CHAPTER VI

MECHANICAL APPLICATIONS

6.1. Membranes and acoustic resonators

In this section we consider the situation when M is a region in $\mathbb{R}^n$ and $A = -\Delta$, where $\Delta = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2 + \ldots + \partial^2/\partial y_n^2$ is the Laplacian in Cartesian coordinates. The boundary condition is either Dirichlet $v|_{\partial M} = 0$, or Neumann $(\partial v/\partial x_n)|_{\partial M} = 0$, where x_n is the distance to ∂M.

The cases $n = 2$ and $n = 3$ are, of course, the ones which have a physical meaning.

1. Membrane. In the case $n = 2$ the eigenvalue problem (1.1.1), (1.1.2) describes the vibrations of an isotropic membrane. Here $\lambda = \omega\sqrt{\frac{\rho}{T}}$, where ω is the vibration frequency, and ρ and T are the surface density and tension of the membrane respectively. Throughout this chapter by "frequency" we mean "circular frequency"; consequently, the number of cycles per second is $\frac{\omega}{2\pi}$.

The function $v(y)$ is the deflection (normal displacement) of the membrane. The Dirichlet boundary condition describes a membrane fixed along its edge, and the Neumann boundary condition a membrane whose edge is free (i.e., there are no forces in the direction normal to the unperturbed surface of the membrane acting on its edge).

Let λ_k be an eigenvalue of (1.1.1), (1.1.2). Then the number $\omega_k = \lambda_k\sqrt{\frac{T}{\rho}}$ is called *eigenfrequency* or *natural frequency*. An eigenfrequency corresponds to a vibration of the type $u(t, y) = e^{-it\omega_k} v_k(y)$ occurring without any external forces.

According to Examples 1.2.3, 1.6.14 and 1.6.15 we have

$$N(\lambda) = \frac{S}{4\pi}\lambda^2 \mp \frac{L}{4\pi}\lambda + o(\lambda), \qquad \lambda \to +\infty,$$

where S is the surface area of M, L is the length of ∂M, and the signs "minus" and "plus" correspond to the Dirichlet and Neumann boundary conditions respectively.

2. Acoustic resonator. In the case $n = 3$ the eigenvalue problem (1.1.1), (1.1.2) describes the vibrations of an acoustic medium occupying a resonator (vessel). Here $\lambda = \frac{\omega}{c}$, where ω is the vibration frequency and c is the speed of sound in the medium.

The function $v(y)$ is the potential of displacements of the acoustic medium. The Dirichlet boundary condition describes a resonator with soft walls (zero pressure on ∂M), and the Neumann boundary condition a resonator with rigid walls (zero normal displacement on ∂M).

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In this case according to Examples 1.2.3 and 1.6.16 we have

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^3 \mp \frac{S}{16\pi}\lambda^2 + o(\lambda^2), \qquad \lambda \to +\infty,$$

where V is the volume of M, S is the surface area of ∂M, and the signs "minus" and "plus" correspond to the Dirichlet and Neumann boundary conditions respectively.

6.2. Elastic plates

1. Statement of result. In this section we consider the situation when M is a region in $\mathbb{R}^2$ and $A = \Delta^2$, where $\Delta = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$ is the Laplacian in Cartesian coordinates. The partial differential equation (1.1.1) in this case describes the flexural vibrations of an isotropic thin elastic plate. Here $\lambda = \left(\frac{12(1-\sigma^2)\rho\omega^2}{Eh^2}\right)^{1/4}$, where ω is the vibration frequency, h is the thickness of the plate, and ρ, E, and $0 < \sigma < \frac{1}{2}$ are the volume density, Young's modulus, and Poisson's ratio of the plate material respectively. The function $v(y)$ is the deflection (normal displacement) of the middle surface of the plate.

In order to describe the boundary conditions we use the local coordinates $x = (x_1, x_2)$ from Example 1.6.14. We denote by $\mathbf{k}(x_1)$ the curvature of ∂M; the sign of the curvature is chosen in such a way that $\mathbf{k} > 0$ when the tangent to ∂M remains outside $\overset{\circ}{M}$ in a small neighbourhood of the point of tangency. By $\mathbf{k}'$ we denote the derivative of $\mathbf{k}$ with respect to x_1. Let us introduce the operators

$$F_0 = 1, \qquad F_1 = \frac{\partial}{\partial x_2}, \qquad F_2 = \frac{\partial^2}{\partial x_2^2} + \sigma\frac{\partial^2}{\partial x_1^2} - \sigma\,\mathbf{k}\frac{\partial}{\partial x_2},$$

$$F_3 = \frac{\partial^3}{\partial x_2^3} + (2-\sigma)\frac{\partial^3}{\partial x_1^2 \partial x_2} + 3\,\mathbf{k}\frac{\partial^2}{\partial x_1^2} + (2-\sigma)\,\mathbf{k}'\frac{\partial}{\partial x_1} - (1+\sigma)\,\mathbf{k}^2\frac{\partial}{\partial x_2} - \mathbf{k}\,F_2\,.$$

We shall consider three types of boundary conditions, corresponding to the following choices of "boundary" operators in (1.1.2).

(1) Clamped edge: $B^{(1)} = F_0$, $B^{(2)} = F_1$.
(2) Hinge supported edge: $B^{(1)} = F_0$, $B^{(2)} = F_2$.
(3) Free edge: $B^{(1)} = F_2$, $B^{(2)} = F_3$.

The clamped edge is what a pure mathematician would call Dirichlet boundary conditions (see Example 1.1.10). The other two types of boundary conditions may appear to be exotic to a pure mathematician, and thus require an explanation. Consider the quadratic form

$$\mathcal{E}(v) := \iint_M \left(|\Delta v|^2 + 2(1-\sigma)(|v_{y_1y_2}|^2 - \mathrm{Re}\,(v_{y_1y_1}\overline{v_{y_2y_2}}))\right) dy_1\,dy_2\,.$$

The functional $\mathcal{E}$ is, up to a constant factor, the potential energy of the plate; see, e.g., [GolLidTo], p. 79, formula (1.9). Perturbing v by δv and integrating by parts we get

$$\delta\mathcal{E} = 2\,\mathrm{Re}\left(\iint_M (\Delta^2 v)\,\overline{\delta v}\,dy_1\,dy_2 + \int_{\partial M}\left((F_3 v)\,\overline{\delta v} - (F_2 v)\frac{\partial\overline{\delta v}}{\partial x_2}\right) dx_1\right).$$

Thus, when we vary $\mathcal{E}(v)$ without any constraints on v we obtain the free boundary conditions, and when we vary $\mathcal{E}(v)$ under the constraint $v|_{\partial M} = 0$ we obtain the hinge supported boundary conditions. Conversely, for any function v satisfying any of the three types of boundary conditions described above we have $(\Delta^2 v, v) = \mathcal{E}(v)$.

Weyl's formula for the biharmonic operator has the form

$$N(\lambda) = \frac{S}{4\pi}\lambda^2 + \frac{\beta L}{4\pi}\lambda + o(\lambda), \qquad \lambda \to +\infty, \tag{6.2.1}$$

where S is the surface area of M, L is the length of ∂M, and β is a dimensionless coefficient the value of which is

$$\beta = -1 - \frac{\Gamma(3/4)}{\sqrt{\pi}\,\Gamma(5/4)} \approx -1.763, \tag{6.2.2}$$

$$\beta = -1, \tag{6.2.3}$$

$$\beta = 4\left(-1 + 4\sigma - 3\sigma^2 + 2(1-\sigma)\sqrt{1 - 2\sigma + 2\sigma^2}\right)^{-1/4} - 1 - \frac{4}{\pi}\int_0^1 \tan^{-1}\left[\left(\frac{1 + (1-\sigma)\xi^2}{1 - (1-\sigma)\xi^2}\right)^2 \sqrt{\frac{1-\xi^2}{1+\xi^2}}\,\right] d\xi \tag{6.2.4}$$

for the cases of clamped, hinge supported, and free edge respectively; here Γ is the Gamma function and $\tan^{-1}$ is the inverse $\tan$. For the case of a free edge the graph of β as a function of Poisson's ratio σ is shown on Figure 13.

Figure 13. The coefficient β in the case of a free edge.

The first asymptotic coefficient of (6.2.1) was evaluated in accordance with Example 1.2.3, whereas the second asymptotic coefficient will be evaluated in the next subsection. Note that the one-term asymptotic formula for the biharmonic operator was first obtained by R. Courant [Cou].

As an example let us consider a circular plate, i.e. $M = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 \le 1\}$. Figures 14–16 show numerical results for the cases of clamped, hinge supported, and free edge respectively.

Figure 14. Eigenvalue distribution for a circular plate with a clamped edge.

Figure 15. Eigenvalue distribution for a circular plate with hinge supported edge.

Figure 16. Eigenvalue distribution for a circular plate with a free edge.

The independent variable, plotted along the horizontal axis, is λ^2; using λ^2 instead of λ is more natural from the mechanical point of view because λ^2 is proportional to the frequency ω. Each plot contains three lines: the stepwise line is the actual counting function $N(\lambda)$, and the two smooth lines are the graphs of the functions $\frac{S}{4\pi}\lambda^2$ (line 1) and $\frac{S}{4\pi}\lambda^2 + \frac{\beta L}{4\pi}\lambda$ (line 2). The actual $N(\lambda)$ was plotted using the numerical results from the handbook [Gon], and, for higher eigenvalues, the numerical results of the authors. Figures 14–16 demonstrate the remarkable effectiveness of the two-term asymptotic formula. On all three graphs Courant's one-term asymptotics $\frac{S}{4\pi}\lambda^2$ lies well away from the actual $N(\lambda)$, giving only a very rough approximation which does not feel the boundary conditions. On the other hand the two-term asymptotics $\frac{S}{4\pi}\lambda^2 + \frac{\beta L}{4\pi}\lambda$ goes right through the actual $N(\lambda)$. The two-term asymptotics feels the boundary conditions through the coefficient β.

Note that the use of the two-term asymptotic formula (6.2.1) in the case of a circular plate is justified because M is convex and ∂M analytic, see subsection 1.3.5.

2. Evaluation of the second asymptotic coefficient. In this subsection we demonstrate that for our problem (biharmonic operator) the second asymptotic coefficient is indeed

$$c_1 = \frac{\beta L}{4\pi}, \tag{6.2.5}$$

with β defined in accordance with (6.2.2)–(6.2.4). We do our calculations using the algorithm described in Section 1.6; the centrepiece of this algorithm is formula (1.6.23). The arguments go along the same lines as in Examples 1.6.14, 1.6.15. By producing explicit calculations for the case of the biharmonic operator we want to show that our algorithm for the calculation of the second asymptotic coefficient is a powerful tool in the analysis of concrete boundary value problems of mechanics and mathematical physics.

Clamped edge. The auxiliary one-dimensional spectral problem in this case is

$$d^4v/dx_2^4 - 2\xi_1^2 d^2v/dx_2^2 + \xi_1^4 v = \nu v, \tag{6.2.6}$$

$$v|_{x_2=0} \;=\; dv/dx_2|_{x_2=0} \;=\; 0\,, \tag{6.2.7}$$

with $v \equiv v(x_2)$, $x_2 \in \mathbb{R}_+$. As in Example 1.6.14, further on in this section we assume $\xi_1 \neq 0$.

The problem (6.2.6), (6.2.7) has no eigenvalues, so

$$\mathbf{N}^+(\nu; x_1, \xi_1) \equiv 0\,. \tag{6.2.8}$$

The problem (6.2.6), (6.2.7) has only one threshold $\nu_1^{\mathrm{st}} = \xi_1^4$, and the continuous spectrum is the semi-infinite interval $[\xi_1^4, +\infty)$. The points $\nu > \xi_1^4$ of the continuous spectrum have multiplicity one, and the corresponding generalized eigenfunctions have the form

$$\begin{aligned} v(x_2) \;&=\; \sin\left(x_2\sqrt{\sqrt{\nu}-\xi_1^2} - \psi\right) \;+\; (\sin\psi)\, e^{-x_2\sqrt{\sqrt{\nu}+\xi_1^2}} \\ &=\; \frac{a_1^-\, e^{ix_2\zeta_1^-(\nu)}}{\sqrt{-2\pi A'(\zeta_1^-(\nu))}} \;+\; \frac{a_1^+\, e^{ix_2\zeta_1^+(\nu)}}{\sqrt{2\pi A'(\zeta_1^+(\nu))}} \;+\; \frac{a_2^+\, e^{ix_2\zeta_2^+(\nu)}}{\sqrt{2\pi A'(\zeta_2^+(\nu))}} \end{aligned}$$

(cf. (1.4.7)), where

$$\psi \;=\; \psi(\nu; \xi_1) \;=\; \tan^{-1}\left(\frac{\sqrt{\sqrt{\nu}-\xi_1^2}}{\sqrt{\sqrt{\nu}+\xi_1^2}}\right) \;\in\; (0, \pi/4)\,, \tag{6.2.9}$$

$\zeta_1^\pm(\nu) = \pm\sqrt{\sqrt{\nu}-\xi_1^2}$, $\zeta_2^+(\nu) = i\sqrt{\sqrt{\nu}+\xi_1^2}$, and

$$a_1^\pm \;=\; \mp i\,\sqrt{2\pi\sqrt{\nu}\sqrt{\sqrt{\nu}-\xi_1^2}}\; e^{\mp i\psi}\;. \tag{6.2.10}$$

As the strong simple reflection condition is satisfied, we can use formulae (1.6.15), (1.6.16). Substituting (6.2.10) into (1.6.15) we obtain for $\nu > \xi_1^4$

$$\arg_0 \det\bigl(i\,R(\nu; x_1, \xi_1)\bigr) \;=\; -\pi/2 - 2\psi(\nu; \xi_1) + 2\pi k \tag{6.2.11}$$

with an unknown integer k. Substituting (6.2.11), (6.2.9) into (1.6.16) we establish that $k = 0$. Thus,

$$\arg_0 \det\bigl(i\,R(\nu; x_1, \xi_1)\bigr) \;=\; \begin{cases} 0\,, & \text{if} \quad \nu \le \xi_1^4\,, \\ -\pi/2 - 2\psi(\nu; \xi_1)\,, & \text{if} \quad \nu > \xi_1^4\,. \end{cases} \tag{6.2.12}$$

According to (1.6.19)

$$c_1 \;=\; \int_0^L \int_{-\infty}^{+\infty} \left(\mathbf{N}^+(1; x_1, \xi_1) \;+\; \frac{\arg_0 \det\bigl(i\,R(1; x_1, \xi_1)\bigr)}{2\pi}\right) đ\xi_1\, dx_1\,. \tag{6.2.13}$$

Substituting (6.2.8), (6.2.12), (6.2.9) into (6.2.13) and evaluating the integral we arrive at (6.2.5), (6.2.2).

Hinge supported edge. The auxiliary one-dimensional spectral problem in this case is described by the equation (6.2.6) with boundary conditions

$$v|_{x_2=0} \;=\; d^2v/dx_2^2\big|_{x_2=0} \;=\; 0\,. \tag{6.2.14}$$

The operator associated with the spectral problem (6.2.6), (6.2.14) is the square of the operator associated with the spectral problem (1.6.22), (1.6.23). Consequently our coefficient c_1 is the same as in Example 1.6.14, which proves (6.2.5), (6.2.3).

Free edge. The auxiliary one-dimensional spectral problem in this case is described by the equation (6.2.6) with boundary conditions

$$\big(d^2v/dx_2^2 - \sigma\xi_1^2 v\big)\big|_{x_2=0} \;=\; \big(d^3v/dx_2^3 - (2-\sigma)\xi_1^2 dv/dx_2\big)\big|_{x_2=0} \;=\; 0\,. \tag{6.2.15}$$

In the following analysis we assume $\sigma \neq 0$. The fact that our formula (6.2.5), (6.2.4) remains valid for $\sigma = 0$ would follow by continuity, see Lemma A.4.1.

The problem (6.2.6), (6.2.15) has one eigenvalue

$$\nu_1 \;=\; \nu_1(\xi_1) \;=\; \Big(-1 + 4\sigma - 3\sigma^2 + 2(1-\sigma)\sqrt{1 - 2\sigma + 2\sigma^2}\Big)\,\xi_1^4\,. \tag{6.2.16}$$

We have $\nu_1 < \xi_1^4$, so this eigenvalue lies below the continuous spectrum. If we look at the dependence of the right-hand side of (6.2.16) on Poisson's ratio σ, we get Taylor's expansion

$$\nu_1 \;=\; (1 - \sigma^4/2 + O(\sigma^5))\,\xi_1^4 \qquad \text{as} \qquad \sigma \to 0\,. \tag{6.2.17}$$

It is remarkable that (6.2.17) contains no quadratic term. This suggests that for realistic values of σ the eigenvalue ν_1 is very close to the threshold ξ_1^4. Say, for $\sigma = \frac{1}{3}$ we get $\nu_1 = \sqrt{\frac{80}{81}}\,\xi_1^4$.

The existence of one eigenvalue (6.2.16) implies

$$\mathbf{N}^+(\nu; x_1, \xi_1) \;=\; \begin{cases} 0 & \text{for} \quad \nu \le \nu_1(\xi_1)\,, \\ 1 & \text{for} \quad \nu > \nu_1(\xi_1)\,. \end{cases} \tag{6.2.18}$$

The generalized eigenfunctions now have the form

$$\begin{aligned} v(x_2) \;&=\; \sin\left(x_2\sqrt{\sqrt{\nu} - \xi_1^2} - \psi\right) \;+\; \frac{\sqrt{\nu} - (1-\sigma)\xi_1^2}{\sqrt{\nu} + (1-\sigma)\xi_1^2}\,(\sin\psi)\,e^{-x_2\sqrt{\sqrt{\nu}+\xi_1^2}} \\ &=\; \frac{a_1^-\,e^{ix_2\zeta_1^-(\nu)}}{\sqrt{-2\pi A'(\zeta_1^-(\nu))}} \;+\; \frac{a_1^+\,e^{ix_2\zeta_1^+(\nu)}}{\sqrt{2\pi A'(\zeta_1^+(\nu))}} \;+\; \frac{a_2^+\,e^{ix_2\zeta_2^+(\nu)}}{\sqrt{2\pi A'(\zeta_2^+(\nu))}}\,, \end{aligned}$$

where

$$\psi \;=\; \psi(\nu;\xi_1) \;=\; \tan^{-1}\left[\left(\frac{\sqrt{\nu} + (1-\sigma)\xi^2}{\sqrt{\nu} - (1-\sigma)\xi^2}\right)^2 \sqrt{\frac{\sqrt{\nu} - \xi^2}{\sqrt{\nu} + \xi^2}}\;\right] \in\; (0, \pi/2)\,, \tag{6.2.19}$$

and the $a_1^\pm$ are given by formula (6.2.10) with our new ψ. Repeating the arguments done in the case of a clamped edge, we conclude that (6.2.12) holds with our new ψ. Substituting (6.2.18), (6.2.16), (6.2.12), (6.2.19) into (6.2.13) we arrive at (6.2.5), (6.2.4).

6.3. Two- and three-dimensional elasticity

1. Statement of result. Let M be a region in $\mathbb{R}^n$, y Cartesian coordinates in $\mathbb{R}^n$, and v an n-component vector-function. We consider the spectral problem for the system of equations

$$-c_t^2\,\Delta v \,-\, (c_l^2 - c_t^2)\,\mathrm{grad}\,\mathrm{div}\, v \,=\, \lambda^2\, v \tag{6.3.1}$$

subject to the Dirichlet boundary conditions $v|_{\partial M} = 0$ (fixed boundary) or the conditions of free boundary. The latter are the variational boundary conditions generated by the quadratic functional

$$\mathcal{E}(v) \,:=\, \int_M \Big((c_l^2 - 2c_t^2)|\mathrm{div}\, v|^2 + \frac{c_t^2}{2} \sum_{i,j} |\partial_{y_j} v_i + \partial_{y_i} v_j|^2 \Big)\, dy\,.$$

The system (6.3.1) describes the vibrations of an isotropic elastic body, see [Lan-Lif, Sect. 22]. Here $\lambda = \omega$ is the vibration frequency, and the constants c_l, c_t are the velocities of longitudinal and transverse waves respectively. They are assumed to satisfy the inequality $c_l/c_t > \sqrt{2}$. Further we denote $\alpha = c_t^2\, c_l^{-2}$.

By considering (6.3.1) we are breaking the promise made in the Preface not to discuss systems in the book. However, we feel it necessary to outline the result for the system of elasticity because this system has played a special role in the development of the subject, see next subsection.

The cases $n = 2$ and $n = 3$ are those of physical interest.

Note that problems in two-dimensional elasticity may arise in two ways: as a result of separation of variables in a infinite three-dimensional elastic cylinder, or in the study of tangential vibrations of a thin elastic plate. The expressions for the velocities c_l, c_t through Young's modulus, volume density and Poisson's ratio σ in these two physical models are different, but $\alpha = 1/2$ always corresponds to $\sigma = 0$.

Weyl's formula for two-dimensional elasticity has the form

$$N(\lambda) \,=\, \frac{(c_l^{-2} + c_t^{-2})S}{4\pi}\,\lambda^2 \,+\, \frac{\beta L}{4\pi c_t}\,\lambda \,+\, o(\lambda)\,, \qquad \lambda \to +\infty\,,$$

where S is the surface area of M, L is the length of ∂M, and β is a dimensionless coefficient. The value of the coefficient β for fixed boundary is

$$\beta \,=\, -1 \,-\, \sqrt{\alpha} \,-\, \frac{4}{\pi} \int_{\sqrt{\alpha}}^{1} \tan^{-1} \sqrt{(1 - \alpha\xi^{-2})(\xi^{-2} - 1)}\; d\xi\,.$$

For free boundary

$$\beta \,=\, 4\gamma^{-1} - 3 \,+\, \sqrt{\alpha} \,+\, \frac{4}{\pi} \int_{\sqrt{\alpha}}^{1} \tan^{-1} \frac{(2 - \xi^{-2})^2}{4\sqrt{(1 - \alpha\xi^{-2})(\xi^{-2} - 1)}}\; d\xi\,,$$

where $0 < \gamma < 1$ is the root of the algebraic equation

$$\gamma^6 - 8\gamma^4 + 8(3 - 2\alpha)\gamma^2 - 16(1 - \alpha) = 0\,.$$

The quantity $c_R = \gamma c_t$ has the physical meaning of the velocity of the Rayleigh wave, see [LanLif, Sect. 24]. The graph of β as a function of α for the cases of fixed and free boundary is shown on Figures 17 and 18 respectively.

Figure 17. The coefficient β in the case of a fixed boundary.

Figure 18. The coefficient β in the case of a free boundary.

Weyl's formula for three-dimensional elasticity has the form

$$N(\lambda) = \frac{(c_l^{-3} + 2c_t^{-3})V}{6\pi^2}\,\lambda^3 + \frac{bS}{16\pi}\,\lambda^2 + o(\lambda^2)\,, \qquad \lambda \to +\infty\,,$$

where V is the volume of M, S is the surface area of ∂M, and

$$b = -\frac{3c_l^4 + c_l^2 c_t^2 + 2c_t^4}{c_l^2 c_t^2 (c_l^2 + c_t^2)}\,, \qquad b = \frac{3c_l^4 - 3c_l^2 c_t^2 + 2c_t^4}{c_l^2 c_t^2 (c_l^2 - c_t^2)} \tag{6.3.2}$$

for the cases of fixed and free boundary, respectively.

Rigorous statements concerning classical two-term asymptotics for systems can be found in [Va4, Sect. 6], [Va6], [SaVa1]. These results deal with the case when the eigenvalues of the principal symbol (which is now a matrix-function) have constant multiplicity. The latter is true for (6.3.1).

Without going into details let us note that the second asymptotic coefficient for systems can be evaluated by the algorithm from Section 1.6 with small modifications described in [Va4, Sect. 6]. Choosing normal coordinates and applying this algorithm to the elasticity system with free boundary it is easy to establish the existence of one eigenvalue $\nu_1(\xi') = c_R^2|\xi'|^2$ corresponding to the Rayleigh wave. In the three-dimensional case this eigenvalue does not appear explicitly in our final expression for the coefficient b only because a contour integration was carried out to simplify the result.

2. Historical background. The one-term asymptotic formula for $N(\lambda)$ in the case of the three-dimensional elasticity operator was obtained by P. Debye [De] in 1912. Debye arrived at this formula by considering the situation when M is a ball, and then extended the result to arbitrary shapes by physical arguments. Debye's analysis involved delicate manipulations with Bessel functions. Note that Debye could not use a cube for his calculations (as Rayleigh did for the Laplacian) because the elasticity problem in a cube does not admit a separation of variables. The difficulty is with the boundary conditions: both the fixed and the free boundary conditions prevent one from separating variables. Later M. Born pointed out that Debye could have simplified matters by dealing with periodic boundary conditions, as in this case the problem in a cube admits a separation variables. One can guess that Debye did not adopt such an approach because periodic boundary conditions do not have a physical meaning for a finite solid body.

Debye's work provided motivation for Weyl's research. Having started with the Laplacian, Weyl eventually produced [We3] in 1915 a rigorous proof of the one-term asymptotics for elasticity.

Debye derived his one-term asymptotic formula for $N(\lambda)$ in order to evaluate the specific heat C of a (three-dimensional) body at low temperatures T:

$$C \approx \mathbf{c}_0 V T^3 , \tag{6.3.3}$$

where $\mathbf{c}_0$ is a constant expressed through the first asymptotic coefficient of $N(\lambda)$ and some physical constants, including Planck's constant. Formula (6.3.3) is known in theoretical physics as *Debye's law*.

Naturally, a two-term asymptotic formula for $N(\lambda)$ would lead to a correction in Debye's law:

$$C \approx \mathbf{c}_0 V T^3 + \mathbf{c}_1 S T^2 . \tag{6.3.4}$$

The correction term $\mathbf{c}_1 S T^2$ becomes noticeable in (6.3.4) if the temperature is sufficiently low and the surface area S is sufficiently large (say, if we are measuring the specific heat of a fine powder). This explains the interest of theoretical physicists in deriving the second asymptotic coefficient of $N(\lambda)$ for the elasticity operator. After a number of publications by different authors producing incorrect formulae for the second asymptotic coefficient, the correct formulae (6.3.2) were obtained by M. Dupuis, R. Mazo and L. Onsager [DupMazOn] in 1960. The argumentation in [DupMazOn] is carried out on a physical level of rigour and the mathematical technique is different from that of Section 1.6.

6.4. Elastic shells

The notation in this and next sections is different from the rest of the book because we had to conform with traditions of shell theory. In particular h is not the Hamiltonian, but the shell thickness.

1. Statement of the problem. Let M be a smooth two-dimensional surface embedded in $\mathbb{R}^3$. Let $x = (x_1, x_2)$ be local coordinates on M, and let the surface be locally given by a three-component radius-vector $\mathbf{r}(x)$. We choose the coordinates x to be orthogonal, so that the first quadratic form of the surface is $d\mathbf{r}^2 = A_1^2 dx_1^2 + A_2^2 dx_2^2$, where $A_i = A_i(x) > 0$, $i = 1, 2$. We set $\mathbf{e}_i = \mathbf{e}_i(x) = A_i^{-1} \mathbf{r}_{x_i}$ (the subscript $_{x_i}$ indicates a partial derivative), and $\mathbf{n} = \mathbf{n}(x) = \mathbf{e}_1 \times \mathbf{e}_2$. Clearly, $\mathbf{e}_i$ is the unit vector in the direction of the coordinate line x_i, $\mathbf{n}$ is the unit normal to M, and the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ form a right triple. We shall assume for simplicity that our coordinate lines coincide with the curvature lines, so that the second quadratic form of the surface is $-\mathbf{n} \cdot d^2\mathbf{r} = R_1^{-1} A_1^2 dx_1^2 + R_2^{-1} A_2^2 dx_2^2$, where $d^2\mathbf{r} = \sum_{i,j} \mathbf{r}_{x_i x_j} dx_i dx_j$, and $R_i^{-1} = R_i^{-1}(x)$, $i = 1, 2$, are the principal curvatures. Our surface M may, of course, have a boundary ∂M which is a smooth one-dimensional curve.

A *shell* is an elastic body occupying the three-dimensional region

$$\{y \in \mathbb{R}^3 : \ y = \mathbf{r}(x) + x_3 \mathbf{n}, \ x \in M, \ |x_3| \le h/2\},$$

where $y = (y_1, y_2, y_3)$ are Cartesian coordinates, and $0 < h \ll 1$ is a parameter called the shell *thickness*. We shall assume the faces $\{x_3 = \pm h/2\}$ to be free, and the edge $\partial M \times [-h/2, h/2]$ to be fixed. The problem of free vibrations of such an elastic body is the one considered in the previous section. However, applying the results of the previous section does not make much sense when h is small: one would get asymptotic formulae which start working only at extremely high frequencies, namely, frequencies $\omega \gg h^{-1}$.

The proper way of telling whether the thickness is small or not is to introduce a characteristic length (or characteristic radius of curvature) R associated with the surface M, and to deal with the *relative* thickness h/R. In most technical applications the relative thickness is very small. It is quite usual to have $h/R \sim 10^{-3}$, whereas a shell with $h/R \sim 10^{-2}$ may be viewed as a rather thick one (this roughly corresponds to the hull of a submarine).

Therefore we shall fix our frequency range and study the behaviour of natural frequencies as $h \to 0$. In this case the elasticity equations (6.3.1) are reduced to the following system of three partial differential equations on M called *shell equations*:

$$\sum_{j=1}^{3} \mathcal{L}_{ij} u_j = \lambda \, u_i , \qquad i = 1, 2, 3. \tag{6.4.1}$$

The $\mathcal{L}_{ij}$ are the linear differential operators of shell theory which have the form

$$\mathcal{L}_{ij} = \frac{h^2}{12} n_{ij} + \ell_{ij} , \qquad i, j = 1, 2, 3. \tag{6.4.2}$$

Here n_{ij} and ℓ_{ij} are the moment and membrane operators, respectively.

We recall explicit expressions for ℓ_{ij}, n_{ij} from [GolLidTo, pp. 77, 78]:

$$\begin{aligned}
\ell_{ii} =& -\frac{1}{1-\sigma^2}\frac{1}{A_i}\frac{\partial}{\partial x_i}\frac{1}{A_iA_j}\frac{\partial}{\partial x_i}A_j - \frac{1}{2(1+\sigma)}\frac{1}{A_j}\frac{\partial}{\partial x_j}\frac{1}{A_iA_j}\frac{\partial}{\partial x_j}A_i - \frac{1}{1+\sigma}R_i{}^{-1}R_j{}^{-1},\\
\ell_{ij} =& -\frac{1}{1-\sigma^2}\frac{1}{A_i}\frac{\partial}{\partial x_i}\frac{1}{A_iA_j}\frac{\partial}{\partial x_j}A_i + \frac{1}{2(1+\sigma)}\frac{1}{A_j}\frac{\partial}{\partial x_j}\frac{1}{A_jA_i}\frac{\partial}{\partial x_i}A_j ,\\
\ell_{i3} =& -\frac{1}{1-\sigma^2}\frac{1}{A_i}\frac{\partial}{\partial x_i}\left(R_i{}^{-1}+R_j{}^{-1}\right) + \frac{1}{1+\sigma}\frac{1}{A_iR_j}\frac{\partial}{\partial x_i},\\
\ell_{3i} =& \frac{1}{1-\sigma^2}\frac{1}{A_iA_j}\left(R_i{}^{-1}+R_j{}^{-1}\right)\frac{\partial}{\partial x_i}A_j - \frac{1}{1+\sigma}\frac{1}{A_iA_j}\frac{\partial}{\partial x_i}\frac{A_j}{R_j},\\
\ell_{33} =& \frac{1}{1-\sigma^2}\left(R_1{}^{-2}+2\sigma R_1{}^{-1}R_2{}^{-1}+R_2{}^{-2}\right)
\end{aligned}$$

(in the above formulae $i,j \le 2$, $i \ne j$),

$$n_{ij} = 0 \qquad \text{for} \qquad i+j < 6\,,$$

$$n_{33} = \frac{1}{1-\sigma^2}\Delta_M^2 + \frac{1}{A_1A_2}\left(\frac{\partial}{\partial x_1} R_1^{-1}R_2^{-1}\frac{A_2}{A_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}R_1^{-1}R_2^{-1}\frac{A_1}{A_2}\frac{\partial}{\partial x_2}\right)\,,$$

where $0 < \sigma < \frac{1}{2}$ is Poisson's ratio and

$$\Delta_M \;=\; \frac{1}{A_1A_2}\left(\frac{\partial}{\partial x_1}\frac{A_2}{A_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{A_1}{A_2}\frac{\partial}{\partial x_2}\right)$$

is the surface Laplacian acting on functions. It can be checked that the matrix differential operators $\left(\ell_{ij}\right)_{i,j=1}^3$ and $\left(n_{ij}\right)_{i,j=1}^3$ are formally self-adjoint and nonnegative with respect to the standard $L_2(M)$-inner product on vector-functions

$$(u,v) \;=\; \iint_M (u_1\overline{v_1} + u_2\overline{v_2} + u_3\overline{v_3})\,dS\,, \tag{6.4.3}$$

$dS = A_1A_2\,dx_1dx_2$.

In (6.4.1) $\lambda = \frac{\rho_s\omega^2}{E}$, where ω is the vibration frequency, and ρ_s and E are the volume density and Young's modulus of the shell material respectively. The vector-function $u(x) = (u_1(x), u_2(x), u_3(x)$ is the displacement of the shell middle surface; its representation in the Cartesian coordinates in $\mathbb{R}^3$ is $\mathbf{u}(x) = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{n}$. It is important to note that u_3 is the displacement in the *normal* direction.

The system of equations (6.4.1) has to be supplemented by the appropriate boundary conditions. As we assumed the original three-dimensional body to be fixed along the edge $\partial M \times [-h/2, h/2]$ these boundary conditions turn out to be

$$u_1|_{\partial M} = u_2|_{\partial M} = u_3|_{\partial M} = \partial u_3/\partial x_1|_{\partial M} = 0\,. \tag{6.4.4}$$

Throughout this section we use near ∂M special local coordinates $x = (x_1, x_2)$ in which $\partial M = \{x_1 = 0\}$. Such a convention is contrary to the rest of the book, but is traditional for shell theory.

Clearly, apart from (6.4.4) there are many other meaningful boundary conditions for the shell equations (6.4.1), but we shall not discuss them for the sake of brevity.

In the special case when the surface M is flat (i.e., $R_1^{-1} \equiv R_2^{-1} \equiv 0$) the shell becomes a plate. It is easy to see that in this case the problem (6.4.1), (6.4.4) separates into two problems which were already considered in Sections 6.2, 6.3.

Further on we use the notation

$$|\xi|_x = \sqrt{A_1^{-2}\xi_1^2 + A_2^{-2}\xi_2^2}\,, \quad K(x,\xi) = |\xi|_x^{-2}(R_1^{-1}A_2^{-2}\xi_2^2 + R_2^{-1}A_1^{-2}\xi_1^2)\,, \tag{6.4.5}$$

where $\xi = (\xi_1, \xi_2) \in T'_xM$. Clearly, $|\xi|_x$ is the principal symbol of Δ_M , and $K(x,\xi)$ is the curvature of the normal section of the surface M at the point x in the direction $\xi_1 dx_1 + \xi_2 dx_2 = 0$.

2. Spectral properties of the shell operator. The shell equations (6.4.1) are simpler than the original elasticity equations (6.3.1) because they contain only two independent variables instead of three. On the other hand, their structure is nontrivial in that different equations of the system (6.4.1) have different orders: the orders are 2, 2, 4, respectively. Such systems should be treated in accordance with the theory developed by S. Agmon, A. Douglis, and L. Nirenberg [AgmDoNir]. It can be shown [AsLid], [GolLidTo] that the problem (6.4.1), (6.4.4) is indeed elliptic in the Agmon–Douglis–Nirenberg sense and generates a self-adjoint operator the spectrum of which is discrete and accumulates to $+\infty$. By $N(h,\lambda)$ we shall denote the counting function, that is, the number of eigenvalues of (6.4.1), (6.4.4) below a given λ. Here we wrote h as a variable to remind of the dependence of the operators (6.4.2) on the small parameter h.

We intend to fix a $\lambda > 0$ and study the asymptotic behaviour of the function $N(h,\lambda)$ as $h \to 0$. Let us stress that fixing h and letting λ tend to infinity would not make mechanical sense: one would get asymptotic formulae which start working for $\lambda \gg h^{-2}$, and this corresponds to frequencies at which one can no longer use the shell equations (6.4.1) and should switch to three-dimensional elasticity (6.3.1).

In studying the problem (6.4.1), (6.4.4) the first impulse is to put $h = 0$. This leads to the so-called membrane problem

$$\sum_{j=1}^{3} \ell_{ij} u_j = \lambda\, u_i\,, \qquad i = 1, 2, 3\,, \tag{6.4.6}$$

$$u_1|_{\partial M} = u_2|_{\partial M} = 0 \tag{6.4.7}$$

(note that the number of boundary conditions is different compared with (6.4.4)). The spectral problem (6.4.6), (6.4.7) is associated with a self-adjoint operator which can be viewed as the limit of the operator associated with (6.4.1), (6.4.4). Namely, let us denote by $E_\lambda(h)$ and E_λ the spectral projections of (6.4.1), (6.4.4) and (6.4.6), (6.4.7), respectively, and let $g = (g_1(x), g_2(x), g_3(x))$ be an arbitrary vector-function from $L_2(M)$. Then, as shown in [AsLid], if $\lambda \in \mathbb{R}$ is not an eigenvalue of (6.4.6), (6.4.7) we have $\lim\limits_{h\to 0} \|(E_\lambda(h) - E_\lambda)g\|_{L_2(M)} = 0$; see also [S.-PaVa].

Despite the convergence of spectral projections the spectra of (6.4.1), (6.4.4) and (6.4.6), (6.4.7) are completely different. In particular, the membrane problem (6.4.6), (6.4.7) always has an essential spectrum. This essential spectrum is a union of two sets: the interval

$$\Big[\min_{T'M} K^2(x,\xi)\,,\ \max_{T'M} K^2(x,\xi)\,\Big]$$

which is the set of λ at which the ellipticity of $\big(\ell_{pq}\big)_{i,j=1}^{3}$ is violated, and the set of λ at which the Shapiro–Lopatinskii condition is violated (this set can also be described explicitly, see [AsLid], [GolLidTo]). Here the ellipticity of ℓ_{ij} and the Shapiro–Lopatinskii condition are understood in the Agmon–Douglis–Nirenberg sense.

3. Spectral asymptotics for the shell operator. Let us fix some $\lambda_{\max} > 0$. Then uniformly over $\lambda \in [0, \lambda_{\max}]$ we have

$$N(h,\lambda) \,=\, a(\lambda)\, h^{-1} \,+\, O(h^{-9/10})\,, \qquad h \to 0\,, \tag{6.4.8}$$

where

$$a(\lambda) \,=\, \varkappa^2 \int\limits_{\xi^4 + K^2(x,\xi) < \lambda} dx\, đ\xi \tag{6.4.9}$$

and $\varkappa = (12(1-\sigma^2))^{1/4}$. The asymptotics $N(h,\lambda) \sim a(\lambda)\, h^{-1}$ with $a(\lambda)$ given by (6.4.9) was initially conjectured by A.L. Gol'denveizer in 1970. The mathematical proof of (6.4.8), (6.4.9) based on a version of Courant's Dirichlet–Neumann bracketing technique is due to A.G. Aslanyan and V.B. Lidskii [AsLid]. Their work was followed by publications by other mathematicians who applied different methods and obtained improved remainder estimates under various assumptions on the level surfaces of $K^2(x,\xi) = \lambda$.

The following result is from [Va9]. Let us fix a $\lambda > 0$ such that this λ is not in the essential spectrum of the membrane problem (6.4.6), (6.4.7) and satisfies $\lambda > \max K^2(x,\xi)$. Then we have

$$N(h,\lambda) \,=\, a(\lambda)\, h^{-1} \,+\, O(h^{-1/2})\,, \qquad h \to 0\,, \tag{6.4.10}$$

If, in addition, the billiard system generated by the Hamiltonian

$$H(\lambda; x, \xi) = \frac{(A_1^{-2}\xi_1^2 + A_2^{-2}\xi_2^2)^{1/2}}{(\lambda - K^2(x,\xi))^{1/4}} \tag{6.4.11}$$

satisfies the nonblocking and nonperiodicity conditions, then

$$N(h,\lambda) \,=\, a(\lambda)\, h^{-1} \,+\, b(\lambda)\, h^{-1/2} \,+\, o(h^{-1/2})\,, \qquad h \to 0\,. \tag{6.4.12}$$

For a clamped edge (6.4.4) the coefficient $b(\lambda)$ is defined as follows. Assume for simplicity that ∂M coincides with the curvature lines, so $R_1^{-1}(x_2) \equiv R_1^{-1}(0,x_2)$ is the curvature of M in the cross section normal to ∂M and $R_2^{-1}(x_2) \equiv R_2^{-1}(0,x_2)$ is that in the cross section normal to M and tangent to ∂M. Assume also that on ∂M we have $R_1^{-1} R_2^{-1} \ge 0$ and $|R_2^{-1}| > |R_1^{-1}|$. Then

$$b(\lambda) \,=\, -\frac{\varkappa}{4\pi} \int\limits_{\partial M} \left((\lambda - R_1^{-2}(x_2))^{1/4} + \frac{4}{\pi} \int\limits_0^{(\lambda - R_1^{-2}(x_2))^{1/4}} \operatorname{Arg} \Delta(\lambda; x_2, \xi_2)\, d\xi_2 \right) dx_2\,,$$

where $0 \le \operatorname{Arg} < 2\pi$,

$$\Delta \,=\, \frac{i\, \det\big(a_{pq}\big)_{p,q=1}^4}{(\zeta_3 - \zeta_2)(\zeta_4 - \zeta_3)(\zeta_2 - \zeta_4)}\,,$$

ζ_q, $q = 1,2,3,4$, are the roots of the algebraic equation

$$(\zeta^2 + \xi_2^2)^4 + (R_2^{-1}\zeta^2 + R_1^{-1}\xi_2^2)^2 = \lambda(\zeta^2 + \xi_2^2)^2$$

specified by the conditions $\operatorname{Im}\zeta_1 = 0$, $\operatorname{Re}\zeta_1 > 0$, and $\operatorname{Im}\zeta_q > 0$, $q = 2,3,4$, and the a_{pq}, $p,q = 1,2,3,4$, are defined as

$$\begin{pmatrix} a_{1q} \\ a_{2q} \end{pmatrix} = \begin{pmatrix} \frac{\zeta_q^2}{1-\sigma^2} + \frac{\xi_2^2}{2(1+\sigma)} & \frac{i\zeta_q\xi_2}{2(1-\sigma)} \\ -\frac{i\zeta_q\xi_2}{2(1-\sigma)} & \frac{\zeta_q^2}{2(1+\sigma)} + \frac{\xi_2^2}{1-\sigma^2} \end{pmatrix}^{-1} \cdot \begin{pmatrix} -\frac{i(R_1^{-1} + \sigma R_2^{-1})\zeta_q}{1-\sigma^2} \\ -\frac{(R_2^{-1} + \sigma R_1^{-1})\xi_2}{1-\sigma^2} \end{pmatrix},$$

$a_{3q} = 1$, $a_{4q} = i\zeta_q$, $q = 1,2,3,4$.

4. Sketch of proof. One can not obtain the sharp one-term asymptotic formula (6.4.10) or the two-term asymptotic formula (6.4.12) by applying the wave equation method directly to the problem (6.4.1), (6.4.4). The proof is based on the following ideas.

Let us denote the value of λ at which we are proving (6.4.12) by λ_{fix}.

First, one may assume without loss of generality that the eigenvalues of the operator $\left(\ell_{pq}\right)_{i,j=1}^{2}$ (upper left block of our full shell operator $\left(\mathcal{L}_{pq}\right)_{i,j=1}^{3}$) subject to the boundary conditions (6.4.7) are greater than λ_{fix}. Moreover, one may assume that our λ_{fix} is not an eigenvalue of the membrane problem (6.4.6), (6.4.7). Both these conditions can be satisfied by a finite rank perturbation independent of h, which may change the counting function only by $O(1)$.

Second, let us "freeze" the spectral parameter λ in the first two equations (6.4.1), that is, set in the first two equations $\lambda = \lambda_{fix}$ and view these two equations with the boundary conditions (6.4.7) as a differential constraint which determines the vector-function (u_1, u_2) given a function u_3. Then we have the "real" spectral parameter λ (which is allowed to vary) only in the third equation (6.4.1). Denote the counting function of this new problem by $N(h, \lambda_{fix}, \lambda)$. It can be shown by variational arguments that $N(h, \lambda_{fix}) = N(h, \lambda_{fix}, \lambda_{fix})$.

Third, resolving the first two equations (6.4.1) with respect to (u_1, u_2) and substituting the result into the third equation we arrive at a scalar spectral problem

$$\frac{h^2}{12} n_{33} u_3 + V_{\lambda_{fix}} u_3 = \lambda u_3 \,, \tag{6.4.13}$$

$$u_3|_{\partial M} = \partial u_3 / \partial x_1|_{\partial M} = 0 \,. \tag{6.4.14}$$

Here $V_{\lambda_{fix}}$ is a bounded scalar operator in $L_2(M)$. This operator is, in fact, a pseudodifferential operator of order 0 with principal symbol $K^2(x,\xi)$, and the subprincipal symbol of $(A_1 A_2)^{1/2}\, V_{\lambda_{fix}} (A_1 A_2)^{-1/2}$ is zero.

Fourth, let us fix the remaining spectral parameter λ in (6.4.13) and declare $\nu = 12h^{-2}$ to be the new spectral parameter. In other words, consider the spectral problem

$$A u_3 = \nu B_{\lambda_{fix}} u_3 \,, \tag{6.4.15}$$

($A = n_{33}$, $B_{\lambda_{fix}} = \lambda_{fix} I - V_{\lambda_{fix}}$) subject to the boundary conditions (6.4.14). Denote by $N_+(\lambda_{fix}, \nu)$ the number of *positive* eigenvalues of (6.4.15), (6.4.14) less than a given ν. Here we had to stress the word "positive" because the operator $B_{\lambda_{fix}}$ is not necessarily semibounded from below. It can be shown by variational arguments that $N(h, \lambda_{fix}, \lambda_{fix}) = N_+(\lambda_{fix}, 12h^{-2})$.

Thus, we have reduced the original spectral problem (6.4.1), (6.4.4) to the spectral problem (6.4.15), (6.4.14). The latter is a scalar problem without a small parameter, and we should be looking for asymptotic formulae for $N_+(\lambda_{fix}, \nu)$ as $\nu \to +\infty$. The only difference between (1.1.1′), (1.1.2) and (6.4.15), (6.4.14) is that in the latter problem we have a pseudodifferential weight $B_{\lambda_{fix}}$.

The problem (6.4.15), (6.4.14) can be handled along the same lines as in Chapters 2–5. In particular, in the interior zone one should construct the wave group by dealing with the "wave" operator $D_t^4 B_{\lambda_{fix}} - A(x, D_x)$.

Note that $B_{\lambda_{fix}}$ is an operator of the Boutet de Monvel type [BdM] and dealing with such an operator directly is inconvenient. This hitch can be overcome if in the process of technical realization of our standard scheme (Chapters 2-5) applied to (6.4.15), (6.4.14) one reintroduces the first two differential equations (6.4.1) (with $\lambda = \lambda_{fix}$) and writes $B_{\lambda_{fix}} u_3$ in terms of (u_1, u_2, u_3). In this way one has to deal only with differential operators.

More details can be found in [Va8], [Va9].

5. Discussion. Let us consider two examples. The first example is a cylindrical shell of radius 1, length 2 and thickness h=0.004. The second is a truncated conical shell with meridian $r = z$, $\frac{1}{\sqrt{2}} \le z \le \sqrt{2}$ (here we use cylindrical coordinates, cf. Example 1.3.9) and thickness h=0.01. In both cases $\sigma = 0.3$ and the edges are clamped. Figures 19 and 20 show numerical results for these two examples.

Figure 19. Eigenvalue distribution for a cylindrical shell.

Figure 20. Eigenvalue distribution for a truncated conical shell.

Each plot contains three lines: the stepwise line is the actual counting function $N(h, \lambda)$, and the two smooth lines are the graphs of the functions $a(\lambda)h^{-1}$ (line 1) and $a(\lambda)h^{-1} + b(\lambda)h^{-1/2}$ (line 2). The actual $N(h, \lambda)$ was plotted using the author's own numerical results (Fig. 19) and numerical results from [AsLid, p. 149] (Fig. 20). As usual, the two-term asymptotic formula shows itself to be very effective.

Let us comment on whether the use of the two-term asymptotic formula is justified in these two examples (see precise conditions in subsection 3). As in both these examples M is a surface of revolution it is not too difficult to analyse the billiard trajectories and establish that we have nonblocking and nonperiodicity. However, we are also supposed to check that λ is not in the essential spectrum of the membrane problem (6.4.6), (6.4.7) and satisfies $\lambda > \max K^2(x, \xi)$. In both examples $\max K^2(x, \xi) = 1$ and the essential spectrum of the membrane problem (6.4.6), (6.4.7) is the union of the interval $[0, 1]$ and the point 1.0012 (the latter is the value of λ at which the Shapiro–Lopatinskii condition for (6.4.6), (6.4.7) fails).

This means that the whole graph on Fig. 19 and two thirds of the graph on Fig. 20 lie in the zone where we are not supposed to use (6.4.12). One can only conclude that our conditions on λ are probably too restrictive, and that it may be possible to give a mathematical proof of (6.4.12) for λ lying on the essential spectrum of the membrane problem, as long as the level surfaces $K^2(x,\xi) = \lambda$ are not too bad.

Shell theory is interesting in that it provides natural examples of periodic Hamiltonian and billiard flows. Indeed, shells in the form of a full sphere or a sufficiently large spherical cap provide such examples because by formula (6.4.11) our Hamiltonian trajectories in this case are geodesics and we have already looked at this situation, see discussion after Lemma 1.3.19. Thus, for a full spherical shell or spherical cap which is greater than or equal to a hemisphere one expects to observe clusters of eigenvalues. Numerical results of F.I. Niordson [Nio] show that this is exactly the case.

In the case of a hemispherical shell of radius R it is interesting to compare the numerical results [Nio] with the asymptotic formula from Example 1.7.13. The problems considered in [Nio] and in Example 1.7.13 are different, but if one replaces in (6.4.13) the pseudodifferential operator $V_{\lambda_{fix}}$ by the operator of multiplication by R^{-2} (its principal symbol) then the differential equations coincide up to a renormalization of the spectral parameter. Elementary calculations show that the asymptotic formula from Example 1.7.13 correctly predicts the wide gaps and the very sharp clusters in the spectrum of the hemispherical shell. The positions of these clusters are, however, slightly shifted as compared with the positions predicted in Example 1.7.13. This shift is explained by two factors:

(1) the boundary of the hemisphere in [Nio] is assumed to be free, not clamped as in Example 1.7.13;
(2) $V_{\lambda_{fix}}$ is not exactly the operator of multiplication by a constant.

Finally, let us elaborate on the analogy between shell equations and the Schrödinger equation. This analogy becomes evident if one reduces shell equations to the scalar problem (6.4.13), (6.4.14). Both (6.4.13), (6.4.14) and the spectral problem for the Schrödinger operator have a small parameter at the higher derivative and both have a potential V. The differences between the two problems are:

(1) the potential in (6.4.13) is pseudodifferential;
(2) the equation (6.4.13) is a fourth order one;
(3) shells are usually compact and require boundary conditions, whereas in the Schrödinger case the problem is normally stated in $\mathbb{R}^n$.

The first difference is probably the most crucial one. The fact that the potential is pseudodifferential leads to an astonishing variety of absolutely different situations, which can be realized by choosing curvatures of different signs, different absolute values, and with different dependence on x. Many of these situations are analysed in [AsLid], [GolLidTo].

6.5. Hydroelasticity

In this section we examine free vibrations of a closed ($\partial M = \varnothing$) shell filled with an ideal compressible fluid.

The surface M divides $\mathbb{R}^3$ into two parts: a bounded domain G_i (interior of the shell) and an unbounded domain G_e (interior of the shell). The spectral problem being considered is

$$\sum_{j=1}^{3} \mathcal{L}_{ij} u_j = \lambda\, u_i\,, \qquad i = 1, 2\,, \tag{6.5.1}$$

$$\sum_{j=1}^{3} \mathcal{L}_{3j} u_j = \lambda\, u_3 + \frac{\rho_f \lambda}{\rho_s h}\, \psi|_M\,, \tag{6.5.2}$$

$$-\frac{c_f^2}{c_s^2} \Delta\psi = \lambda\, \psi \quad \text{in} \quad G_i\,, \tag{6.5.3}$$

$$\left.\frac{\partial\psi}{\partial n}\right|_M = u_3\,. \tag{6.5.4}$$

Here

(1) $\mathcal{L}_{ij}$, u_j, λ, ρ_s, E, σ and h are as in the previous section;
(2) $\psi = \psi(y)$ is the potential of displacements of the fluid, i.e., $\operatorname{grad}\psi$ is the vector-function of fluid displacements;
(3) $y = (y_1, y_2, y_3)$ are Cartesian coordinates in $\mathbb{R}^3$;
(4) $\Delta = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2 + \partial^2/\partial y_3^2$;
(5) ρ_f and c_f are the fluid density and the speed of sound in the fluid, respectively;
(6) $c_s := \sqrt{E/\rho_s}$ (characteristic speed of sound in the shell material);
(7) $\partial/\partial n$ is the derivative along the exterior normal to M.

The term $\frac{\rho_f\lambda}{\rho_s h}\, \psi|_M$ in (6.5.2) describes the dynamical pressure of the fluid acting on the shell, and (6.5.4) is the so-called non-penetration condition (the fluid can not go through the shell).

The first difficulty with (6.5.1)–(6.5.4) is that this system contains an "extra" occurrence of the spectral parameter λ, that is, the one in the term $\frac{\rho_f\lambda}{\rho_s h}\, \psi|_M$. So it is not *a priori* clear whether (6.5.1)–(6.5.4) is a spectral problem for a self-adjoint operator in some Hilbert space. In order to overcome this difficulty let us rewrite (6.5.2) in equivalent form

$$\sum_{j=1}^{3} \mathcal{L}_{3j} u_j + \frac{\rho_f c_f^2}{\rho_s c_s^2 h}\, (\Delta\psi)|_M = \lambda\, u_3 \tag{6.5.2$'$}$$

(here we used (6.5.3)). The system (6.5.1), (6.5.2′), (6.5.3) can be written as

$$Af = \lambda f\,, \tag{6.5.5}$$

where A is the 4×4 matrix operator

$$A = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} & 0 \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} & \frac{\rho_f c_f^2}{\rho_s c_s^2 h}\, (\Delta(\,\cdot\,))|_M \\ 0 & 0 & 0 & -\frac{c_f^2}{c_s^2}\Delta \end{pmatrix}$$

acting on quadruples $f = \big(u_1(x_1,x_2), u_2(x_1,x_2), u_3(x_1,x_2), \psi(y_1,y_2,y_3)\big)$. It is easy to see that on quadruples satisfying the non-penetration condition (6.5.4) the spectral problem (6.5.5) is formally self-adjoint and nonnegative with respect to the inner product

$$\iint_M (u_1\overline{v_1} + u_2\overline{v_2} + u_3\overline{v_3})\,dS + \frac{\rho_f}{\rho_s h}\iiint_{G_i} \operatorname{grad}\psi\cdot\overline{\operatorname{grad}\varphi}\,dV \tag{6.5.6}$$

(cf. (6.4.3)), $dV = dy_1dy_2dy_3$. It can be shown [AsLidVa] that the problem (6.5.5), (6.5.4) generates a self-adjoint operator in the Hilbert space with inner product (6.5.6), and that the spectrum of this operator is discrete and accumulates to $+\infty$. By $N(h,\lambda)$ we shall denote the counting function, that is, the number of eigenvalues of (6.5.1)–(6.5.4) below a given λ.

The following result is from [AsLidVa], [Va2]. Let is fix some $0 < \lambda_{\min} \le \lambda_{\max}$. Then uniformly over $\lambda \in [\lambda_{\min}, \lambda_{\max}]$ we have

$$N(h,\lambda) = a(\lambda)\,h^{-6/5} + b(\lambda)\,h^{-4/5} + O(h^{-3/5}), \qquad h\to 0 \tag{6.5.7}$$

(cf. (6.4.10). If, in addition, the geodesic flow on M satisfies the nonperiodicity condition, then uniformly over $\lambda \in [\lambda_{\min}, \lambda_{\max}]$

$$N(h,\lambda) = a(\lambda)\,h^{-6/5} + b(\lambda)\,h^{-4/5} + c(\lambda)\,h^{-3/5} + o(h^{-3/5}), \qquad h\to 0 \tag{6.5.8}$$

(cf. (6.4.12)). Here

$$a(\lambda) = \frac{S}{4\pi}\left(\frac{12(1-\sigma^2)\rho_f\lambda}{\rho_s}\right)^{2/5},$$

$$b(\lambda) = \frac{6(1-\sigma^2)^{3/5}}{5\pi}\left(\frac{12\rho_f\lambda}{\rho_s}\right)^{-2/5}\left(2\pi + \lambda S - \frac{3}{2}\iint_M \left(\frac{R_1^{-1}+R_2^{-1}}{2}\right)^2 dS\right),$$

$$c(\lambda) = \frac{1}{20\pi}\left(\frac{12(1-\sigma^2)\rho_f\lambda}{\rho_s}\right)^{1/5}\iint_M \frac{R_1^{-1}+R_2^{-1}}{2}\,dS\,,$$

with S being the surface area of M.

The proof of the asymptotic formulae (6.5.7), (6.5.8) is based on the reduction of the original spectral problem (6.5.1)–(6.5.4) to the scalar pseudodifferential spectral problem

$$\frac{h^2}{12}n_{33}u_3 + V_{\lambda_{fix}}u_3 = \lambda u_3 + \frac{\rho_f\lambda_{fix}}{\rho_s h}F_{\lambda_{fix}}u_3 \tag{6.5.9}$$

on M (cf. (6.4.13)), where $F_{\lambda_{fix}} : \partial\psi/\partial n|_M \to \psi|_M$ is the Neumann–Dirichlet operator for the Helmholtz equation (6.5.3). The latter is a pseudodifferential operator of order -1 with principal symbol $|\xi|_x^{-1}$, and the subprincipal symbol of $(A_1A_2)^{1/2}F_{\lambda_{fix}}(A_1A_2)^{-1/2}$ is $|\xi|_x^{-2}K(x,\xi)/2$ (see (6.4.5) for notation). The equation (6.5.9) is simpler than (6.4.13) because now the manifold has no boundary, but at the same time it is more complicated because the small parameter h comes into the equation twice and there is no obvious way of excluding it. Note that the main (according to their contribution to $N(h,\lambda)$) terms in (6.5.9) are $\frac{h^2}{12}n_{33}u_3$

and $\frac{\rho_f \lambda_{fix}}{\rho_s h} F_{\lambda_{fix}} u_3$, so we are dealing with the unusual situation when the lower order term can not be disregarded. This happens because this lower order term contains a negative power of the small parameter h .

A special method for dealing with pseudodifferential operators with parameters was developed in [Va1], [Va5]. The application of this general result to (6.5.9) gives (6.5.7), (6.5.8). See [AsLidVa], [Va2] for more details.

It may seem strange that we are able to obtain up to three asymptotic terms for $N(h,\lambda)$. The explanation is that the small parameter h comes into our problem in a complicated way, and the term $b(\lambda)\,h^{-4/5}$ should really be viewed as a correction to the first asymptotic term. The second term proper is $c(\lambda)\,h^{-3/5}$. Its appearance is caused by the fact that that the Neumann–Dirichlet operator has a nontrivial subprincipal symbol.

Suppose now that the fluid is not inside, but outside the shell. In other words, suppose that (6.5.2), (6.5.3) have been replaced by

$$\sum_{j=1}^{3} \mathcal{L}_{3j} u_j = \lambda\, u_3 - \frac{\rho_f \lambda}{\rho_s h}\, \psi|_M \ , \tag{6.5.10}$$

$$-\frac{c_f^2}{c_s^2}\Delta\psi = \lambda\,\psi \quad \text{in} \quad G_e\,. \tag{6.5.11}$$

In this case one can also associate a self-adjoint operator with the system (6.5.1), (6.5.10), (6.5.11), (6.5.4), and the spectrum of this self-adjoint operator is $\mathbb{R}_+$ (consequence of the unboundedness of G_e). The fact that the spectrum of the exterior problem fills the nonnegative half-line (or, in terms of frequency ω , the whole real line) is not very informative from the mechanical point of view. This inconvenience can be overcome by modifying our choice of function spaces and extending the resolvent by analyticity through the (continuous) spectrum onto the whole complex ω-plane, see [LtinVa3] and [S.-HuS.-Pa., Chap. 9]. The extended resolvent is meromorphic as a function of ω with poles at some complex ω_k which are called *resonances*. It turns out that a massive group of resonances lies close (in terms of the small parameter h) to the real line, and the real parts of the corresponding λ_k are distributed in accordance with (6.5.7), (6.5.8). The only difference is that in the case of the exterior problem the coefficient $c(\lambda)$ changes sign (this happens because the fluid is on the other side of the shell).

A detailed mathematical analysis of both the interior and exterior problems was carried out in [Va2], [Va8]. See also the review paper [LtinVa3].

REMARK 6.5.1. Apart from the relative thickness h/R the hydroelasticity problem contains two other dimensionless parameters, namely, ρ_f/ρ_s and c_f/c_s . In the derivation of (6.5.7), (6.5.8) we assumed these two parameters to be fixed. In mechanical terms this means that we assumed these parameters to be of the order of 1. However, in reality this is not exactly the case. Say, for the pair water–Duralumin we have $\rho_f/\rho_s \approx 0.357$ and $c_f/c_s \approx 0.288$, and for water–steel $\rho_f/\rho_s \approx 0.127$ and $c_f/c_s \approx 0.286$. The influence of these two additional parameters is such that they make the asymptotic convergence in (6.5.7), (6.5.8) not as good as one would have hoped. Basically, this implies that formula (6.5.8) is sufficiently accurate only in a relatively narrow frequency range. Things can be improved if one rewrites (6.5.8) in the form

$$N(h,\lambda) = \int dx\, đ\xi \,+\, o(h^{-3/5})\,, \qquad h \to 0\,, \tag{6.5.12}$$

where integration is carried out over all $(x,\xi) \in T'M$ such that

$$\frac{h^2|\xi|_x^4}{12(1-\sigma^2)} + K^2(x,\xi) < \lambda + \frac{\rho_f \lambda}{\rho_s h|\xi|_x}\left(1 \pm \frac{K(x,\xi)}{2|\xi|_x}\right), \tag{6.5.13}$$

with sign "plus" for the interior problem and "minus" for exterior; note that (6.5.13) is obtained from (6.5.9) if one replaces equality by inequality and operators by their principal and subprincipal symbols. From the purely mathematical point of view (6.5.12), (6.5.13) is equivalent to (6.5.8). However, in reality (6.5.12), (6.5.13) works in a much wider frequency range because we have avoided additional errors caused by resolving (6.5.13) in powers of h.