PREFACE

Spectral asymptotics for partial differential operators have been the subject of extensive research for over a century. It has attracted the attention of many outstanding mathematicians and physicists.

As a characteristic example let us consider the following spectral problem

$$-\Delta v = \lambda^2 v \quad \text{in} \quad M\,, \qquad v|_{\partial M} = 0\,, \tag{0.0.1}$$

where M is a bounded domain in $\mathbb{R}^3$, and Δ is the Laplace operator. The problem (0.0.1) has nontrivial solutions v only for a discrete set of $\lambda = \lambda_k$, which are called eigenvalues. Let us enumerate the eigenvalues in increasing order: $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ In the general case the eigenvalues λ_k can not be evaluated explicitly. Moreover, for large k it is difficult to evaluate them numerically. So it is natural to look for asymptotic formulae for λ_k as $k \to \infty$.

However, for a number of reasons it is traditional in such problems to deal with the matter the other way round, i.e., to study the sequential number k as a function of λ. Namely, let us introduce the counting function $N(\lambda)$ defined as the number of eigenvalues λ_k less than a given λ. Then our asymptotic problem is reformulated as the study of the asymptotic behaviour of $N(\lambda)$ as $\lambda \to +\infty$. The derivation of asymptotic formulae for $N(\lambda)$ is the main subject of this book.

It is well known that for the problem (0.0.1)

$$N(\lambda) = \frac{V}{6\pi^2}\,\lambda^3 + o(\lambda^3)\,, \qquad \lambda \to +\infty\,, \tag{0.0.2}$$

where V is the volume of M. The asymptotic formula (0.0.2) has been known for a long time, it appeared already in the works of Rayleigh. Written in a slightly different form it is known in theoretical physics as the Rayleigh–Jeans law.

Rayleigh [Ra] arrived at (0.0.2) by considering the case when the domain M is a cube of side a. Then, solving the problem (0.0.1) by separation of variables one obtains

$$N(\lambda) = \#\{\vec{q} \in \mathbb{N}^3 : |\vec{q}| < R\}\,,$$

where $R = a\lambda/\pi$. In other words, $N(\lambda)$ is the number of integer lattice points in an octant of a ball of radius R. Clearly, for large R we have

$$N(\lambda) \approx \frac{1}{8}\,(4\pi R^3/3) = \frac{a^3}{6\pi^2}\,\lambda^3 = \frac{V}{6\pi^2}\,\lambda^3\,.$$

Now physical arguments suggest that the same formula should hold for a domain of arbitrary shape.

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Formula (0.0.2) is remarkable not only for its role in the development of theoretical physics, but also for the fact that Rayleigh made a mistake by writing it without the coefficient $1/8$. This mistake was corrected by J.H. Jeans. As pointed out in [Ja], Jeans's contribution to the "Rayleigh–Jeans" law was only the statement: "It seems to me that Lord Rayleigh has introduced an unnecessary factor 8 by counting negative as well as positive values of his integers", [Je, p. 98].

The first rigorous proof of (0.0.2) was given by H. Weyl [We1]. Later R. Courant and D. Hilbert included a proof of (0.0.2) in their classical textbook [CouHilb], which stimulated the study of asymptotic formulae of this type. The list of mathematicians who have contributed to this field includes S. Agmon, V.M. Babich, P.H. Bérard, M.S. Birman, T. Carleman, Y. Colin de Verdiére, J. Duistermaat, B.V. Fedosov, L. Gårding, V.W. Guillemin, L. Hörmander, V.Ya. Ivrii, M. Kac, B.M. Levitan, R.B. Melrose, G. Métivier, Å. Pleijel, R.T. Seeley, M.A. Shubin, M.Z. Solomyak, A. Weinstein, and many others. An extensive bibliographical review can be found in [RoSoSh]. Physicists also worked on spectral asymptotics and have made essential contributions. Being less familiar with the physical literature we shall only mention the names of M.V. Berry, P. Debye, L. Onsager; see also [BaHilf] for further bibliography.

The asymptotic formula (0.0.2) is remarkably simple: the asymptotic coefficient is determined only by the volume of the domain and is independent of its shape. Moreover, a similar one-term asymptotic formula has been established in a very general setting, namely, for an elliptic self-adjoint partial differential operator with variable coefficients acting on a manifold subject to reasonably good boundary conditions.

However, this simplicity and high degree of generality indicate the weaknesses of (0.0.2) and its analogues. First, such formulae involve only the most basic geometric characteristics of M: say, the eigenvalues of the problem (0.0.1) for a cube and a long narrow parallelepiped of the same volume are obviously quite different, but (0.0.2) does not feel this difference. Secondly, one-term asymptotic formulae do not depend on the boundary conditions: say, if we replace in (0.0.1) the Dirichlet boundary condition by the Neumann one the eigenvalues will change substantially, which can not be noticed from (0.0.2). These deficiencies motivated the search for sharper results.

In 1913 H. Weyl put forward [We2] a conjecture concerning the existence of a second asymptotic term. Namely, he predicted that for the problem (0.0.1)

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^3 - \frac{S}{16\pi}\lambda^2 + o(\lambda^2), \qquad \lambda \to +\infty, \tag{0.0.3}$$

where S is the surface area of ∂M. Formula (0.0.3) became known as *Weyl's conjecture*. It was finally justified, under a certain condition on periodic billiard trajectories, by V.Ya. Ivrii [Iv1] and R.B. Melrose [Me] only in 1980. This revived interest to such problems. In particular, in subsequent years Ivrii extended his result on two-term asymptotics to much more general classes of boundary value problems. As our book does not aim to provide a full bibliographic review and reflects the research interests of its authors, we refer only to Ivrii's publications [Iv2]–[Iv4] where the reader can find further references.

Our contribution to the problem concerns the following aspects.

First, we are interested in deriving two-term asymptotic formulae for higher order differential operators.

Secondly, we study the case when the condition on periodic billiard trajectories, which guarantees the existence of a classical second term in Weyl's formula, fails. In this case the second asymptotic term may contain an oscillating function, which depends on the structure of the set of periodic billiard trajectories.

Thirdly, we obtain two-term asymptotic formulae for the spectral function. In this case one has to deal with loops instead of periodic billiard trajectories.

The basic idea which we use for the derivation of spectral asymptotics is due to B.M. Levitan [Ltan]. It involves the study of the singularities of the corresponding evolutionary problem (say, in the case of (0.0.1) this would be the wave equation), and the subsequent application of Fourier Tauberian theorems. This approach produces the sharpest possible results. Levitan's method was developed by L. Hörmander, J.J. Duistermaat, V.W. Guillemin, and R.B. Melrose (see [Hö1], [DuiGui], [DuiGuiHö], [Me]). The most advanced version of this method is due to V.Ya. Ivrii [Iv1]–[Iv4]. Our approach, however, is somewhat different from that of Ivrii, even in the case of the Laplace operator.

We tried to make the book self-contained and all our constructions explicit. The main results are collected in Chapter 1. Chapter 2 introduces the reader to the main technical tools; it can be regarded as a brief introduction to microlocal analysis. Chapters 3–5 are devoted to the proofs of our main results. Chapter 6 lists the basic mechanical applications; it is intended mostly for applied mathematicians and does not require a sophisticated mathematical background. The book also has a number of appendices. Some of them can be read separately from the main text, others contain cumbersome proofs. Appendix A was written by A. Holst, and Appendix B by M. Levitin.

We do not aim at achieving the highest possible degree of generality in our book. In particular, we do not discuss

(1) systems, see [Sa4], [Sa5], [SaVa1], [Va4], [Va6];
(2) piecewise smooth boundaries, see [Va6];
(3) very non-smooth (fractal) boundaries, see [FlLtinVa1], [FlLtinVa2], [FlVa1], [FlVa2], [LtinVa1], [LtinVa2], [Va10].

This book was preceded by survey papers [GolVa], [Sa7], [SaVa2] describing our main results.

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