n \xrightarrow{\rho} f$ implies $f_n \xrightarrow{\sigma} f$. However, the inverse mapping is not continuous. Indeed, the sequence $f_n(x) = n^{-1} \sin(n^2 x)$ converges to the zero function with respect to the metric σ but does not converge with respect to the metric ρ.

Definition 7.14. We say that a (real or complex-valued) function f defined on a metric space $f : (X, \rho)$ is *uniformly continuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| \leqslant \varepsilon$ whenever $\rho(x, y) \leqslant \delta$.

Obviously, a uniformly continuous function is continuous.

Theorem 7.15. *If (X, ρ) is a compact metric space then any continuous function f on (X, ρ) is uniformly continuous.*

Proof. Let $\varepsilon > 0$. Since f is continuous, for every point $x \in X$ there exists $\delta_x > 0$ such that

$$|f(y) - f(x)| \leqslant \varepsilon/2 \quad \text{whenever} \quad \rho(y, x) \leqslant \delta_x. \tag{7.1}$$

Let $J_x = B_{\delta_x/2}(x)$. Since $x \in J_x$, the collection of open balls $\{J_x\}_{x\in X}$, is an open cover of X. Since X is compact, it has a finite subcover, that is, there exists a finite collection of points $x_1, x_2, \dots, x_k$ such that $X = \cup_{n=1}^{k} J_{x_k}$. Denote $\delta = \frac{1}{2} \min\{\delta_{x_1}, \dots \delta_{x_k}\}$. Since the number of points x_n is finite, we have $\delta > 0$.

Let $x, y \in X$ and $\rho(x,y) \leqslant \delta$. Since $X = \cup_{n=1}^{k} J_{x_n}$, there exists n such that $x \in J_{x_n}$, that is, $\rho(x, x_n) \leqslant \delta_{x_n}/2$. By the triangle inequality

$$\rho(y, x_n) \leqslant \rho(x, x_n) + \rho(x, y) \leqslant \delta_{x_n}/2 + \delta \leqslant \delta_{x_n}$$

and, in view of (7.1),

$$|f(y) - f(x)| \leqslant |f(y) - f(x_n)| + |f(x_n) - f(x)| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus we have proved that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| \leqslant \varepsilon$ whenever $\rho(x, y) \leqslant \delta$. □

Further on in this section, for the sake of simplicity, we shall deal only with real-valued functions. Theorems 7.17 and 7.19 can be easily generalized to complex-valued functions by considering their real and imaginary parts separately.

Definition 7.16. If (K, ρ) is a metric space then $C(K)$ denotes the linear space of continuous functions $f : K \to \mathbb{R}$ provided with the norm $\|f\| = \sup_{x \in K} |f(x)|$.

Theorem 7.17 (Weierstrass' approximation theorem). *Let $\mathcal{P}$ be the set of polynomials in $C[a, b]$, where $[a, b]$ is a finite closed interval. Then $\overline{\mathcal{P}} = C[a, b]$.*

Proof. The theorem follows from Theorem 7.19 (see below).

There are many equivalent ways to state the above theorem. For example:

(1) Any continuous function on a closed bounded interval can be uniformly approximated by polynomials.
(2) Let f be continuous on $[a, b]$. Given any $\varepsilon > 0$ there exists a polynomial p such that $|p(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$.
(3) The polynomials are dense in $C[a, b]$.

Clearly, if $f, g \in C(K)$ then $fg \in C(K)$. A linear space with this property is called an algebra.

Definition 7.18. A set of functions $\mathcal{P} \in C(K)$ is said to be a subalgebra of $C(K)$ if $\mathcal{P}$ is a linear space and $fg \in \mathcal{P}$ whenever $f, g \in \mathcal{P}$.

Theorem 7.19 (Stone–Weierstrass). *Let K be a compact metric space and $\mathcal{P}$ be a subalgebra of $C(K)$. If*

(1) *$\mathcal{P}$ contains the constant functions and*
(2) *for every pair of points $x, y \in K$ there exists a function $f \in \mathcal{P}$ such that $f(x) \neq f(y)$ (in other words, $\mathcal{P}$ separates the points of K)*

then $\overline{\mathcal{P}} = C(K)$.

Proof. See Appendix.

Example 7.20. Finite sums of the form

$$c + \sum_{i=1}^{k} a_i \sin(ix) + \sum_{j=1}^{l} b_j \cos(jx),$$

where c, a_i, b_j are some constants, are called trigonometric polynomials. The Stone–Weierstrass theorem implies that the trigonometric polynomials are dense in $C[a,b]$ for any closed interval $[a,b]$, provided that $b-a<2\pi$. Indeed,

(1) from the equalities

$$\begin{aligned} 2\sin(nx)\cos(mx) &= \sin((n+m)x)+\sin((n-m)x)\,,\\ 2\sin(nx)\sin(mx) &= \cos((n-m)x)-\cos((n+m)x)\,,\\ 2\cos(nx)\cos(mx) &= \cos((n-m)x)+\cos((n+m)x) \end{aligned}$$

it follows that the set of trigonometric polynomials is a subalgebra;

(2) if $\sin(nx)=\sin(ny)$ and $\cos(nx)=\cos(ny)$ for all n then the equalities

$$\begin{aligned} 0=\sin(nx)-\sin(ny) &= 2\cos\frac{n(x+y)}{2}\,\sin\frac{n(x-y)}{2}\,,\\ 0=\cos(ny)-\cos(nx) &= 2\sin\frac{n(x+y)}{2}\,\sin\frac{n(x-y)}{2} \end{aligned}$$

imply that $\sin\frac{n(x-y)}{2}=0$ for all n. This is only possible if $\frac{(x-y)}{2}=k\pi$ for some integer k, which implies that $x=y$ (since $b-a<2\pi$).

8. Integration

1. Step functions.

Definition 8.1. We say that a complex-valued function ψ defined on a finite interval $[a,b]$ is a *step function* if there exists a finite collection of intervals $I_k\subset[a,b]$, $k=1,2,\dots,N$, such that

(1) $\cup_{k=1}^N I_k=[a,b]$;
(2) if $j\neq k$ then $I_j\cap I_k=\varnothing$;
(3) ψ is constant on each interval I_k.

In the above definition the intervals I_k are not necessarily open or closed and may be degenerate. Every step function ψ is determined by the collection of intervals I_k and constants $c_k=\psi|_{I_k}$. We shall write $\psi\sim\{I_k,c_k\}$ if ψ is constant on the intervals I_k and takes the value c_k on I_k.

Lemma 8.2. *The step functions form a linear space, that is,*

(1) *if ψ is a step function and $\lambda\in\mathbb{C}$ then $\lambda\psi$ is a step function;*
(2) *if ψ_1 and ψ_2 are step functions then $\psi_1+\psi_2$ is a step function.*

Proof.
(1) If $\psi\sim\{I_k,c_k\}$ then $\lambda\psi\sim\{I_k,\lambda c_k\}$, so $\lambda\psi$ is a step function.
(2) Let $\psi_1\sim\{I_j,c_j\}$, $j=1,\dots,N_1$, and $\psi_2\sim\{\widetilde I_k,\widetilde c_k\}$, $k=1,\dots,N_2$. The intersections $I_j\cap\widetilde I_k$ are disjoint intervals and $I_j\cap\widetilde I_k\subset[a,b]$. Moreover, since $\cup_{j=1}^{N_1}I_j=[a,b]$ and $\cup_{k=1}^{N_2}\widetilde I_k=[a,b]$, we have

$$\cup_{j=1}^{N_1}\cup_{k=1}^{N_2}(I_j\cap\widetilde I_k)=[a,b]\,.$$

Thus the collection of intervals $\{I_j\cap\widetilde I_k\}$, $j=1,\dots,N_1$, $k=1,\dots,N_2$, satisfies the conditions of Definition 8.1. Clearly, the function $\psi_1+\psi_2$ is constant on each interval $I_j \cap \widetilde{I}_k$ and takes the value $c_j + \widetilde{c}_k$. Therefore $\psi_1 + \psi_2$ is a step function and $\psi_1 + \psi_2 \sim \{I_j \cap \widetilde{I}_k, c_j + \widetilde{c}_k\}$. □

Let $B[a,b]$ be the linear space of bounded functions on $[a,b]$. The step functions are bounded and therefore the set of all step functions is a subset of $B[a,b]$. We denote by $R[a,b]$ the closure of this subset with respect to the standard metric on $B[a,b]$ (Example 1.6). By Corollary 3.20 $f \in R[a,b]$ if and only if there exists a sequence of step functions ψ_n converging to f in $B[a,b]$, that is,

$$\sup_{x\in[a,b]} |f(x) - \psi_n(x)| \to 0\,, \qquad n \to \infty\,.$$

Roughly speaking, $R[a,b]$ consists of all step functions and the functions f which can be approximated by step functions. The functions $f \in R[a,b]$ are called *Riemann integrable* functions.

Theorem 8.3. $C[a,b] \subset R[a,b]$.

Proof. Let $f \in C[a,b]$. Then, by Theorem 7.15, f is uniformly continuous, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x)-f(y)| \leqslant \varepsilon$ whenever $|x-y| \leqslant \delta$. In particular, we can choose $\delta_n > 0$ in such a way that $|f(x) - f(y)| \leqslant 1/n$ whenever $|x - y| \leqslant \delta_n$. Let us split the interval $[a,b]$ into the union of non-overlapping intervals I_k whose lengths are not greater than δ_n, and set $c_k = f(x_k)$ where x_k is an arbitrary point lying in I_k. Let $\psi_n \sim \{I_k, c_k\}$ be the corresponding step function. If $x \in [a,b]$ then $|x - x_k| \leqslant \delta_n$ for some k and, consequently,

$$|f(x) - \psi_n(x)| = |f(x) - f(x_k)| \leqslant 1/n\,.$$

Therefore $|f(x) - \psi_n(x)| \leqslant 1/n$ for all $x \in [a,b]$. This implies that

$$\sup_{x\in[a,b]} |f(x) - \psi_n(x)| \leqslant 1/n \to 0\,, \qquad n \to \infty\,,$$

so $f \in R[a,b]$. □

Definition 8.4. We say that a complex-valued function f defined on a finite interval $[a,b]$ is *piecewise continuous* if there exists a finite collection of intervals $I_n \subset [a,b]$, $n = 1,2,\dots,N$, such that

(1) $\cup_{k=1}^N I_k = [a,b]$;
(2) if $j \neq k$ then $I_j \cap I_k = \varnothing$;
(3) ψ is continuous on each interval I_k and has finite limits at the end points of the interval I_k.

Corollary 8.5. *If f is piecewise continuous then $f \in R[a,b]$.*

Proof. By Theorem 8.3, for every interval we can find a sequence of step functions $\psi_n^{(k)}$ such that

$$\sup_{x\in I_k} |f(x) - \psi_n^{(k)}(x)| \to 0\,, \qquad n \to \infty\,. \tag{8.1}$$

Let us extend $\psi_n^{(k)}$ by zero to the whole interval $[a,b]$ and define

$$\psi_n(x) = \sum_{k=1}^N \psi_n^{(k)}(x).$$

Then ψ_n is a step function defined on $[a,b]$ and, in view of (8.1),

$$\sup_{x\in[a,b]} |f(x) - \psi_n(x)| \to 0\,, \qquad n \to \infty\,,$$

so $f \in R[a,b]$. □

Definition 8.6. If ψ is a step function and $\psi \sim \{I_k, c_k\}_{k=1,\dots,N}$ then we define

$$\int_a^b \psi(x)\,dx = \sum_{k=1}^{N} c_k\,\mu(I_k)\,,$$

where $\mu(I_k)$ is the length of the interval I_k.

Lemma 8.7.

(1) *If ψ is a step function and $\lambda \in \mathbb{C}$ then*

$$\int_a^b \lambda\psi(x)\,dx = \lambda \int_a^b \psi(x)\,dx\,.$$

(2) *If ψ_1, ψ_2 are step functions then*

$$\int_a^b (\psi_1(x) + \psi_2(x))\;dx = \int_a^b \psi_1(x)\,dx + \int_a^b \psi_2(x)\,dx\,.$$

Proof. Let $\psi \sim \{I_k, c_k\}$. Then (1) immediately follows from the definition of the integral and the fact that $\lambda\psi \sim \{I_k, \lambda c_k\}$.

If $\psi_1 \sim \{I_j, c_j\}_{j=1,\dots,N}$ and $\psi_2 \sim \{\widetilde{I}_k, \widetilde{c}_k\}_{k=1,\dots,\widetilde{N}}$ then

$$\psi_1 + \psi_2 \;\sim\; \{I_j \cap \widetilde{I}_k, c_j + \widetilde{c}_k\}$$

(see the proof of Lemma 8.2). Therefore

$$\begin{aligned}\int_a^b (\psi_1(x) + \psi_2(x))\;dx &= \sum_{k=1}^{\widetilde{N}} \sum_{j=1}^{N} (c_j + \widetilde{c}_k)\,\mu(I_j \cap \widetilde{I}_k) \\ &= \sum_{j=1}^{N} \left(c_j \sum_{k=1}^{\widetilde{N}} \mu(I_j \cap \widetilde{I}_k) \right) + \sum_{k=1}^{\widetilde{N}} \left(\widetilde{c}_k \sum_{j=1}^{N} \mu(I_j \cap \widetilde{I}_k) \right).\end{aligned}$$

In view of conditions (1), (2) of Definition 8.1 we have

$$\sum_{k=1}^{\widetilde{N}} \mu(I_j \cap \widetilde{I}_k) = \mu(I_j)\,, \quad \sum_{j=1}^{N} \mu(I_j \cap \widetilde{I}_k) = \mu(\widetilde{I}_k)\,,$$

and therefore

$$\begin{aligned}&\int_a^b (\psi_1(x) + \psi_2(x))\;dx \\ &\qquad = \sum_{k=1}^{N} \widetilde{c}_k\,\mu(\widetilde{I}_k) + \sum_{j=1}^{\widetilde{N}} c_j\,\mu(I_j) = \int_a^b \psi_1(x)\,dx + \int_a^b \psi_2(x)\,dx\,.\end{aligned}$$

This proves (2). □

Lemma 8.8. *If ψ is a step function then*

$$| \int_a^b \psi(x)\, dx \,| \leqslant (b-a) \sup_{x\in[a,b]} |\psi(x)|\,.$$

Proof. Let $\psi \sim \{I_k, c_k\}$. Then

$$| \int_a^b \psi(x)\, dx| = | \sum_{k=1}^{N} c_k\, \mu(I_k)| \leqslant \sum_{k=1}^{N} |c_k|\, \mu(I_k)$$
$$\leqslant \sup_{x\in[a,b]} |\psi(x)| \sum_{k=1}^{N} \mu(I_k) = (b-a) \sup_{x\in[a,b]} |\psi(x)|\,.$$

2. Definition and basic properties of integrals.

Definition 8.9. Let $f \in R[a,b]$ and ψ_n be a sequence of step functions converging to f in $B[a,b]$. We define

$$\int_a^b f(x)\, dx = \lim \int_a^b \psi_n(x)\, dx\,, \qquad n\to\infty, \tag{8.2}$$

and $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$.

In order to show that our definition is correct we have to prove that the limit exists and is independent of the choice of sequence ψ_n.

Assume that $\psi_n \to f$ in $B[a,b]$. Then ψ_n is a Cauchy sequence in $B[a,b]$, that is,

$$\sup_{x\in[a,b]} |\psi_n(x) - \psi_m(x)| \to 0\,, \qquad m,n\to\infty\,.$$

Therefore, in view of Lemmas 8.7 and 8.8, we have

$$| \int_a^b \psi_n(x)\, dx - \int_a^b \psi_m(x)\, dx \,| = | \int_a^b (\psi_n(x) - \psi_m(x))\, dx \,|$$
$$\leqslant (b-a) \sup_{x\in[a,b]} |\psi_n(x) - \psi_m(x)| \to 0$$

as $m,n\to\infty$. This implies that the sequence of complex numbers $\int_a^b \psi_n(x)\, dx$ form a Cauchy sequence, so the limit in (8.2) exists. If $\widetilde{\psi}_n$ is another sequence of step functions converging to f in $B[a,b]$ then

$$\sup_{x\in[a,b]} |\psi_n(x) - \widetilde{\psi}_n(x)| \leqslant \sup_{x\in[a,b]} |\psi_n(x) - f(x)| + \sup_{x\in[a,b]} |f(x) - \widetilde{\psi}_n(x)| \to 0$$

as $n\to\infty$. Therefore

$$| \int_a^b \psi_n(x)\, dx - \int_a^b \widetilde{\psi}_n(x)\, dx \,| = | \int_a^b \Big(\psi_n(x) - \widetilde{\psi}_n(x)\Big)\, dx \,|$$
$$\leqslant (b-a) \sup_{x\in[a,b]} |\psi_n(x) - \widetilde{\psi}_n(x)| \to 0$$

as $n\to\infty$, which implies that the limit does not depend on the choice of sequence ψ_n.

Lemma 8.10. *For all $f \in R[a,b]$ we have*

$$|\int_a^b f(x)\,dx\,| \leqslant (b-a)\sup_{x\in[a,b]}|f(x)|\,.$$

Proof. Let ψ_n be a sequence of step functions converging to f. By Lemma 8.8 we have

$$|\int_a^b \psi_n(x)\,dx\,| \leqslant (b-a)\sup_{x\in[a,b]}|\psi_n(x)|$$

for all n. By the triangle inequality

$$\sup_{x\in[a,b]}|\psi_n(x)| \leqslant \sup_{x\in[a,b]}|f(x)| + \sup_{x\in[a,b]}|\psi_n(x)-f(x)|\,.$$

Combining these two inequalities we obtain

$$\begin{aligned}|\int_a^b f(x)\,dx\,| &= |\lim_{n\to\infty}\int_a^b \psi_n(x)\,dx\,| \\ &= \lim_{n\to\infty}|\int_a^b \psi_n(x)\,dx\,| \leqslant (b-a)\lim_{n\to\infty}\sup_{x\in[a,b]}|\psi_n(x)| \\ &\leqslant (b-a)\sup_{x\in[a,b]}|f(x)| + (b-a)\lim_{n\to\infty}\sup_{x\in[a,b]}|\psi_n(x)-f(x)|\,.\end{aligned}$$

Since $\psi_n \to f$ in $B[a,b]$, the second term on the right hand side is equal to zero, which proves the lemma. □

If $a=b$ then any function defined on the degenerate interval $[a,b]=\{a\}$ is a step function (because it takes only one value). Lemma 8.10 immediately implies that $\int_a^a f(x)\,dx = 0$.

Definition 8.11. If X is a linear space then a linear map $F: X \to \mathbb{C}$ is said to be a *linear functional* on X.

Theorem 8.12. *The map*

$$f \to \int_a^b f(x)\,dx \tag{8.3}$$

is a linear uniformly continuous functional on $R[a,b]$.

Proof. In order to prove that the functional (8.3) is linear it is sufficient to show that

$$\int_a^b \lambda\left(f(x)+g(x)\right)\,dx = \lambda\int_a^b f(x)\,dx + \lambda\int_a^b g(x)\,dx \tag{8.4}$$

for all $f, g \in R[a,b]$ and $\lambda \in \mathbb{C}$. Let ψ_n and φ_n be sequences of step functions converging in $B[a,b]$ to f and g respectively. Then, by Lemma 8.7,

$$\int_a^b \lambda\left(\psi_n(x)+\varphi_n(x)\right)\,dx = \lambda\int_a^b \psi_n(x)\,dx + \lambda\int_a^b \varphi_n(x)\,dx$$

for every n. Taking the limits on the right and left hand sides and using the definition of integral we obtain (8.4).

In order to prove that the functional (8.3) is uniformly continuous we have to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\int_a^b (f(x) - g(x))\,dx\,| \leqslant \varepsilon$ whenever $\sup_{x\in[a,b]} |f(x) - g(x)| \leqslant \delta$. By Lemma 8.10

$$|\int_a^b (f(x) - g(x))\,dx\,| \leqslant (b-a) \sup_{x\in[a,b]} |f(x) - g(x)|\,. \tag{8.5}$$

Therefore, given $\varepsilon > 0$ we can take $\delta = (b-a)^{-1}\varepsilon$. □

If f is a function on $[a, b]$, let us denote

$$f_+(x) = \begin{cases} f(x)\,, & \text{if } f(x) \geqslant 0\,, \\ 0\,, & \text{if } f(x) < 0\,, \end{cases} \qquad f_-(x) = \begin{cases} 0\,, & \text{if } f(x) \geqslant 0\,, \\ -f(x)\,, & \text{if } f(x) < 0\,. \end{cases}$$

Clearly, f_+ and f_- are non-negative functions on $[a, b]$ and

$$f(x) = f_+(x) - f_-(x)\,, \quad |f(x)| = f_+(x) + f_-(x)\,, \quad f_+(x)f_-(x) = 0$$

for all $x \in [a, b]$.

Any real-valued function f is represented as a linear combination of non-negative functions f_+ and f_-. Any complex-valued function f is the linear combination of real-valued functions $\operatorname{Re} f$ and $\operatorname{Im} f$, and therefore it can also be represented as a linear combination of non-negative functions. Since the integral is a linear functional, one can always reduce the the study of the integral of a complex function to the study of integrals of non-negative real-valued functions.

Proposition 8.13. *If $f \in R[a, b]$ is a non-negative function then $\int_a^b f(x)\,dx \geqslant 0$.*

Proof. Let ψ_n be a sequence of step functions converging to f. Since

$$|f(x) - (\psi_n)_+(x)| \leqslant |f(x) - \psi_n(x)|\,, \qquad \forall x \in [a, b]\,,$$

we have

$$\sup_{x\in[a,b]} |f(x) - (\psi_n)_+(x)| \leqslant \sup_{x\in[a,b]} |f(x) - \psi_n(x)| \to 0$$

as $n \to \infty$. This implies that the sequence of step functions $(\psi_n)_+$ also converges to f. Obviously, $\int_a^b (\psi_n)_+(x)\,dx \geqslant 0$ for all n. Taking the limit we obtain that $\int_a^b f(x)\,dx \geqslant 0$. □

Proposition 8.14. *If $f \in R[a, b]$ then $|f| \in R[a, b]$ and*

$$|\int_a^b f(x)\,dx\,| \leqslant \int_a^b |f(x)|\,dx\,. \tag{8.6}$$

Proof. If ψ_n is a sequence of step functions converging to f in $B[a, b]$ then the sequence of step functions $|\psi_n|$ converges in $B[a, b]$ to $|f|$. Therefore $|f| \in R[a, b]$ whenever $f \in R[a, b]$.

Clearly, the estimate (8.6) holds for each step function ψ_n. Taking the limit, we obtain (8.6) for an arbitrary function $f \in R[a, b]$. □

Proposition 8.15. *If $f \in R[a,b]$ and $c \in [a,b]$ then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,. \tag{8.7}$$

Proof. It is sufficient to prove (8.7) for the step functions, after that the general result is obtained by taking the limit.

Assume that ψ is a step function and $\psi \sim \{I_k, c_k\}$. Let $c \in I_{k_0}$. Denote $I'_{k_0} = \{x \in I_{k_0} : x < c\}$ and $I''_{k_0} = \{x \in I_{k_0} : x \geqslant c\}$. Then

$$\begin{aligned}\int_a^b \psi(x)\,dx &= \sum_k c_k\,\mu(I_k) = \sum_{k<k_0} c_k\,\mu(I_k) + c_{k_0}\,\mu(I_{k_0}) + \sum_{k>k_0} c_k\,\mu(I_k)\\ &= \sum_{k<k_0} c_k\,\mu(I_k) + c_{k_0}\,\mu(I'_{k_0}) + c_{k_0}\,\mu(I''_{k_0}) + \sum_{k>k_0} c_k\,\mu(I_k)\\ &= \int_a^c \psi(x)\,dx + \int_c^b \psi(x)\,dx\,.\end{aligned}$$

Lemma 8.10 plays the key role in the theory of Riemann integrals. It implies the fundamental theorem of calculus and all other standard results on integrals (see Appendix).

3. Unbounded functions and unbounded intervals.

Definition 8.16. Let $c \in (a,b)$ and $f(x)$ is bounded for all $x \in [a,c) \cup (c,b]$. If $f \in R[a, c-\varepsilon]$ and $f \in R[c+\delta, b]$ for all positive ε, δ then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\int_a^{c-\varepsilon} f(x)\,dx + \lim_{\delta\to 0}\int_{c+\delta}^b f(x)\,dx\,. \tag{8.8}$$

It may happen that one of the limits in the right hand side of (8.8) does not exist (or is equal to ∞ or $-\infty$), and then we say that $\int_a^b f(x)\,dx$ is not defined.

Remark 8.17. If the interval $[a,b]$ contains a finite collection of points at which the function f is unbounded, we split $[a,b]$ into a union of intervals each of which contains only one such point and then define the integral using Proposition 8.15.

Example 8.18. Let $f(x) = x^{-1}$ and $a < 0 < b$. Then

$$\int_a^{-\varepsilon} f(x)\,dx = \log(\varepsilon) - \log(-a) \to -\infty\,, \qquad \varepsilon \to 0\,,$$

$$\int_\delta^b f(x)\,dx = \log b - \log\delta \to \infty\,, \qquad \delta \to 0\,,$$

so the integral $\int_a^{-\varepsilon} f(x)\,dx$ is not defined. However,

$$\lim_{\varepsilon\to 0}\left(\int_a^{-\varepsilon} f(x)\,dx + \int_\varepsilon^b f(x)\,dx\right) = \lim_{\varepsilon\to 0}\left(\log b - \log(-a)\right) = \log b - \log(-a)$$

exists and finite. Therefore it is important that in (8.8) we consider separate limits $\lim\limits_{\varepsilon\to 0}\int_a^{c-\varepsilon} f(x)\,dx$ and $\lim\limits_{\delta\to 0}\int_{c+\delta}^b f(x)\,dx$.

Definition 8.19. If $f \in R[a,b]$ for all $-\infty < a < b < \infty$ then we define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx\,,$$
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx\,,$$
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx\,.$$

Again, it is important that in the definition of $\int_{-\infty}^\infty f(x)\,dx$ we take separate limits; the limit $\lim\limits_{b\to\infty}\int_{-b}^b f(x)\,dx$ may exist even if the integral $\int_{-\infty}^\infty f(x)\,dx$ is not well defined.

Clearly, $\int_a^b f(x)\,dx = \int_{-\infty}^\infty \widetilde{f}(x)\,dx$, where $\widetilde{f}(x) = f(x)$ if $x \in [a,b]$ and $\widetilde{f}(x) = 0$ otherwise. Therefore any integral over an interval can be written as an integral over the whole real line.

Remark 8.20. If $[a,b]$ is an unbounded interval and $R[a,b]$ is the space of bounded integrable functions on $[a,b]$ then the map $f \to \int_a^b f(x)\,dx$ is not a continuous functional on $R[a,b]$ with respect to the standard metric $\rho(f,g) = \sup\limits_{x\in[a,b]}|f(x) - g(x)|$. Indeed, the sequence of functions

$$f_n(x) = \begin{cases} 1/n\,, & -n \leqslant x \leqslant n\,,\\ 0\,, & \text{otherwise}\,,\end{cases}$$

converges to the zero function in $B(-\infty,\infty)$ but $\int_{-\infty}^\infty f_n(x)\,dx = 2$ for all n.

4. Integrals depending on a parameter.

Theorem 8.21. *If $[a,b]$ and $[\alpha,\beta]$ are finite intervals and $f(x,t)$ is a continuous function on $[a,b]\times[\alpha,\beta]$ then $\int_a^b f(x,t)\,dx$ is a continuous function on $[\alpha,\beta]$.*

Proof. The set $[a,b]\times[\alpha,\beta]$ is compact. Therefore f is uniformly continuous on $[a,b]\times[\alpha,\beta]$ (Theorem 7.15), that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup\limits_{x\in[a,b]}|f(x,t_0) - f(x,t)| \leqslant \varepsilon$ whenever $|t_0 - t| \leqslant \delta$. This implies that

$$\sup_{x\in[a,b]}|f(x,t_0) - f(x,t)| \to 0\,, \quad \text{as } t \to t_0\,.$$

By (8.5)

$$\Big|\int_a^b (f(x,t_0) - f(x,t))\,dx\,\Big| \leqslant (b-a)\sup_{x\in[a,b]}|f(x,t_0) - f(x,t)| \to 0$$

as $t \to t_0$. $\square$

Theorem 8.22. *If $[a,b]$ and $[\alpha,\beta]$ are finite intervals and $f(x,t)$ is a continuous function on $[a,b]\times[\alpha,\beta]$ which is continuously differentiable in t then*

$$\frac{d}{dt}\left(\int_a^b f(x,t)\,dx\right)=\int_a^b \frac{\partial}{\partial t} f(x,t)\,dx\,.$$

Proof. We have

$$\frac{d}{dt}\left(\int_a^b f(x,t)\,dx\right)=\lim_{\delta\to 0}\delta^{-1}\left(\int_a^b f(x,t+\delta)\,dx-\int_a^b f(x,t)\,dx\right)$$

and, by the mean value theorem,

$$\delta^{-1}\left(\int_a^b f(x,t+\delta)\,dx-\int_a^b f(x,t)\,dx\right)$$
$$=\int_a^b \delta^{-1}\left(f(x,t+\delta)-f(x,t)\right)\,dx=\int_a^b f'(x,t+\delta^*)\,dx\,,$$

where $0<\delta^*\leqslant\delta$. If $\delta\to 0$ then $\delta^*\to 0$ and, by Theorem 8.21,

$$\lim_{\delta^*\to 0}\int_a^b f'(x,t+\delta^*)\,dx=\int_a^b f'(x,t)\,dx\,.$$

This completes the proof. □

Theorem 8.23. *If $[a,b]$ is a bounded interval, $f_n\in R[a,b]$ and*

$$\sup_{x\in[a,b]}|f(x)-\sum_{n=1}^k f_n(x)|\underset{k\to\infty}{\longrightarrow}0$$

then

$$\sum_{n=1}^{\infty}\left(\int_a^b f_n(x)\,dx\right)=\int_a^b f(x)\,dx$$

(in other words, we can integrate the series term by term).

Proof. Since the integral is a linear continuous functional on $R[a,b]$, we have

$$\int_a^b f(x)\,dx=\int_a^b\left(\lim_{k\to\infty}\sum_{n=1}^k f_n(x)\right)dx$$
$$=\lim_{k\to\infty}\sum_{n=1}^k\left(\int_a^b f_n(x)\,dx\right)=\sum_{n=1}^{\infty}\left(\int_a^b f_n(x)\,dx\right)$$

(in the first line $\lim\limits_{k\to\infty}$ denotes the limit in $B[a,b]$, in the second line $\lim\limits_{k\to\infty}$ stands for the limit in $\mathbb{C}$). □

Corollary 8.24. *Let $[a,b]$ be a bounded interval and f_n be continuously differentiable functions on $[a,b]$ such that*

(1) $\sum_{n=1}^\infty f_n(x) = f(x)$ *for every* $x \in [a,b]$;
(2) *the series* $\sum_{n=1}^\infty f_n'$ *is uniformly convergent.*

Then the function f is continuously differentiable and

$$\sum_{n=1}^\infty f_n'(x) = f'(x)\,, \qquad \forall x \in [a,b]$$

(in other words, we can differentiate the series term by term).

Proof. Let $\sum_{n=1}^\infty f_n' = \widetilde{f}$. Since the series $\sum_{n=1}^\infty f_n'$ converges uniformly (that is, converges in $C[a,b]$), the function $\widetilde{f}$ is continuous. Theorem 8.23 and the fundamental theorem of calculus imply that

$$\int_a^x \widetilde{f}(t)\,\mathrm{d}t \;=\; \sum_{n=1}^\infty \int_a^x f_n'(t)\,\mathrm{d}t \;=\; \sum_{n=1}^\infty (f_n(x) - f_n(a)) \;=\; f(x) - f(a)$$

for all $x \in [a,b]$. Now the fundamental theorem of calculus implies that f is continuously differentiable and $f' = \widetilde{f}$. $\square$

5. Picard's Existence Theorem for First Order Differential Equations.

Let f be a real-valued function defined on an open domain $\Omega \in \mathbb{R}^2$. Consider the ordinary (non-linear) differential equation

$$\frac{d\varphi}{dx} \;=\; f(x, \varphi(x)) \tag{8.9}$$

with the initial condition $\varphi(x_0) = \varphi_0$, where x is a one dimensional variable, φ is a function of x and φ_0 is some constant.

Theorem 8.25 (Picard's theorem). *Let $(x_0, \varphi_0) \in \Omega$ and f be a continuous function satisfying the Lipschitz condition*

$$|f(x,y_1) - f(x,y_2)| \;\leqslant\; c\,|y_1 - y_2|\,, \tag{8.10}$$

where c is some constant. Then the equation (8.9) with the initial condition $\varphi(x_0) = \varphi_0$ has a unique solution on some interval $[x_0 - \delta, x_0 + \delta]$.

Proof. Since f is continuous, we have $|f(x,y)| \leqslant R$ whenever (x,y) lie in a sufficiently small ball B about the point (x_0, φ_0). Let us choose a small positive constant δ such that

(1) $(x,y) \in B$ whenever $|x - x_0| \leqslant \delta$ and $|y - \varphi_0| \leqslant R\delta$;
(2) $c\delta < 1$.

Denote by C^* the closed ball of radius $R\delta$ centre φ_0 in the space $C[x_0-\delta, x_0+\delta]$; in other words, C^* is the set of all continuous functions ψ on the interval $[x_0-\delta, x_0+\delta]$ satisfying the estimate

$$\sup_{x\in[x_0-\delta,x_0+\delta]} |\psi(x) - \varphi_0| \;\leqslant\; R\delta\,.$$

By Theorem 5.6, the space C^* provided with the metric (1.1) is complete.

By the fundamental theorem of calculus, the equation (8.9) with the initial condition $\varphi(x_0) = \varphi_0$ is equivalent to the integral equation

$$\varphi(x) \;=\; \varphi_0 + \int_{x_0}^{x} f(t, \varphi(t))\, dt\,. \tag{8.11}$$

Consider the map $T : C^* \to C[x_0 - \delta, x_0 + \delta]$ defined by

$$T\psi(x) \;=\; \varphi_0 + \int_{x_0}^{x} f(t, \psi(t))\, dt\,, \quad \text{where } x \in [x_0 - \delta, x_0 + \delta]\,.$$

Then (8.11) is equivalent to the identity $T\varphi = \varphi$. If $\psi \in C^*$ then, by Lemma 8.10, we have

$$|T\psi(x) - \varphi_0| \;\leqslant\; |\int_{x_0}^{x} f(t, \psi(t))\, dt| \;\leqslant\; R\delta\,, \qquad \forall x \in [x_0 - \delta, x_0 + \delta]\,,$$

which implies that $T : C^* \to C^*$. Lemma 8.10 and (8.10) also imply that

$$|T\psi_1(x) - T\psi_2(x)| \leqslant \int_{x_0}^{x} |f(t, \psi_1(t)) - f(t, \psi_2(t))|\, dt \leqslant c\delta \sup_{t \in [x_0 - \delta, x_0 + \delta]} |\psi_1(t) - \psi_2(t)|\,.$$

Since $c\delta < 1$, the above inequality means that the map $T : C^* \to C^*$ is a contraction. Now Picard's theorem follows from Theorem 5.18.