DISTRIBUTIONS, FOURIER TRANSFORMS AND MICROLOCAL ANALYSIS

LTCC LECTURE COURSE FOR PHD STUDENTS

Basic details

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Core Audience: 1st year, pure mathematics

Course Format: 10 hours at 2 hours per week

Keywords: Fourier transform, tempered distributions, singularities, pseudodifferential operators, the elliptic regularity theorem.

Prerequisites: functions of several real variables, partial derivatives, Riemann integrals and their basic properties.

Electronic lecture notes will be available

Syllabus

1. Rapidly decreasing functions.

- Definition of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.
- Convergence in the space $\mathcal{S}(\mathbb{R}^n)$.
- Fourier transform in $\mathcal{S}(\mathbb{R}^n)$ and its basic properties.
- Praceval's formula.
- The inversion formula.

2. The space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

- Definition and examples.
- Operators in the space of distributions $\mathcal{S}'(\mathbb{R}^n)$.
- Supports of distributions.
- Fourier transform in the space $\mathcal{S}'(\mathbb{R}^n)$.
- Divergent integrals.
- The Schwartz kernel theorem.

3. Pseudodifferential operators.

- Oscillatory integrals.
- Definition of pseudodifferential operators (PDOs).
- Amplitudes and symbols of PDOs.
- Continuity of PDOs in the Schwartz spaces.

4. Solving partial differential equations with the use of PDOs.

- Stationary equations with constant coefficients.
- Non-stationary equations with constant coefficients. The heat and wave equations.
- Composition of PDOs.
- Elliptic (pseudo)differential operators and their approximate inverses.

5. Singularities of functions and distributions.

- The singular support and the wave front set.
- The elliptic regularity theorem.
- Propagation of singularities.

Brief description of the course

Suppose that we want to describe singularities of a function $f(x)$ on $\mathbb{R}^n$. In classical analysis one only deals with the variables x, and typical statements look like "the function f has a singularity at the point x_0" or "f is smooth in a neighbourhood of x_0". However, the function may be smooth in one direction and non-smooth in another direction, so such statements contain a limited information about singularities. More detailed description of singularities must involve an additional variable $\xi \in \mathbb{R}^n$ specifying the directions in which the function is not smooth. In other words, the set of singularities should be a subset of $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$, and then we say that "$f$ is not smooth at the point $(x,\xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$" if f is not smooth at the point x in the direction ξ. This is the main idea of microlocal analysis; the word 'microlocal' simply means that we conduct analysis of functions in the space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ of dimension $2n$, even though the functions themselves are defined on the n-dimensional space.

The variable ξ is usually called the dual variable. It is naturally associated with the Fourier transform, which intertwines multiplication by ξ with differentiation with respect to x. The first part of the course will discuss the Fourier transform in the space $\mathcal{S}(\mathbb{R}^n)$ consisting of infinitely smooth functions rapidly decaying at infinity.

Elements of the dual space $\mathcal{S}'(\mathbb{R}^n)$ are called tempered distributions. They are not proper functions but they possess many similar properties: one can define the support of a distribution, differentiate distributions, multiply them by smooth functions. More generally, any linear operator in $\mathcal{S}(\mathbb{R}^n)$ whose transposed is continuous can be extended to $\mathcal{S}'(\mathbb{R}^n)$. In particular, the Fourier transform and partial differential operators with smooth coefficients are well defined in the space $\mathcal{S}'(\mathbb{R}^n)$. The latter is important, since there are partial differential equations which do not have classical solutions but may be solved in the space $\mathcal{S}'(\mathbb{R}^n)$.

It turns out that some divergent integrals of the form $\int F(x,y)\,\mathrm{d}y$ can be understood as distributions. Possibly, the most famous result on distributions is the Schwartz kernel theorem, according to which for any linear operator K in the space of smooth functions there exists a distribution $\mathcal{K}(x,y)$ such that

$$Ku\,(x) \;=\; \int \mathcal{K}(x,y)\,u(y)\,\mathrm{d}y$$

where the integral is understood in the distributional sense. The distribution $\mathcal{K}$ is called the Schwartz kernel of the operator K. Roughly speaking, the theorem means that all linear operators can be regarded as integral operators. Therefore, instead of studying a linear operator K, it is sufficient to consider the corresponding distribution $\mathcal{K}$.

An operator A is said to be *pseudodifferential* if its Schwartz kernel is given by an integral of the form

$$(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\,a(x,y,\xi)\,\mathrm{d}\xi\,,$$

where a is a smooth function on $\mathbb{R}^n_x\times\mathbb{R}^n_y\times\mathbb{R}^n_\xi$ whose derivatives admit certain estimates. Such integrals are called *oscillatory integrals*, and the function a in an oscillatory integral is called an *amplitude.* One can show that an oscillatory integral can be rewritten in the same form but with another amplitude $\sigma_A(x,\xi)$ independent of y. The amplitude σ_A is said to be the *symbol* of the corresponding PDO A. Note that oscillatory integrals are usually divergent and must be understood in the sense of distributions.

If a is a polynomial of ξ then the corresponding PDO is a differential operator; in particular, if a is identically equal to 1 then $A=I$. The other way round, the Schwartz of every differential operator can be represented by an oscillatory integral with an amplitude polynomially depending on ξ. This explains why operators defined by oscillatory integrals are called ‘pseudodifferential’.

One of the main results in the theory of pseudodifferential operators is the composition theorem, according to which the composition AB of two PDOs A and B is equal to the PDO with symbol $\sigma_A\circ\sigma_B$ modulo a negligible term, where

$$\sigma_A\circ\sigma_B \;=\; \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!}\,\partial_\xi^\alpha\sigma_A\,\partial_x^\alpha\sigma_B$$

and the series is understood in an appropriate sense. Under certain conditions on σ_B, one can choose a symbol σ_A satisfying $\sigma_A\circ\sigma_B=1$, which implies that $AB=I$ modulo a negligible term. This allows one to construct approximate inverses to (pseudo)differential operators, and thus to find approximate solutions to (pseudo)differential equations. Note that for operators with constant coefficients (whose symbols do not depend on x), the ‘negligible’ term in the composition theorem is zero and $\sigma_A\circ\sigma_B$ coincides with the usual product of the symbols. Therefore the above construction gives precise solutions for differential equations with constant coefficients.

There are many versions of the theory of pseudodifferential operators which are distinguished by the notion of ‘negligible’ terms. The classical theory has been developed to study singularities of functions and distributions. In this theory the ‘negligible’ objects are infinitely smooth functions, operators with infinitely smooth Schwartz kernels and symbols decaying faster than any negative power of $|\xi|$ as $|\xi|\to\infty$.

Obviously, the set of singularities of a function $f : \mathbb{R}^n \mapsto \mathbb{C}$ can be described as follows: $x_0 \in \mathbb{R}^n$ is not a singular point of f if there exists a smooth function ρ such that

(1) ρ does not vanish in a neighbourhood of x_0,
(2) the product ρf is an infinitely smooth function.

In a similar manner, using a PDO instead the cut-off function ρ, one can define microlocal singularities of f. Namely, we say that the point (x_0, ξ_0) is not a singular point of f if there exists a PDO A such that

(1) its symbol σ_A does not vanish in a conic with respect to ξ neighbourhood of (x_0, ξ_0),
(2) Af is an infinitely smooth function.

The set of microlocal singularities of f is called the wave front set and is denoted by $\mathrm{WF}f$.

Thus we see that PDOs play the same role in microlocal analysis as cut-off function in classical analysis. In particular, using PDOs, one can introduce a microlocal partition of unity, speak about functions 'micro-localized' in a small neighbourhood of a given point (x_0, ξ_0), etc. These ideas are very useful in the theory of partial differential equations, as they often allow one to describe properties of solutions to the equation $Au = f$ in terms of local properties of the symbol σ_A. One of the most famous results in this direction is the following theorem.

Elliptic regularity theorem. Let A be a (pseudo)differential operator with homogeneous in ξ symbol σ_A, and let $Au = f$. Assume that $\sigma_A(x_0, \xi_0) \neq 0$. Then $(x_0, \xi_0) \in \mathrm{WF}u$ if and only if $(x_0, \xi_0) \in \mathrm{WF}f$.

Rougly speaking, this means that under the condition $\sigma_A \neq 0$ singularities of solutions u stay at the same points as singularities of f. If $\sigma_A(x, \xi) = 0$ at some points $(x, \xi) \in \mathbb{R}_x \times \mathbb{R}_\xi$ then the singularities can move away from $\mathrm{WF}f$. This effect is known as propagation of singularities. It will be briefly discussed in the end of the course.

Recommended reading

M. Shubin, *Pseudodifferential Operators and Spectral Theory,* Springer–Verlag, 1987.

Additional optional reading

M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical Limit.* Cambridge University Press, 1999 (LMS Lecture Notes Series, vol. 268).

L. Hörmander, *The Analysis of Linear Partial Differential Operators,* I–IV. Springer–Verlag, 1984.

M. Taylor, *Pseudodifferential operators,* Princeton University Press, Princeton, New Jersey, 1981.

Yu. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators,* American Mathematical Society, 1996.