

HOMOLOGICAL MIRROR SYMMETRY FOR HIGHER DIMENSIONAL PAIRS OF PANTS

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ABSTRACT. Using Auroux's description of Fukaya categories of symmetric products of punctured surfaces, we compute the partially wrapped Fukaya category of the complement of $k + 1$ generic hyperplanes in $\mathbb{C}P^n$, for $k \geq n$, with respect to certain stops in terms of the endomorphism algebra of a generating set of objects. The stops are chosen so that the resulting algebra is formal. In the case of the complement of $(n + 2)$ -generic hyperplanes in $\mathbb{C}P^n$ (n -dimensional pair-of-pants), we show that our partial wrapped Fukaya category is equivalent to a certain categorical resolution of the derived category of the singular affine variety $x_1x_2\dots x_{n+1} = 0$. By localizing, we deduce that the (fully) wrapped Fukaya category of n -dimensional pants is equivalent to the derived category of $x_1x_2\dots x_{n+1} = 0$.

1. INTRODUCTION

Originally homological mirror symmetry was conceived by Kontsevich as an equivalence of the Fukaya category of a compact symplectic manifold with the bounded derived category of coherent sheaves on a mirror dual compact complex variety. Since then it grew into a vast program connecting Fukaya categories of several kinds associated with not necessarily compact symplectic manifolds with derived categories of several kinds attached to possibly singular algebraic varieties.

In [15], we proved a version of homological mirror symmetry relating Fukaya categories of punctured Riemann surfaces with some derived categories attached to stacky nodal curves. More precisely, for a punctured Riemann surface Σ and a line field η on Σ , we choose a certain set of stops Λ , and consider the following sequence of pre-triangulated categories, related by quasi-functors

$$\mathcal{F}(\Sigma, \eta) \rightarrow \mathcal{W}(\Sigma, \Lambda, \eta) \rightarrow \mathcal{W}(\Sigma, \eta)$$

where $\mathcal{F}(\Sigma, \eta)$ is the compact Fukaya category ([26]), $\mathcal{W}(\Sigma, \Lambda, \eta)$ is the partially wrapped Fukaya category ([2], [11]), and $\mathcal{W}(\Sigma, \eta)$ is the (fully) wrapped Fukaya category ([1]). The first functor is full and faithful, and the second functor is a localization functor corresponding to dividing by the full subcategory of Lagrangians supported near Λ .

On the mirror side, we consider a nodal stacky curve C obtained by attaching copies of weighted projective lines at their orbifold points (see [15] for details), and we again have a sequence of categories

$$\text{Perf}(C) \rightarrow D^b(\mathcal{A}_C) \rightarrow D^b\text{Coh}(C)$$

where \mathcal{A}_C is a sheaf of algebras, called the Auslander sheaf over C that was previously studied in [5]. We again have that the first functor is full and faithful, and the second

functor is a localization. The main result of [15] is an equivalence of homologically smooth and proper, pre-triangulated categories

$$\mathcal{W}(\Sigma, \Lambda, \eta) \simeq D^b(\mathcal{A}_C). \quad (1.1)$$

It is proved by constructing a generating set of objects on each side and matching their endomorphism algebras. The main point is that these algebras turn out to be formal (in fact, concentrated in degree 0), which means that we only need to prove an isomorphism of the usual associative algebras and do not have to worry about higher products.

One then deduces an equivalence $\mathcal{W}(\Sigma, \eta) \simeq D^b\text{Coh}(C)$ by identifying the subcategories on both sides of the equivalence (1.1) with respect to which to take quotient. The equivalence $\mathcal{F}(\Sigma, \eta) \simeq \text{Perf}(C)$ is deduced by characterizing both sides as subcategories of the two sides of (1.1). Note that considering the same generators in the localized categories leads to dg-algebras which are far from formal. Note also that the embedding $\text{Perf}(C) \hookrightarrow D^b(\mathcal{A}_C)$ is a simple example of categorical resolutions considered in [13].

Let us explain this in more detail in a simple case. Let Σ be the pair-of-pants, that is, a 3-punctured sphere, Λ be 2 stops at the outer boundary as drawn in Figure 1. We also choose a line field η on Σ which has rotation number 2 around the outer boundary and 0 along the interior boundary components (see [16] for a recent detailed study of line fields on punctured surfaces).

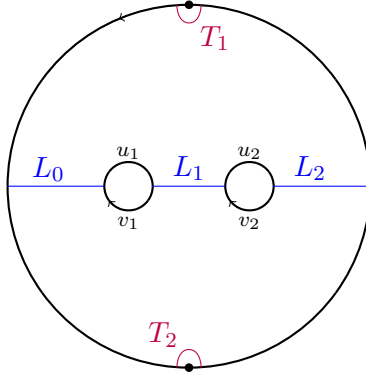


FIGURE 1. Pair-of-pants

The partially wrapped Fukaya category $\mathcal{W}(\Sigma, \Lambda, \eta)$ is generated by the Lagrangians L_0, L_1, L_2 drawn on Figure 1, and their endomorphism algebra is easily computed to be given by the following quiver with relations:

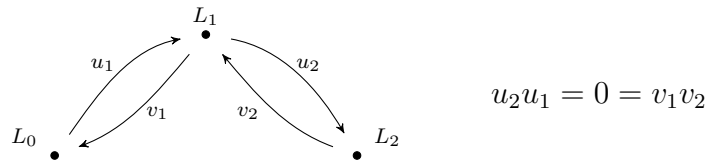


FIGURE 2. Endomorphism algebra of a generating set

On the B-side, the mirror is given by the Auslander order A over the node algebra

$$R = \mathbf{k}[x_1, x_2]/(x_1x_2),$$

where \mathbf{k} is a commutative ring (which we assume to be regular). The Auslander order in this case is simply,

$$A = \text{End}_R(R/(x_1) \oplus R/(x_2) \oplus R).$$

One can directly see that A is isomorphic to the quiver algebra given in Figure 2. Note that R is a Cohen-Macaulay algebra and the modules $R, R/(x_1), R/(x_2)$ comprise the set of indecomposable maximal Cohen-Macaulay modules of R , and they generate $D^b(A)$ as a triangulated category.

Now, the wrapped Fukaya category $\mathcal{W}(\Sigma, \eta)$ is the localization of $\mathcal{W}(\Sigma, \Lambda, \eta)$ given by dividing out by the subcategory generated by the objects T_1, T_2 supported near the stops. We can express them in terms of L_0, L_1, L_2 as follows:

$$\begin{aligned} T_1 &\simeq \{L_0 \xrightarrow{u_1} L_1 \xrightarrow{u_2} L_2\} \\ T_2 &\simeq \{L_2 \xrightarrow{v_2} L_1 \xrightarrow{v_1} L_0\} \end{aligned}$$

Similarly, $D^b(R)$ is the localization of $D^b(A)$ obtained by dividing out by the corresponding subcategory, and this allows one to establish an equivalence

$$\mathcal{W}(\Sigma, \eta) \simeq D^b(R).$$

1.1. New results. In this paper, we apply the above strategy to prove homological mirror symmetry for the higher-dimensional pair-of-pants,

$$\mathcal{P}_n = \text{Sym}^n(\mathbb{P}^1 \setminus \{p_0, p_1, \dots, p_{n+1}\}),$$

where $\text{Sym}^n(\Sigma) = \Sigma^n/\mathfrak{S}_n$, see Section 2 for a brief review of symplectic topology of these spaces. In other words, n -dimensional pair-of-pants is the complement of $(n + 2)$ generic hyperplanes in \mathbb{P}^n .

On the A-side, we first introduce a stop $\Lambda = \Lambda_1 \cup \Lambda_2$ where $\Lambda_i = \{q_i\} \times \text{Sym}^{n-1}(\Sigma)$ for some base points q_1, q_2 . We pick a grading structure η and consider the partially wrapped Fukaya $\mathcal{W}(\mathcal{P}_n, \Lambda, \eta)$, where we use some commutative ring \mathbf{k} as coefficients. There is a natural generating set of objects $\{L_S : S \subset \{0, 1, \dots, n + 1\}, |S| = n\}$ in this category and our first result is an explicit computation of the algebra of morphisms between these objects. In fact, we do this more generally for the symplectic manifolds

$$M_{n,k} = \text{Sym}^n(\mathbb{P}^1 \setminus \{p_0, p_1, \dots, p_k\}),$$

see Theorem 3.2.5 for the precise description of the resulting algebra.

Next, we specialize to the case $k = n + 1$. In this case, on the B-side we consider the categorical resolutions of the algebra

$$R = \mathbf{k}[x_1, x_2, \dots, x_{n+1}]/(x_1x_2 \dots x_{n+1})$$

given by

$$\mathcal{B}^\circ := \text{End}_R(R/(x_1) \oplus R/(x_{[1,2]}) \oplus \dots \oplus R/(x_{[1,n]}) \oplus R)$$

and

$$\mathcal{B}^{\circ\circ} := \text{End}_R\left(\bigoplus_{I \subset [1, n+1], I \neq \emptyset} R/(x_I)\right),$$

where the summation is over all nonempty subintervals of $[1, n+1]$, and for $I \subset [1, n+1]$ we use the notation $x_I := \prod_{i \in I} x_i$.

By explicit computations we prove the following theorem (see Theorem 5.0.2).

Theorem 1.1.1. *Assume that \mathbf{k} is a regular ring. There exists a grading structure η on \mathcal{P}_n such that we have an equivalence of pre-triangulated categories over \mathbf{k} ,*

$$\mathcal{W}(\mathcal{P}_n, \Lambda, \eta) \simeq D^b(\mathcal{B}^{\circ\circ}).$$

We next analyze the localization of these categories corresponding to dividing out by objects supported near the stops to deduce homological mirror symmetry for the (fully) wrapped Fukaya categories:

Corollary 1.1.2. *For the same grading structure η we have equivalences of pre-triangulated categories over \mathbf{k} :*

$$\mathcal{W}(\mathcal{P}_n, \Lambda_1, \eta) \simeq D^b(\mathcal{B}^{\circ}) \quad \text{and} \quad \mathcal{W}(\mathcal{P}_n, \eta) \simeq D^b\text{Coh}(x_1 x_2 \dots x_{n+1} = 0).$$

1.2. Relation to other works. Homological mirror symmetry for pair-of-pants is a much studied subject. However, a complete proof of Corollary 1.1.2 has not appeared in writing until this paper. A motivation for studying these particular examples of mirror symmetry comes from a theorem of Mikhalkin [19] that a hypersurface in $\mathbb{C}P^{n+1}$ admits a decomposition into several \mathcal{P}_n , much like a Riemann surface admits a decomposition into several \mathcal{P}_1 .

In [27], Sheridan identifies the mirror (immersed) Lagrangian in \mathcal{P}_n corresponding to \mathcal{O}_0 , the structure sheaf of the origin in the triangulated category of singularities of the normal crossing divisor $x_0 x_1 x_2 \dots x_{n+1} = 0$ in \mathbb{C}^{n+2} . By a theorem of Orlov [22], the latter category is quasi-equivalent to the matrix factorization category $\text{mf}(\mathbb{C}^{n+2}, x_0 x_1 x_2 \dots x_{n+1})$. Note that the latter category (or more precisely, its \mathbb{G}_m -equivariant version where x_1, \dots, x_{n+1} have weight 0, and x_0 has weight 2) should be naturally quasi-equivalent to the derived category of $x_1 x_2 \dots x_{n+1} = 0$ (a similar result was proved by Isik [12] and Shipman [28]). Under this equivalence, \mathcal{O}_0 corresponds to a perfect object supported at the origin.

In [21] instead of wrapped Fukaya category of \mathcal{P}_n , Nadler studies the \mathbb{Z}_2 -graded category of wrapped microlocal sheaves associated to a skeleton of \mathcal{P}_n (see also the follow-up paper by Gammage and Nadler [7]). It is then verified that this category agrees with the \mathbb{Z}_2 -graded category of matrix factorizations $\text{mf}(\mathbb{C}^{n+2}, x_0 x_1 x_2 \dots x_{n+1})$. It is expected and in certain cases proved that wrapped microlocal sheaves category is equivalent to wrapped Fukaya category - see [10] for the case of cotangent bundles. However, such an equivalence for arbitrary Weinstein manifold is not yet accomplished. Nonetheless, in view of the works of Ganatra-Pardon-Shende [8], [9],[10] such an equivalence in the case of \mathcal{P}_n seems to be within reach (see in particular the discussion in [10, Section 6.6] which outlines a proof depending on a work-in-progress). Establishing such an equivalence for \mathcal{P}_n would give another confirmation for Corollary 1.1.2 (at least, in the \mathbb{Z}_2 -graded case).

The closest to our work is [4], where Auroux sketches a proof of homological mirror symmetry for \mathcal{P}_n which depends on certain conjectures about generation by an explicit collection of Lagrangians and a classification of A_∞ -structures on their cohomology.

We also mention that partially wrapped Fukaya categories of symmetric products appear predominantly in Heegaard Floer homology [17], [2], [3] (see also [14]). In particular, our computations of $\mathcal{W}(M_{n,k})$ give an alternative viewpoint for knot Floer homology. We defer this to a future work.

The paper is organized as follows. After reviewing some background material on partially wrapped Fukaya categories of symmetric powers of Riemann surfaces in Sec. 2, we present the computation of the algebra of morphisms between generating Lagrangians in $M_{n,k}$ in Sec. 3. Then in Sec. 4 we deal with the B-side of the story: we study the derived categories of modules over \mathcal{B}° and $\mathcal{B}^{\circ\circ}$. In particular, we construct semiorthogonal decompositions of these categories and obtain the localization results similar to those corresponding to the removal of a stop on the A-side. Finally, in Sec. 5 we prove the equivalences of categories on the A-side and on the B-side (see Theorem 5.0.2).

Conventions. Our base commutative ring \mathbf{k} is assumed to be regular, i.e., Noetherian and of finite global dimension. When we write complexes of modules in the form $[\dots \rightarrow \cdot]$, we assume that the rightmost term sits in degree 0. By Hom we denote cohomology of the morphism spaces hom in dg-categories.

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2. A BRIEF REVIEW OF FUKAYA CATEGORIES OF SYMMETRIC PRODUCTS OF RIEMANN SURFACES

Let Σ be a Riemann surface, for each $n > 0$, there exists a smooth n -dimensional complex algebraic variety

$$\text{Sym}^n(\Sigma) := \Sigma^n / \mathfrak{S}_n \tag{2.1}$$

where \mathfrak{S}_n is the permutation group which acts by permuting the components of the product.

Let $\pi : \Sigma^n \rightarrow \text{Sym}^n(\Sigma)$ be the branched covering map. Fix an area form ω on Σ . In [23, Section 7], Perutz explains how to smoothen the closed current $\pi_*(\omega^{\times n})$ on $\text{Sym}^n(\Sigma)$ to a Kähler form Ω by modifying it in an arbitrarily small analytic neighborhood of the (big) diagonal. In particular, outside this neighborhood we have $\Omega = \pi_*(\omega^{\times n})$. Throughout, we will view $\text{Sym}^n(\Sigma)$ as a symplectic manifold equipped with such a Kähler form Ω .

If we write $g = g(\Sigma)$ for the genus of Σ , the first Chern class of such a variety is given by

$$c_1(\text{Sym}^n(\Sigma)) = (n + 1 - g)\eta - \theta$$

where η and θ are the Poincaré duals of the class $\{pt\} \times \text{Sym}^{n-1}(\Sigma)$ and the theta divisor, respectively. These two cohomology classes span the invariant part of $H^2(\text{Sym}^n(\Sigma))$ under the action of the mapping class group of Σ . Moreover, we have that $[\Omega] = \eta$.

In particular, when

$$\Sigma = \mathbb{P}^1 \setminus \{p_0, p_1, \dots, p_k\} \quad (2.2)$$

$\text{Sym}^n(\Sigma)$ is an exact symplectic manifold with $c_1 = 0$. Such symplectic manifolds are sometimes referred to as symplectically Calabi-Yau manifolds, and their Fukaya categories can be \mathbb{Z} -graded [25]. From the point of view of symplectic topology, the positions of the points do not matter, so let us introduce the notation

$$M_{n,k} = \text{Sym}^n(\mathbb{P}^1 \setminus \{p_0, p_1, \dots, p_k\}) \quad (2.3)$$

to denote the exact symplectic manifold with $c_1 = 0$. The grading structures are given by homotopy classes of trivializations of the bicanonical bundle, and there is an effective $H^1(\text{Sym}^n(\Sigma)) \simeq H^1(\Sigma)$ worth of choices.

Recall the well-known isomorphism of algebraic varieties

$$\text{Sym}^n(\mathbb{P}^1) \simeq \mathbb{P}^n \quad (2.4)$$

given by sending an effective divisor of degree n on \mathbb{P}^1 to its homogeneous equation defined up to rescaling.

Therefore, one can think of $M_{n,k}$ as the complement of $k+1$ generic hyperplanes in \mathbb{P}^n . This provides an alternative way to equip $M_{n,k}$ with a symplectic structure by viewing it as an affine variety but we will not pursue this any further, as we prefer to emphasize the structure of $M_{n,k}$ as a symmetric product on a punctured genus 0 surface. The two symplectic structures are equivalent as they both tame the standard complex structure $J = \text{Sym}^n(j)$ on $M_{n,k}$ induced from \mathbb{P}^n (see [23, Prop. 1.1]). This also makes it clear that for $0 \leq k < n$, $M_{n,k} = \mathbb{C}^{n-k} \times (\mathbb{C}^*)^k$ which is a subcritical Stein manifold, so our main interest will be for $k \geq n$.

We will also equip $M_{n,k}$ with stops Λ_Z corresponding to choice of symplectic hypersurfaces of the form $\{p\} \times \text{Sym}^{n-1}(\Sigma)$ for $p \in Z$, where Z is finite set of points. The set Z will be indicated by choosing stops in the ideal boundary of Σ . More precisely, by removing cylindrical ends, we view Σ as a 2-dimensional surface with boundary and the set Z will be chosen as a finite set of points on $\partial\Sigma$.

We write $\mathcal{W}(M_{n,k}, \Lambda_Z)$ for the partially wrapped Fukaya category. Motivated by bordered Heegaard Floer homology [17], these categories were originally constructed by Auroux in the papers [2], [3]. These works provide foundational results on these categories, as well as some very useful results about generating objects and existence of certain exact triangles.

All of the Lagrangians that we use will be of the form $L_1 \times L_2 \times \dots \times L_n$ where $L_i \subset \Sigma$ are pairwise disjoint Lagrangian arcs in Σ , which can be considered as objects in $\mathcal{W}(\Sigma, Z)$. Auroux explains how the Floer cohomology between such product Lagrangians can be

computed. As vector spaces, the morphism spaces are defined by

$$\begin{aligned} & \text{hom}(L_{i_1} \times L_{i_2} \times \dots \times L_{i_n}, L_{j_1} \times L_{j_2} \times \dots \times L_{j_n}) \\ &= \bigoplus_{\sigma} \text{hom}(L_{i_1}, L_{\sigma(i_1)}) \otimes \text{hom}(L_{i_2}, L_{\sigma(i_2)}) \otimes \dots \otimes \text{hom}(L_{i_n}, L_{\sigma(i_n)}) \end{aligned}$$

where σ runs through bijections $\{i_1, \dots, i_n\} \rightarrow \{j_1, \dots, j_n\}$.

Following [17], we can represent these generators via strand diagrams as follows. First, the endpoints of the Lagrangians $L_{i_1}, \dots, L_{i_n}, L_{j_1}, \dots, L_{j_n}$ are grouped into equivalence classes according to which boundary component of Σ they end on. Note that the set of endpoints lying on each boundary component has a cyclic order induced by the orientation of the boundary. Thus, given a morphism (f_1, \dots, f_n) , we can assume that it is of the form $(f_{i_1,1}, \dots, f_{i_{r_1},1}, f_{i_1,2}, \dots, f_{i_{r_2},2}, \dots, f_{i_1,k}, \dots, f_{i_{r_k},k})$ where $f_{i_1,s}, \dots, f_{i_{r_s},s}$ are Reeb chords along the s^{th} boundary component $\partial\Sigma_s \subset \partial\Sigma$, where the Reeb flow is simply the rotation along the orientation of the boundary. Thus, each $f_{i_j,s}$ either represents the idempotent of the corresponding Lagrangian $L_{i_j,s}$ or goes in the strictly positive direction along $\partial\Sigma_s$. Thus, the set of Reeb chords $f_{i_1,s}, \dots, f_{i_{r_s},s}$ can be represented in $\mathbb{R} \times [0, 1]$ as upward veering strands from $\mathbb{R} \times \{0\}$ to $\mathbb{R} \times \{1\}$, or as a straight horizontal line if it corresponds to an idempotent. Here, \mathbb{R} is the universal cover of the component $\partial\Sigma_s$ and $[0, 1]$ is the time direction, see Figure 3.

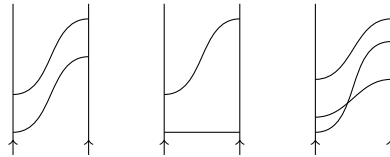


FIGURE 3. A strand diagram with 3 boundary components

In the case when a boundary component contains stops the Reeb chords are not allowed to pass through the stops. Hence, instead of using the universal cover \mathbb{R} , one cuts along the stops and uses the subintervals to draw the strand diagram. We do not elaborate on the notation to describe this.

Now, the product in $\mathcal{W}(M_{n,k}, \Lambda_Z)$ is induced by the composition in $\mathcal{W}(\Sigma, Z)$. Namely, we have

$$(f_1 \otimes \dots \otimes f_n) \circ (g_1 \otimes \dots \otimes g_n) = (f_1 g_{\sigma(1)} \otimes f_2 g_{\sigma(2)} \otimes \dots \otimes f_n g_{\sigma(n)})$$

if there exists a $\sigma \in \mathfrak{S}_n$ such that all the compositions $f_i g_{\sigma(i)}$ in $\mathcal{W}(\Sigma, Z)$ are non-zero, and with the additional important condition that in the strand representation no two strands of the concatenated diagram cross more than once; otherwise the product is set to be zero, see Figure 4

The differential on the space of morphisms is defined as the sum of all the ways of resolving one crossing of the strand diagram excluding resolutions in which two strands intersect twice, see Figure 5.

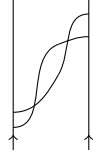


FIGURE 4. A strand diagram with two strands crossing more than once.

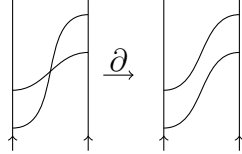


FIGURE 5. Resolution of strand diagram

In this way we get a structure of a dg-category on $\mathcal{W}(M_{n,k}, \Lambda_Z)$ (there are no higher products). In what follows, we use these results without further explanations. The reader is referred to [2] for details.

We can choose a line field to give $\mathcal{W}(\Sigma, Z)$ a \mathbb{Z} -grading (see [16] for a recent study of this structure). There are effectively $H^1(\Sigma)$ worth of choices for the line field. The set of grading structures for $M_{n,k}$ is a torsor for an isomorphic group $H^1(\text{Sym}^n(\Sigma)) \cong H^1(\Sigma)$. However, the relation between grading structures on Σ and on $\text{Sym}^n(\Sigma)$ seems to be quite subtle: it is easy to see that the grading of a morphism $(f_1 \otimes \dots \otimes f_n)$ in $\mathcal{W}(M_{n,k}, \Lambda_Z)$ cannot be given by the sum of the gradings of morphisms f_i in $\mathcal{W}(\Sigma, Z)$. For example, we will encounter objects L_1, L_2 and morphisms $u \in \text{hom}(L_1, L_2)$ and $v \in \text{hom}(L_2, L_1)$ such that

$$\partial(\text{id}_{L_1} \otimes uv) = \partial(vu \otimes \text{id}_{L_2}) = v \otimes u \in \text{hom}(L_1 \times L_2, L_1 \times L_2).$$

This makes direct determination of the gradings in $\mathcal{W}(M_{n,k}, \Lambda_Z)$ difficult. Instead, we are able to pin down the grading structures on $M_{n,k}$ using an explicit calculation of the endomorphism algebra of a generating set of Lagrangians.

Auroux proves in [3, Theorem 1] that given a set of Lagrangians L_0, L_1, \dots, L_k such that their complement in Σ is a disjoint union of disks with at most 1 stop in their boundary then for $1 \leq n \leq (k+1)$, the corresponding partially wrapped Fukaya category of $\text{Sym}^n(\Sigma)$ is generated by $\binom{k+1}{n}$ product Lagrangians $L_{i_1} \times \dots \times L_{i_n}$ where (i_1, \dots, i_n) runs through size n subsets of $\{0, 1, \dots, k\}$. Notice that this generation result only depends on the configuration of L_i on Σ and is independent of n .

Finally, we recall a basic exact triangle from [2, Lemma 5.2]. Let us consider the Lagrangians $L = L_1 \times L_2 \times \dots \times L_n$, $L' = L'_1 \times L_2 \times \dots \times L_n$, and $L'' = L''_1 \times L_2 \times \dots \times L_n$, where L''_1 is the arc obtained by sliding L_1 along L'_1 . Then L , L' and L'' fit into an exact triangle

$$L \xrightarrow{u \otimes \text{id}} L' \rightarrow L'' \rightarrow L[1] \tag{2.5}$$

coming from an exact triangle

$$L_1 \xrightarrow{u} L'_1 \rightarrow L''_1 \rightarrow L_1[1]$$

in $\mathcal{W}(\Sigma, Z)$

Similarly, if $L = L_1 \times L_2 \times \dots \times L_n$ and $L' = L'_1 \times L_2 \times \dots \times L_n$, where L'_1 is obtained by sliding L_1 along L_2 , then L and L' are isomorphic in the category $\mathcal{W}(M_{n,k}, \Lambda_Z)$. Indeed, in this situation one can show that L and L' are Hamiltonian isotopic (see [23], [2]).

3. A-SIDE

Throughout, we will work over a commutative ring \mathbf{k} . We consider the sphere Σ_k with $(k + 1)$ holes and 2 stops $Z = q_1 \cup q_2$ on one of the boundary components. We have a generating set of Lagrangians L_0, L_1, \dots, L_k , which connect i^{th} hole to $(i + 1)^{\text{th}}$ hole for $i \in \mathbb{Z}/(k + 1)$, see Figure 6 for $k = 3$. As in Figure 6, we view Σ_k as a k -holed disk. We call the punctures that lie in the interior of the disk the *interior punctures* of Σ_k and label them with $1, 2, \dots, k$ from left to right. We call the unique puncture that corresponds to the boundary of the disk, the *exterior puncture* of Σ_k and label it with 0.

Let $M_{n,k} = \text{Sym}^n(\Sigma_k)$ and $\Lambda_Z = \Lambda_1 \cup \Lambda_2$ be the corresponding stops. Thus, $\Lambda_i = q_i \times \text{Sym}^{n-1}(\Sigma_k)$ are symplectic hypersurfaces in $M_{n,k}$.

The objects L_0, \dots, L_k generate the partially wrapped Fukaya category $\mathcal{W}(\Sigma, Z)$. Furthermore, by Auroux's theorem [3, Theorem 1], the category $\mathcal{W}(M_{n,k}, \Lambda_Z)$ is generated by the Lagrangians

$$L_S = L_{i_1} \times L_{i_2} \times \dots \times L_{i_n}$$

where $i_j \in S$ and S is a subset of $[0, k]$ of size n .

Below we are going to describe the algebra

$$\mathcal{A}^{\circ\circ} = \bigoplus_{S, S'} \text{Hom}_{\mathcal{W}(M_{n,k}, \Lambda_Z)}(L_S, L_{S'}) \tag{3.1}$$

It will turn out that for $n < k$, $\mathcal{A}^{\circ\circ}$ is in fact an R -algebra, where

$$R = \mathbf{k}[x_1, \dots, x_k]/(x_1 \dots x_k). \tag{3.2}$$

Here x_i will correspond to the closed Reeb orbit around the i^{th} interior puncture of Σ_k .

At the i^{th} interior puncture, we write u_i, v_i for the two primitive Reeb chords

$$u_i \in \text{hom}_{\mathcal{W}(\Sigma_k, Z)}(L_{i-1}, L_i), \quad v_i \in \text{hom}_{\mathcal{W}(\Sigma_k, Z)}(L_i, L_{i-1}),$$

as in Figure 6.

3.1. Case of 2-dimensional pairs-of-pants. As a warm-up, let us consider the special case $n = 2, k = 3$. The symplectic manifold $M_{2,3}$ is also known as the 2-dimensional pair-of-pants. The category $\mathcal{W}(M_{2,3}, \Lambda_Z)$ is generated by $\binom{4}{2} = 6$ Lagrangians, and the following proposition computes all the morphisms between them.

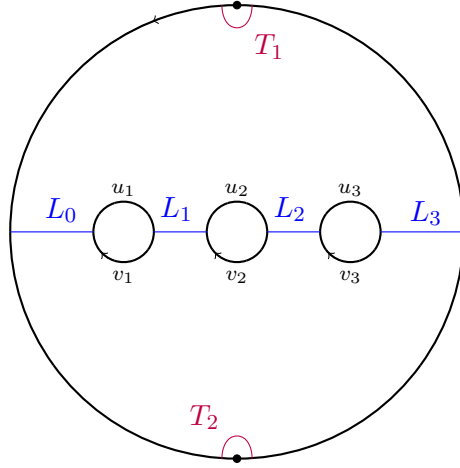


FIGURE 6. Sphere with 4 holes, 2 stops, a generating set of Lagrangians (blue) and certain other Lagrangians supported near stops (brown)

Proposition 3.1.1. *We have natural identifications*

$$\text{End}(L_2 \times L_3) = R/(x_1)$$

$$\text{End}(L_0 \times L_3) = R/(x_2)$$

$$\text{End}(L_0 \times L_1) = R/(x_3)$$

$$\text{End}(L_1 \times L_3) = R/(x_1x_2)$$

$$\text{End}(L_0 \times L_2) = R/(x_2x_3)$$

$$\text{End}(L_1 \times L_2) = R/(x_1x_2x_3)$$

The morphisms between these objects are encoded by the following quiver over R with relations:

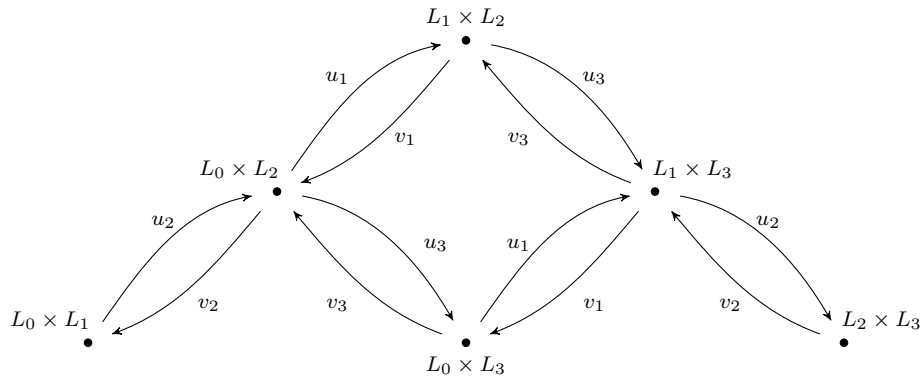


FIGURE 7. Morphisms between a generating set of objects.

where

$$u_i v_i = x_i = v_i u_i, \quad u_3 u_2 = v_2 v_3 = u_2 u_1 = v_1 v_2 = 0$$

and

$$u_3 u_1 = u_1 u_3, v_3 v_1 = v_1 v_3, u_3 v_1 = v_1 u_3, u_1 v_3 = v_3 u_1$$

Proof. When K and L do not have endpoints on the same boundary component of Σ , the generators for $\text{end}(K \times L)$ are just given by $k \otimes \text{id}, \text{id} \otimes l$ where $k \in \text{End}(K)$ and $l \in \text{End}(L)$. The differential on $\text{end}(K \times L)$ is zero, and k and l commute. For example,

$$\text{End}(L_0 \times L_3) = \mathbf{k}\langle \text{id} \otimes x_3, x_1 \otimes \text{id} \rangle = R/(x_2).$$

On the other hand, when both K and L have an end in the i th boundary component of Σ , we have that $\text{end}(K \times L)$ contains the vector subspace spanned by the elements

$$(v_i u_i)^m \otimes (u_i v_i)^n, u_i (v_i u_i)^m \otimes v_i (u_i v_i)^n, \text{ for } m, n \geq 0. \quad (3.3)$$

To understand the algebra structure let us set

$$a_i = v_i u_i \otimes \text{id}_L, \quad b_i = \text{id}_K \otimes u_i v_i, \quad c_i = u_i \otimes v_i.$$

Then we have the relations

$$a_i b_i = b_i a_i = 0, \quad a_i c_i = c_i b_i, \quad b_i c_i = c_i a_i,$$

where the first relation comes from the product rule explained in Figure 4.

The quadratic algebra with these relations has the Gröbner bases

$$(c_i^n, a_i^m c_i^n, b_i^m c_i^n)_{n \geq 0, m > 0},$$

which is exactly the elements (3.3). The differential is given by

$$\partial(a_i) = -\partial(b_i) = c_i, \quad \partial(c_i) = 0$$

and extended by the graded Leibniz rule, where we have $\deg(a_i) = \deg(b_i) = 0$, $\deg(c_i) = 1$. It is easy to check that the relations are preserved. This also determines the signs. Furthermore, we have

$$\partial(c_i^n) = 0, \quad \partial(a_i c_i^n) = -\partial(b_i c_i^n) = c_i^{n+1}, \quad \partial(a_i^m c_i^n) = -\partial(b_i^m c_i^n) = (a_i^{m-1} + b_i^{m-1}) c_i^{n+1},$$

where $m \geq 2$.

If K and L end at the i^{th} boundary component of Σ , we let

$$x_i = a_i + b_i = v_i u_i \otimes \text{id}_L + \text{id}_K \otimes u_i v_i.$$

One can see by a straightforward calculation from the explicit description of the chain complex given above, that the contribution to cohomology $\text{End}(K \times L)$ from the i^{th} boundary components comes from x_i and its positive powers. This determines the cohomology. For example,

$$\text{End}(L_1 \times L_2) = \mathbf{k}\langle u_1 v_1 \otimes \text{id}, v_2 u_2 \otimes \text{id} + \text{id} + u_2 v_2, \text{id} \otimes v_3 u_3 \rangle = R/(x_1 x_2 x_3)$$

The morphisms between different Lagrangians are calculated in the same way (see Theorem 3.2.5 below for a more general calculation). \square

Let us record one simple computation used above.

Lemma 3.1.2. *Let us consider the subcomplex of $\text{end}(K \times L)$ spanned by the elements*

$$((v_i u_i)^m \otimes (u_i v_i)^n)_{m \geq 0, n > 0}, \quad (u_i (v_i u_i)^m \otimes v_i (u_i v_i)^n)_{m \geq 0, n \geq 0}.$$

Then this subcomplex is exact.

Proof. In terms of the generators a_i, b_i, c_i , our subcomplex is spanned by the elements

$$(a_i^m c_i^n)_{m \geq 0, n > 0}, \quad (b_i^m c_i^n)_{m > 0, n \geq 0}.$$

This complex splits into a direct sum of subcomplexes with fixed total degree (given by $m + n$). The subcomplex C^\bullet corresponding to the degree $n > 0$ has terms

$$\begin{aligned} C^0 &= \langle b_i^n \rangle, \quad C^1 = \langle a_i^{n-1} c_i, b_i^{n-1} c_i \rangle, \quad C^2 = \langle a_i^{n-2} c_i^2, b_i^{n-2} c_i^2 \rangle, \dots, \\ C^{m-1} &= \langle a_i c_i^{n-1}, b_i c_i^{n-1} \rangle, \quad C^m = \langle c_i^n \rangle. \end{aligned}$$

Now we see that for $m \in [1, n-1]$, one has

$$\ker(d : C^m \rightarrow C^{m+1}) = \langle (a_i^{n-m} + b_i^{n-m}) c_i^m \rangle = \text{im}(d : C^{m-1} \rightarrow C^m),$$

while $\ker(d : C^0 \rightarrow C^1) = 0$ and $\text{im}(d : C^{n-1} \rightarrow C^n) = C^n$. \square

3.2. General n, k with $k \geq n$. We now describe the computation for arbitrary k, n with $k \geq n$. Let L_0, \dots, L_{k-1}, L_k be the arcs that generate $\mathcal{W}(\Sigma, Z)$ as before. The generators of $\mathcal{W}(M_{n,k}, \Lambda_Z)$ are given by the $\binom{k+1}{n}$ Lagrangians. Let

$$L_S = L_{i_1} \times L_{i_2} \times \dots \times L_{i_n}, \quad \text{with } S = \{i_1 < i_2 < \dots < i_n\} \subset [0, k].$$

In order to understand morphisms between L_S and $L_{S'}$ we need the following combinatorial statement.

Proposition 3.2.1. *Let $S, S' \subset [0, k]$ be a pair of size n subsets, and let $g : S \rightarrow S'$ be a bijection such that for every $i \in S$ one has $g(i) \in \{i-1, i, i+1\}$. Then there exists a collection of disjoint subintervals $I_1, \dots, I_r \subset [0, k]$ such that $S \setminus \sqcup_j I_j = S' \setminus \sqcup_j I_j$; $g(i) = i$ for $i \in S \setminus \sqcup_j I_j$; and for each subinterval I_j one of the following holds:*

- (1) $I_j = [i, i+1] \subset S \cap S'$ and g swaps i with $i+1$;
- (2) $I_j = [a, b]$, $S \cap I_j = [a, b-1]$, $S' \cap I_j = [a+1, b]$, and $g(i) = i+1$ for $i \in S \cap I_j$;
- (3) $I_j = [a, b]$, $S' \cap I_j = [a, b-1]$, $S \cap I_j = [a+1, b]$, and $g(i) = i-1$ for $i \in S \cap I_j$.

We need some preparations before giving a proof. For a pair of size n subsets $S, S' \subset [0, k]$, let us set

$$T = T(S, S') := \{i \mid \#(S \cap [0, i]) = \#(S' \cap [0, i])\}.$$

We can write T as the union of disjoint intervals,

$$T = [s_1, t_1] \sqcup [s_2, t_2] \sqcup \dots \sqcup [s_r, t_r],$$

where $t_r = k$, $s_i \leq t_i$ and $t_i + 1 < s_{i+1}$.

Lemma 3.2.2. (i) Let $g : S \rightarrow S'$ be a bijection such that for every $i \in S$ one has $g(i) \in \{i-1, i, i+1\}$. Then g induces bijections

$$S \cap [s_i+1, t_i] \xrightarrow{\sim} S' \cap [s_i+1, t_i], 2 \leq i \leq r, \quad S \cap [t_i+1, s_{i+1}] \xrightarrow{\sim} S' \cap [t_i+1, s_{i+1}], 1 \leq i \leq r-1.$$

In addition, if $s_1 = 0$ then

$$g(S \cap [0, t_1]) = S \cap [0, t_1] = S' \cap [0, t_1],$$

and if $s_1 > 0$ then

$$g(S \cap [0, s_1]) = S' \cap [0, s_1], \quad g(S \cap [s_1+1, t_1]) = S' \cap [s_1+1, t_1].$$

Furthermore, we have

$$S \cap [s_i+1, t_i] = S' \cap [s_i+1, t_i]$$

(note that these intervals could be empty). On the other hand, each restriction

$$g : S \cap [t_i+1, s_{i+1}] \rightarrow S' \cap [t_i+1, s_{i+1}]$$

(resp., $g : S \cap [0, s_1] \rightarrow S' \cap [0, s_1]$ if $s_1 > 0$) is given either by $g(j) = j-1$ or $g(j) = j+1$.

(ii) Let $g : S \rightarrow S$ be a permutation such that for every $i \in S$ one has $g(i) \in \{i-1, i, i+1\}$. Then there exists a subset $S_0 \subset S$ of the form

$$S_0 = \{i_1, i_1+1, i_2, i_2+1, \dots, i_r, i_r+1\},$$

where $i_s+1 < i_{s+1}$, such that g swaps i_s and i_{s+1} for $s = 1, \dots, r$, and $g(i) = i$ for every $i \in S \setminus S_0$.

Proof. (i) Note that for every i we have $g(S \cap [0, i]) \subset S' \cap [0, i+1]$ and $g^{-1}(S' \cap [0, i]) \subset S \cap [0, i+1]$. Now, since $t_i \in T$ and $t_i+1 \notin T$, we have either $t_i+1 \in S \setminus S'$ or $t_i+1 \in S' \setminus S$. In the former case we have

$$g(S \cap [0, t_i]) \subset S' \cap [0, t_i+1] = S' \cap [0, t_i],$$

while in the latter case we have

$$g^{-1}(S' \cap [0, t_i]) \subset S \cap [0, t_i+1] = S \cap [0, t_i].$$

Thus, we get $g(S \cap [0, t_i]) = S' \cap [0, t_i]$.

Next, we have $s_i-1 \notin T$ and $s_i \in T$, so either $s_i \in S \setminus S'$ or $s_i \in S' \setminus S$. In the former case we have

$$g(S \cap [s_i+1, k]) \subset S' \cap [s_i, k] = S' \cap [s_i+1, k],$$

while in the latter case we have

$$g^{-1}(S' \cap [s_i+1, k]) \subset S \cap [s_i, k] = S \cap [s_i+1, k].$$

Since $\#(S \cap [s_i+1, k]) = \#(S' \cap [s_i+1, k])$, we deduce that $g(S \cap [s_i+1, k]) = S' \cap [s_i+1, k]$. This implies our first assertion.

Assume that $s_i+1 \leq t_i$. Since both s_i and s_i+1 are in T , either the element s_i+1 belongs to $S \cap S'$, or it does not belong to both S and S' . Proceeding in the same way we see that $S \cap [s_i+1, t_i] = S' \cap [s_i+1, t_i]$.

Next, let us consider the interval $[t_i+1, s_{i+1}]$. For every i in this interval, except for the right end, we have $i \notin T$. Assume first that $t_i+1 \in S$. Then, since $t_i \in T$ and $t_i+1 \notin T$,

we should have $t_i + 1 \notin S'$, so $g(t_i + 1) = t_i + 2 \in S'$. If $t_i + 2 < s_{i+1}$ we can continue in the same way: since $\#(S \cap [0, t_i + 1]) = \#(S' \cap [0, t_i + 1]) + 1$ and $t_i + 2 \in S' \setminus T$, we should have $t_i + 2 \in S$, and so $g(t_i + 2) = t_i + 3$, etc. In the case $t_i + 1 \notin S$, we should have $t_i + 1 \in S'$, so we can apply the same argument with S and S' swapped and g replaced by g^{-1} .

(ii) We use induction on the cardinality of S . Let i_0 be the minimal element of S . Then we have either $g(i_0) = i_0$ or $g(i_0) = i_0 + 1$. In the former case we have $g(S \setminus \{i_0\}) = S \setminus \{i_0\}$, so we can apply the induction assumption to $S \setminus \{i_0\}$. Now let us assume that $g(i_0) = i_0 + 1$. Then $g^{-1}(i_0) \neq i_0$, so we should have $g^{-1}(i_0) = i_0 + 1$. Thus, in this case g swaps i_0 and $i_0 + 1$. Now we can replace S by $S \setminus \{i_0, i_0 + 1\}$ and apply the induction assumption. \square

Proof of Proposition 3.2.1. By Lemma 3.2.2(i), we can partition $[0, k]$ into subintervals of two kinds: those for which one of the conditions (2) or (3) of Proposition 3.2.1 holds, and subintervals I such that $S \cap I = S' \cap I$ and $g(S \cap I) = S' \cap I$. It remains to apply Lemma 3.2.2(ii) to all subintervals of the second kind. \square

Definition 3.2.3. We say that two size n subsets $S, S' \subset [0, k]$ are *close* if there exists a bijection $g : S \rightarrow S'$ such that $g(i) \in \{i - 1, i, i + 1\}$ for every i .

Corollary 3.2.4. *If size n subsets $S, S' \subset [0, k]$ are close then there exists a unique collection of disjoint subintervals of $[0, k]$, $(I_1, \dots, I_r, J_1, \dots, J_s)$, such that*

- $([0, k] \setminus \sqcup_a I_a \sqcup \sqcup_b J_b) \cap S = ([0, k] \setminus \sqcup_a I_a \sqcup \sqcup_b J_b) \cap S'$;
- for each $a = 1, \dots, r$, if $I_a = [i, j]$ then $I_a \cap S = [i, j - 1]$, $I_a \cap S' = [i + 1, j]$;
- for each $b = 1, \dots, s$, if $J_b = [i, j]$ then $J_b \cap S' = [i, j - 1]$, $J_b \cap S = [i + 1, j]$.

For every proper subinterval $[i, j] \subset [0, k]$, let us set

$$\mathcal{A}_{[i,j]} = \begin{cases} \mathbf{k}[x_i, \dots, x_{j+1}] / (x_i \dots x_{j+1}) & \text{if } i > 0, j < k, \\ \mathbf{k}[x_1, \dots, x_{j+1}] & \text{if } i = 0, j < k, \\ \mathbf{k}[x_i, \dots, x_k] & \text{if } i > 0, j = k, \end{cases}$$

$$\mathcal{A}'_{[i,j]} = \mathbf{k}[x_{i+1}, \dots, x_j].$$

Finally, for each interval $[i, j]$ we consider the elements

$$u_{[i,j]} := u_{i+1} \otimes u_{i+2} \otimes \dots \otimes u_j \in \text{hom}(L_i, L_{i+1}) \otimes \text{hom}(L_{i+1}, L_{i+2}) \otimes \dots \otimes \text{hom}(L_{j-1}, L_j),$$

$$v_{[i,j]} := v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_j \in \text{hom}(L_{i+1}, L_i) \otimes \text{hom}(L_{i+2}, L_{i+1}) \otimes \dots \otimes \text{hom}(L_j, L_{j-1}).$$

Theorem 3.2.5. *(i) For every k -tuple of integers $(d_1, \dots, d_k) \in \mathbb{Z}^k$ there exists a unique grading structure on $M_{n,k}$ such that for $S = [i_1, j_1] \sqcup [i_2, j_2] \sqcup \dots \sqcup [i_r, j_r]$ with $j_s + 1 < i_{s+1}$, one has a natural isomorphism of graded algebras*

$$\text{End}(L_S) \simeq \mathcal{A}(S, S) := \mathcal{A}_{[i_1, j_1]} \otimes \mathcal{A}_{[i_2, j_2]} \otimes \dots \otimes \mathcal{A}_{[i_r, j_r]},$$

where we have $\deg(x_i) = d_i$.

(ii) For a pair of size n subsets $S, S' \subset [0, k]$, one has $\text{Hom}(L_S, L_{S'}) = 0$ if S and S' are not close. If S and S' are close then there is a natural identification

$$\text{Hom}(L_S, L_{S'}) \simeq \mathcal{A}(S, S') \cdot f_{S, S'}, \quad \text{with } \mathcal{A}(S, S') := \mathcal{A}(S_0, S_0) \otimes \bigotimes_{a=1}^r \mathcal{A}'_{I_a} \otimes \bigotimes_{b=1}^s \mathcal{A}'_{J_b}, \quad (3.4)$$

where we use the subintervals $(I_a), (J_b)$ from Corollary 3.2.4, and set

$$S_0 = S \setminus (\sqcup_a I_a \sqcup \sqcup_b J_b) = S' \setminus (\sqcup_a I_a \sqcup \sqcup_b J_b),$$

$$f_{S, S'} := \bigotimes_{a=1}^r u_{I_a} \otimes \bigotimes_{b=1}^s v_{J_b} \otimes \text{id} \in \text{Hom}(L_S, L_{S'})$$

Furthermore, (3.4) is an isomorphism of $\text{End}(L_{S'}) - \text{End}(L_S)$ -bimodules, where the structure of a $\mathcal{A}(S', S') - \mathcal{A}(S, S)$ -bimodule on $\mathcal{A}(S, S')$ is induced by the surjective homomorphisms of \mathbf{k} -algebras

$$\mathcal{A}(S, S) \rightarrow \mathcal{A}(S, S'), \quad \mathcal{A}(S', S') \rightarrow \mathcal{A}(S, S')$$

sending each x_i either to x_i or to 0 (if x_i is absent in $\mathcal{A}(S, S')$).

(iii) The compositions are uniquely determined by the bimodule structures on $\text{Hom}(L_S, L_{S'})$ together with the rule

$$f_{S', S''} f_{S, S'} = \begin{cases} (\prod_{[i-1, i] \subset T(S, S', S'')} x_i) \cdot f_{S, S''}, & S \text{ and } S'' \text{ are close,} \\ 0, & \text{otherwise.} \end{cases} \quad (3.5)$$

Here $(I_a), (J_b)$ (resp., $(I'_a), (J'_b)$) are the subintervals of Corollary 3.2.4 for the pair S, S' (resp., S', S''), and

$$T(S, S', S'') = ((\sqcup_a I_a) \cap (\sqcup_{b'} J'_{b'})) \cup ((\sqcup_{a'} I'_{a'}) \cap (\sqcup_b J_b)).$$

which can be empty in which case we put $f_{S', S''} f_{S, S'} = f_{S, S''}$.

(iv) The dg-algebra $\bigoplus_{S, S'} \text{hom}(L_S, L_{S'})$ contains a quasi-isomorphic dg-subalgebra with the trivial differential, in particular, it is formal.

For a given assignment of degrees, $\deg(x_i) = d_i, i = 1, \dots, k$, there is a unique \mathbb{Z} -grading on the algebra

$$\mathcal{A}^{\circ\circ} = \bigoplus_{S, S'} \text{Hom}(L_S, L_{S'})$$

coming from some choices of $\deg(f_{S, S'}) = d_{S, S'}$, up to a transformation of the form $d_{S, S'} \mapsto d_{S, S'} + d_{S'} - d_S$.

Proof. (i) Recall that a grading structure on $M_{n, k}$ is given by a fiberwise universal cover of the Lagrangian Grassmannian of the tangent bundle ([25]). Such a cover exists since $2c_1(M_{n, k}) = 0$, and the set of possible gradings structures up to homotopy is a torsor over $H^1(M_{n, k}, \mathbb{Z}) \simeq \mathbb{Z}^k$. As all our Lagrangians L_S are contractible, they can be graded (uniquely up to a shift by \mathbb{Z}). Changing the homotopy class of the grading by some cohomology class $c \in H^1(M_{n, k}; \mathbb{Z}) \simeq H^1(M_{n, k}, L; \mathbb{Z})$, changes the degree of a Reeb chord $x \in \text{end}(L_S)$ by $\langle c, [x] \rangle$ where $[x] \in H_1(M_{n, k}, L_S; \mathbb{Z})$ (see [1, Sec. 9a]).

Now we use the fact that away from the big diagonal $\Delta \subset M_{n,k}$ the Reeb flow is just the product of the Reeb flows on the surface. In addition, we observe that the homotopy class of any Reeb loop of the form $(pt. \times \dots \times pt. \times \gamma_i \times pt. \times \dots \times pt.)$ in $M_{n,k} \setminus \Delta$, coming from the loop $\gamma_i = u_i v_i$ in Σ , for fixed i does not depend on the choice of points in the other components. It follows that the degree of the generator of $\text{end}(L_S)$ corresponding to such a loop is independent of S (it depends only on i). Hence, the fact that grading structures on $M_{n,k}$ form a torsor over $H^1(M_{n,k}, \mathbb{Z})$ implies that for a given $(d_1, d_2, \dots, d_k) \in \mathbb{Z}^k$, there exists a unique grading structure such that $|x_i| = d_i$.

Let us first calculate $\text{End}(L_S)$ in the case $S = [i, j]$, where $i > 0$, $j < k$. Let us set

$$E[i, j] := \text{end}(L_{[i, j]}).$$

First, we claim that there is a natural quasiisomorphism of dg-algebras

$$k[x_i, \dots, x_{j+1}]/(x_i \dots x_{j+1}) \rightarrow E[i, j], \quad (3.6)$$

such that

$$\begin{aligned} x_i &\mapsto (u_i v_i) \otimes \text{id} \otimes \dots, \\ x_s &\mapsto \dots \otimes \text{id} \otimes (v_s u_s) \otimes \text{id} \otimes \dots + \dots \otimes \text{id} \otimes (u_s v_s) \otimes \text{id} \otimes \dots \quad \text{for } i < s \leq j, \\ x_{j+1} &\mapsto \dots \otimes \text{id} \otimes (v_{j+1} u_{j+1}). \end{aligned}$$

Here we view the source as a complex with zero differential. It is easy to check that the morphisms corresponding to x_s are closed, pairwise commute and their product is zero, so the map (3.6) is well defined.

Next, we want to prove that (3.6) is a quasi-isomorphism. We proceed by induction on $j - i$. In the base case $j = i$ this is easy to see. For the induction step we start with a decomposition into a direct sum of subcomplexes,

$$E[i, j] = \bigoplus_{n \geq 0} E[i, j](n).$$

Here for $n > 0$, $E[i, j](n)$ is spanned by elements of the form $(u_i v_i)^n \otimes \dots$, while $E[i, j](0)$ is spanned by the remaining basis elements.

It is easy to see that for $n > 0$, $E[i, j](n)$ is isomorphic (by multiplication with x_i^n) with $E[i + 1, j]$, so we know its cohomology from the induction assumption. More precisely, we know that the map

$$x_i^n \cdot k[x_{i+1}, \dots, x_{j+1}]/(x_{i+1} \dots x_{j+1}) \rightarrow E[i, j](n)$$

is a quasiisomorphism.

It remains to deal with $E[i, j](0)$. Namely, we want to prove that the natural map

$$k[x_{i+1}, \dots, x_{j+1}] \rightarrow E[i, j](0)$$

is a quasiisomorphism.

Let us define subcomplexes $C(0) \subset C(1) \subset E[i, j](0)$ as follows. We define $C(0)$ to be the span of all elements of the form

$$\begin{aligned} & ((v_{i+1}u_{i+1})^m \otimes (u_{i+1}v_{i+1})^n \otimes \text{id}_{L_{i+2}} \otimes \dots)_{m \geq 0, n > 0}, \\ & (u_{i+1}(v_{i+1}u_{i+1})^m \otimes v_{i+1}(u_{i+1}v_{i+1})^n \otimes \text{id}_{L_{i+2}} \otimes \dots)_{m \geq 0, n \geq 0}. \end{aligned}$$

By Lemma 3.1.2, the complex $C(0)$ is acyclic.

Next, we define $C(1)$ to be spanned by $C(0)$ and by all elements of the form $\text{id}_{L_i} \otimes \dots$. We have an isomorphism

$$E[i+1, j](0) \xrightarrow{\sim} C(1)/C(0)$$

induced by the natural embedding of $E[i+1, j](0)$ into $C(1)$. Thus, by the induction assumption, the natural map

$$k[x_{i+2}, \dots, x_{j+1}] \rightarrow C(1)/C(0)$$

is a quasiisomorphism.

Finally, the quotient $E[i, j](0)/C(1)$ splits into a direct sum of subcomplexes $E(n)$ numbered by $n > 0$, where $E(n)$ is spanned by elements of the form $(v_{i+1}u_{i+1})^n \otimes \dots$ (modulo $C(1)$). Note that we have

$$x_{i+1}^n \equiv (v_{i+1}u_{i+1})^n \otimes \text{id} \otimes \dots \text{ mod } C(1),$$

and it is easy to see that the multiplication by x_{i+1}^n gives an isomorphism of complexes,

$$E[i+1, j](0) \xrightarrow{\sim} E(n).$$

Therefore, we deduce that the natural map

$$x_{i+1}^n k[x_{i+2}, \dots, x_{j+1}] \rightarrow E(n)$$

is a quasiisomorphism for every $n > 0$.

Thus, we have a morphism of exact triples of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & k[x_{i+2}, \dots, x_{j+1}] & \longrightarrow & k[x_{i+1}, \dots, x_{j+1}] & \longrightarrow & \bigoplus_{n>0} x_{i+1}^n k[x_{i+2}, \dots, x_{j+1}] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C(1)/C(0) & \longrightarrow & E[i, j](0)/C(0) & \longrightarrow & E[i, j](0)/C(1) & \longrightarrow & 0 \end{array}$$

in which the left and right vertical arrows are quasiisomorphisms, hence, the middle one is also a quasiisomorphism.

Next, we can slightly modify the above argument for the cases $i = 0$ and $j < k$ (resp., $i > 0$ and $j = k$). In these cases we still claim that the natural maps

$$k[x_1, \dots, x_{j+1}] \rightarrow E[0, j], \quad k[x_i, \dots, x_k] \rightarrow E[i, k]$$

are quasiisomorphisms. Namely, in the case of $E[0, j]$ we skip the first step and just deal with this complex exactly as with the complexes $E[i, j](0)$, considering the subcomplexes analogous to $C(0)$ and $C(1)$. In the case of $E[i, k]$, we repeat exactly the same steps as for

$E[i, j]$, considering first the subcomplexes $E[i, k](n)$ and using the induction assumption to see that the maps

$$x_i^n \cdot k[x_{i+1}, \dots, x_k] \rightarrow E[i, k](n)$$

are quasiisomorphisms.

Now for $S = [i_1, j_1] \sqcup [i_2, j_2] \sqcup \dots \sqcup [i_r, j_r]$ with $j_s + 1 < i_{s+1}$, we have a natural identification

$$\text{end}(L_S) \simeq \text{end}(L_{[i_1, j_1]}) \otimes \dots \otimes \text{end}(L_{[i_r, j_r]}),$$

and so we have a similar decomposition of $\text{End}(L_S)$.

(ii) Now let us compute $\text{Hom}(L_S, L_{S'})$ for a pair of size n subsets $S, S' \subset [0, k]$. If S and S' are not close then we have $\text{hom}(L_S, L_{S'}) = 0$. Otherwise, we have a decomposition

$$\text{hom}(L_S, L_{S'}) \simeq \text{end}(L_{([0, k] \setminus \sqcup_a I_a \sqcup \sqcup_b J_b) \cap S}) \otimes \bigotimes_{a=1}^r \text{hom}(L_{I_a \cap S}, L_{I_a \cap S'}) \otimes \bigotimes_{b=1}^s \text{hom}(L_{J_b \cap S}, L_{J_b \cap S'}), \quad (3.7)$$

where we use the subintervals $(I_a), (J_b)$ from Corollary 3.2.4.

It remains to identify the complexes $\text{hom}(L_{I_a \cap S}, L_{I_a \cap S'})$ and $\text{hom}(L_{J_b \cap S}, L_{J_b \cap S'})$. First, let $I = I_a = [i, j]$, so that $I \cap S = [i, j - 1]$ and $I \cap S' = [i + 1, j]$. We have

$$\text{hom}(L_{[i, j-1]}, L_{[i+1, j]}) = \bigotimes_{p \in [i, j-1]} \text{hom}(L_p, L_{p+1}),$$

with the basis given by

$$u_{i+1}(v_{i+1}u_{i+1})^{m_1} \otimes \dots \otimes u_j(v_ju_j)^{m_{j-i}}, \quad m_1 \geq 0, \dots, m_{j-i} \geq 0.$$

It is easy to check that we have

$$u_{i+1}(v_{i+1}u_{i+1})^{m_1} \otimes \dots \otimes u_j(v_ju_j)^{m_{j-i}} = (x_{i+1}^{m_1} \dots x_j^{m_{j-i}})u_{[i, j]} = u_{[i, j]}(x_{i+1}^{m_1} \dots x_j^{m_{j-i}}),$$

where we view the monomial in x_1, \dots, x_k as an endomorphism of $L_{[i, j-1]}$ or of $L_{[i+1, j]}$. In particular, the differential on $\text{hom}(L_{[i, j-1]}, L_{[i+1, j]})$ is zero.

Similarly, we see that for $J = J_b = [i, j]$, the space $\text{hom}(L_{[i+1, j]}, L_{[i, j-1]})$ is spanned by

$$(x_{i+1}^{m_1} \dots x_j^{m_{j-i}})v_{[i, j]} = v_{[i, j]}(x_{i+1}^{m_1} \dots x_j^{m_{j-i}}),$$

and has zero differential.

(iii) Formula (3.5) can be checked directly. To compose arbitrary elements in $\text{Hom}(L_S, L_{S'})$ and $\text{Hom}(L_{S'}, L_{S''})$ we can use (iii) to write them in the form $f_{S, S'}p(x)$ and $q(x)f_{S', S''}$, with $p(x)$ and $q(x)$ some polynomials in x_1, \dots, x_k , and then use (3.5).

(iv) The first assertion is clear from our previous computations: the subalgebra in question is generated by $f_{S, S'}$ and by $x_i \in \text{end}(L_S)$.

By (i), we know that for each choice $\text{deg}(x_i) = d_i$, there exists a \mathbb{Z} -grading on \mathcal{A}° coming from a grading structure on $M_{n, k}$. Thus, it is enough to prove uniqueness of $(d_{S, S'})$ up to adding $d_{S'} - d_S$. The numbers $d_{S, S'}$ are constrained by equations (3.5), corresponding to triples (S, S', S'') of pairwise close subsets. Thus, a difference between two systems $(d_{S, S'})$ is a 1-cocycle for the following simplicial complex X . The set of vertices, X_0 , is the set of all size n subsets in $[0, k]$. The set of edges, X_1 , is given by pairs (S, S') of subsets which are close, and the set of 2-simplices, X_2 , is the set of triples (S, S', S'') of pairwise close

subsets. Our uniqueness statement would follow from the vanishing of $H^1(X, \mathbb{Z})$. We claim that in fact the geometric realization $|X|_{\text{real}}$ is simply connected. The proof is by induction on k . The case $k = 1$ is straightforward (for $n = 0$ we get a point and for $n = 1$ we get a segment). Another special case we have to include to make our proof work is $n = k + 1$, when X reduces to a point.

Let us write $X = X(n, k)$. We have natural embeddings as simplicial subcomplexes,

$$\begin{aligned} i_k : X(n, k-1) &\hookrightarrow X(n, k) : S \subset [0, k-1] \mapsto S, \\ j_k : X(n-1, k-1) &\hookrightarrow X(n, k) : S \subset [0, k-1] \mapsto S \sqcup \{k\}, \end{aligned}$$

so that every vertex of $X(n, k)$ is either in the image of i_k or in the image of j_k .

Let us in addition consider the subcomplex $Y \subset X(n, k)$ spanned by all vertices of the form $S \sqcup \{k-1\}$ and $S \sqcup \{k\}$, with $S \subset [0, k-2]$, $|S| = n-1$. Note that whenever S and S' are close subsets of $[0, k-2]$, all four subsets

$$S \sqcup \{k-1\}, \quad S \sqcup \{k\}, \quad S' \sqcup \{k-1\}, \quad S' \sqcup \{k\}$$

are pairwise close. This implies that $|Y|_{\text{real}}$ can be identified with the 2-skeleton of $|X(n-1, k-2)|_{\text{real}} \times I$, where I is a segment. In particular, by the induction assumption, $|Y|_{\text{real}}$ is simply connected.

Now assume that we have an edge e between a vertex S in $i_k(X(n, k-1))$, where $S \subset [0, k-1]$ and a vertex $S' \sqcup \{k\}$ in $j_k(X(n-1, k-1))$. This means that S and $S' \sqcup \{k\}$ are close, which can happen only if $S = T \sqcup \{k-1\}$, with T and S' close. In this case the triangle

$$(S = T \sqcup \{k-1\}, T \sqcup \{k\}, S' \sqcup \{k\})$$

gives a homotopy between e and a segment $(T \sqcup \{k-1\}, T \sqcup \{k\})$ in Y , followed by a segment in $j_k(X(n-1, k-1))$. Since $i_k(X(n, k-1))$, $j_k(X(n-1, k-1))$ and Y are simply connected, this implies that $X(n, k)$ is also simply connected. \square

Corollary 3.2.6. *The algebra $\mathcal{A}^{\circ\circ}$ has a structure of $\mathbf{k}[x_1, \dots, x_k]$ -algebra. In the case $n < k$, it has a structure of an algebra over $R = \mathbf{k}[x_1, \dots, x_k]/(x_1 \dots x_k)$.*

3.3. Localization by stops. By the general stop removal theorem [9, Theorem 1.16], removing a stop corresponds to localizing by the subcategory of Lagrangians supported near the image of the stop. In our case the stops Λ_1 and Λ_2 are neighborhoods of the symplectic submanifolds $M_{n-1, k}$ and using the generators of the partially wrapped Fukaya category of $M_{n-1, k}$, we obtain that these subcategories are given by

$$\begin{aligned} \mathcal{D}_1 &= \langle T_1 \times X : X = L_{i_1} \times \dots \times L_{i_{n-1}} \rangle \\ \mathcal{D}_2 &= \langle T_2 \times X : X = L_{i_1} \times \dots \times L_{i_{n-1}} \rangle \end{aligned}$$

where $\{i_1, \dots, i_{n-1}\}$ runs through size $(n-1)$ subsets of $[0, k]$. Thus, we have the following result.

Theorem 3.3.1. *We have equivalences of pre-triangulated categories*

$$\begin{aligned} \mathcal{W}(M_{n, k}, \Lambda_1) &\simeq \mathcal{W}(M_{n, k}, \Lambda_Z) / \mathcal{D}_2 \\ \mathcal{W}(M_{n, k}) &\simeq \mathcal{W}(M_{n, k}, \Lambda_Z) / \langle \mathcal{D}_1, \mathcal{D}_2 \rangle \end{aligned}$$

Remark 3.3.2. If we only consider one stop, say q_1 , then the partially wrapped Fukaya category $\mathcal{W}(\Sigma, q_1)$ is generated by the Lagrangians L_1, \dots, L_k (i.e., we do not need L_0), as in this case the complement of L_1, \dots, L_k is a disk which contains at most 1 stop in its boundary. Furthermore, by Auroux's theorem [3, Theorem 1], the category $\mathcal{W}(M_{n-1,k}, \Lambda_1)$ is generated by $L_{i_1} \times \dots \times L_{i_{n-1}}$ with $\{i_1, \dots, i_{n-1}\} \subset [1, k]$. This allows to reduce the number of generators in the definition of \mathcal{D}_1 and \mathcal{D}_2 above. For the same reason, it is enough to use only products of L_1, \dots, L_k when generating the fully wrapped Fukaya category of $M_{n,k}$.

We next give an explicit description of the objects in \mathcal{D}_1 and \mathcal{D}_2 in terms of the generators of $\mathcal{W}(M_{n,k}, \Lambda_Z)$.

Proposition 3.3.3. *Let $X = L_{i_1} \times \dots \times L_{i_{n-1}}$ and let $j_1 < j_2 \dots < j_{k-n+2}$ be the complement of $\{i_1, \dots, i_{n-1}\}$ in $[0, k]$, then in $\mathcal{W}(M_{n,k}, \Lambda_Z)$ we have the following equivalences*

$$\begin{aligned} T_1 \times X &\simeq \{L_{j_1} \times X \xrightarrow{u_{[j_1, j_2]}} L_{j_2} \times X \rightarrow \dots \xrightarrow{u_{[j_{k-n+1}, j_{k-n+2}]} } L_{j_{k-n+2}} \times X\}, \\ T_2 \times X &\simeq \{L_{j_{k-n+2}} \times X \xrightarrow{v_{[j_{k-n+1}, j_{k-n+2}]} } L_{j_{k-n+1}} \times X \rightarrow \dots \xrightarrow{v_{[j_1, j_2]}} L_{j_1} \times X\} \end{aligned}$$

The first equivalence also holds in $\mathcal{W}(M_{n,k}, \Lambda_1)$.

Proof. This is obtained by repeatedly applying the exact triangle given by (2.5). More precisely, first as in Figure 8, we slide $L_{j_{k+1}}$ along the components of X until one of its legs gets next to L_{j_k} . This gives an isomorphism

$$L_{j_{k+1}} \times X \simeq \tilde{L}_{j_{k+1}} \times X.$$

Next, let us consider

$$C := \text{cone}(\tilde{L}_{j_{k+1}} \times X \rightarrow L_{j_k} \times X).$$

Because of the above isomorphism we also have an exact triangle

$$L_{j_{k+1}} \times X \rightarrow L_{j_k} \times X \rightarrow C \rightarrow \dots$$

which is shown in Figure 9.

Thus, applying this kind of triangle repeatedly, we can express $\tilde{T}_2 \times X$ drawn in Figure 10 as

$$\tilde{T}_2 \times X \simeq \{L_{j_{k-n+2}} \times X \rightarrow L_{j_{k-n+1}} \times X \rightarrow \dots \rightarrow L_{j_1} \times X\}$$

where \tilde{T}_2 has the property that all objects L_0, \dots, L_i to the left of the left leg of \tilde{T}_2 are in X and all the objects L_j, \dots, L_k to the right of the right leg of \tilde{T}_2 are in X . Therefore, sliding \tilde{T}_2 over these Lagrangians, we exhibit a Hamiltonian isotopy between $\tilde{T}_2 \times X$ and $T_2 \times X$, which completes the proof of the proposition for $T_2 \times X$. The case of $T_1 \times X$ is considered similarly. \square

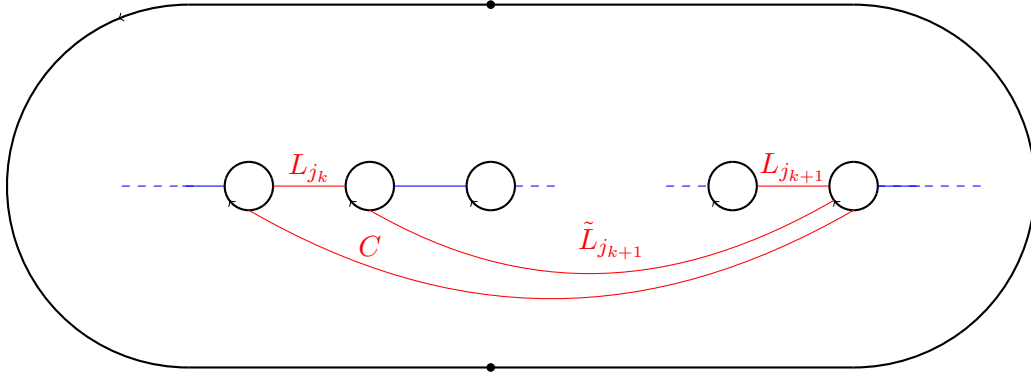


FIGURE 8. Hamiltonian isotopy of $L_{j_{k+1}} \times X \rightarrow \tilde{L}_{j_{k+1}} \times X$

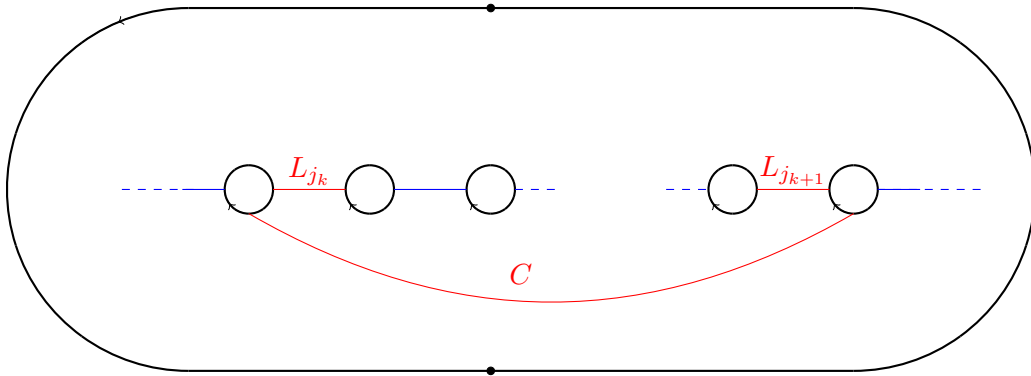


FIGURE 9. Exact triangle

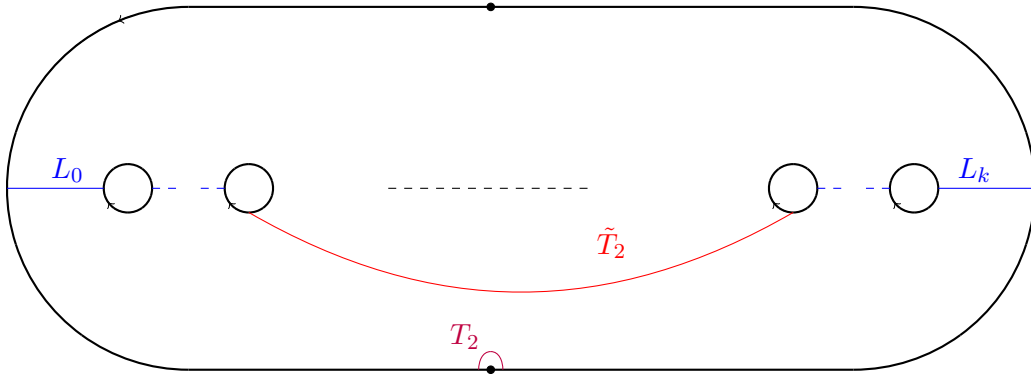


FIGURE 10. Hamiltonian isotopy $T_2 \times X \rightarrow \tilde{T}_2 \times X$

4. B-SIDE

4.1. **Two categorical resolutions on the B-side.** Recall that we work over a base commutative ring \mathbf{k} , which we assume to be regular. Let $R = R_{[1,k]}$ denote the ring

$\mathbf{k}[x_1, \dots, x_k]/(x_1 \dots x_k)$. Here we define two categorical resolutions of the category $\text{Perf}(R)$. Both are given using modules over certain finite R -algebras.

For every subset $I \subset [1, k]$, let us set $x_I := \prod_{i \in I} x_i$. We consider two R -algebras,

$$\mathcal{B}^\circ = \mathcal{B}_{[1,k]}^\circ := \text{End}_R(R/(x_1) \oplus R/(x_{[1,2]}) \oplus \dots \oplus R/(x_{[1,k-1]}) \oplus R),$$

$$\mathcal{B}^{\circ\circ} = \mathcal{B}_{[1,k]}^{\circ\circ} := \text{End}_R\left(\bigoplus_{I \subset [1,k], I \neq \emptyset} R/(x_I)\right),$$

where the summation is over all nonempty subintervals of $[1, k]$. Note that both \mathcal{B}° and $\mathcal{B}^{\circ\circ}$ are finitely generated as R -modules, in particular, they are Noetherian.

For each nonempty subinterval $I \subset [1, k]$ (resp., for $I = [1, m]$) we denote by P_I the natural projective $\mathcal{B}^{\circ\circ}$ -module (resp., \mathcal{B}° -module) corresponding to the summand $R/(x_I)$. When we need to distinguish the \mathcal{B}° -module $P_{[1,m]}$ from the $\mathcal{B}^{\circ\circ}$ -module with the same name, we will write $P_{[1,m]}^\circ$ (resp., $P_{[1,m]}^{\circ\circ}$) to denote the \mathcal{B}° -module (resp., $\mathcal{B}^{\circ\circ}$ -module). It is easy to see that we have natural identifications

$$\text{Hom}(P_I, P_J) \simeq \text{Hom}_R(R/(x_J), R/(x_I)),$$

compatible with composition. In particular, $\text{End}(P_I) = R/(x_I)$, and

$$(\mathcal{B}^\circ)^{op} = \text{End}_{\mathcal{B}^{\circ\circ}}\left(\bigoplus_{m=1}^k P_{[1,m]}\right).$$

Thus, if we set $P = P_{[1,k]}$, then we get natural faithful functors

$$i_R^{\mathcal{B}^\circ} : \text{Perf}(R) \rightarrow D^b(\mathcal{B}^\circ) : M \mapsto P \otimes_R M,$$

$$i_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}} : \text{Perf}(\mathcal{B}^\circ) \rightarrow D^b(\mathcal{B}^{\circ\circ}) : M \mapsto \left(\bigoplus_{m=1}^k P_{[1,m]}\right) \otimes_{\mathcal{B}^\circ} M,$$

which are left adjoint to the restriction functors

$$r_R^{\mathcal{B}^\circ} : D^b(\mathcal{B}^\circ) \rightarrow D^b(R) : M \mapsto \text{Hom}_{\mathcal{B}^\circ}(P, M),$$

$$r_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}} : D^b(\mathcal{B}^{\circ\circ}) \rightarrow D^b(\mathcal{B}^\circ) : M \mapsto \text{Hom}_{\mathcal{B}^{\circ\circ}}\left(\bigoplus_{m=1}^k P_{[1,m]}, M\right). \quad (4.1)$$

Here we denote by $D^b(\mathcal{B}^\circ)$ the bounded derived category of finitely generated \mathcal{B}° -modules, while $\text{Perf}(\mathcal{B}^\circ)$ is the full subcategory on bounded complexes of finitely generated projective modules (and similarly for other algebras). We also consider composed functors $i_R^{\mathcal{B}^{\circ\circ}} = i_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}} \circ i_R^{\mathcal{B}^\circ}$ and

$$r_R^{\mathcal{B}^{\circ\circ}} = r_R^{\mathcal{B}^\circ} \circ r_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}}. \quad (4.2)$$

We are going to show that the algebras $\mathcal{B}^{\circ\circ}$ and \mathcal{B}° have finite global dimension and that the restriction functors $r_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}}$ and $r_R^{\mathcal{B}^{\circ\circ}}$ are localization functors with respect to explicit subcategories of $D^b(\mathcal{B}^{\circ\circ})$, whereas $i_R^{\mathcal{B}^\circ}$ and $i_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}}$ are fully faithful, so that $\mathcal{B}^{\circ\circ}$ and \mathcal{B}° provide categorical resolutions of R .

For a subinterval $I \subset [1, k]$, let us denote by R_I , $\mathcal{B}_I^{\circ\circ}$ and \mathcal{B}_I° the algebras defined in the same way as $R_{[1,k]}$, $\mathcal{B}_{[1,k]}^{\circ\circ}$ and $\mathcal{B}_{[1,k]}^\circ$, but with the variables x_1, \dots, x_k replaced by $(x_i)_{i \in I}$.

We will need to use some other induction functors, in addition to $i_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ\circ}}$ and $i_R^{\mathcal{B}^{\circ}}$. Let us consider the natural (exact) restriction functors

$$r_{\mathcal{B}_{[1,k-1]}^{\circ\circ}[x_k]}^{\mathcal{B}^{\circ\circ}} : \mathcal{B}^{\circ\circ} - \text{mod} \rightarrow \mathcal{B}_{[1,k-1]}^{\circ\circ}[x_k] - \text{mod} : M \mapsto \text{Hom}_{\mathcal{B}^{\circ\circ}}\left(\bigoplus_{J \subset [1,k-1], J \neq \emptyset} P_J, M\right),$$

$$r_{\mathcal{B}_{[2,k]}^{\circ\circ}[x_1]}^{\mathcal{B}^{\circ\circ}} : \mathcal{B}^{\circ\circ} - \text{mod} \rightarrow \mathcal{B}_{[2,k]}^{\circ\circ}[x_1] - \text{mod} : M \mapsto \text{Hom}_{\mathcal{B}^{\circ\circ}}\left(\bigoplus_{J \subset [2,k], J \neq \emptyset} P_J, M\right),$$

where we use the identifications

$$R_{[1,k]}/x_{[1,k-1]} = R_{[1,k-1]}[x_k], \quad R_{[1,k]}/x_{[2,k]} = R_{[2,k]}[x_1].$$

We have $M_J = \text{Hom}_{\mathcal{B}^{\circ\circ}}(P_J, M)$, so in both cases the restricted module has the decomposition into components (which are R -modules) of the form

$$r_{\mathcal{B}_I^{\circ\circ}}^{\mathcal{B}^{\circ\circ}}(M) = \bigoplus_{J \subset I, J \neq \emptyset} M_J.$$

Lemma 4.1.1. (i) *One has natural isomorphisms*

$$i_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ\circ}}(P_{[1,i]}^{\circ\circ}) \simeq P_{[1,i]}^{\circ\circ}, \quad i_R^{\mathcal{B}^{\circ}}(R) \simeq P = P_{[1,k]},$$

where $1 \leq i \leq k$. The canonical adjunction maps

$$\text{Id} \rightarrow r_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ\circ}} \circ i_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ\circ}}, \quad \text{Id} \rightarrow r_R^{\mathcal{B}^{\circ}} \circ i_R^{\mathcal{B}^{\circ}}$$

are isomorphisms on perfect derived categories and on abelian categories.

(ii) *The restriction functor $r_{\mathcal{B}_{[2,k]}^{\circ\circ}[x_1]}^{\mathcal{B}^{\circ\circ}}$ has a left adjoint functor*

$$i_{\mathcal{B}_{[2,k]}^{\circ\circ}[x_1]}^{\mathcal{B}^{\circ\circ}} : \mathcal{B}_{[2,k]}^{\circ\circ} - \text{mod} \rightarrow \mathcal{B}^{\circ\circ} - \text{mod}$$

such that $r_{\mathcal{B}_{[2,k]}^{\circ\circ}[x_1]}^{\mathcal{B}^{\circ\circ}} \circ i_{\mathcal{B}_{[2,k]}^{\circ\circ}[x_1]}^{\mathcal{B}^{\circ\circ}} \simeq \text{Id}$ and

$$i_{\mathcal{B}_{[2,k]}^{\circ\circ}[x_1]}^{\mathcal{B}^{\circ\circ}}(M)_I = \begin{cases} M_{I \cap [2,k]} & I \neq [1], \\ 0 & I = [1]. \end{cases}$$

In particular, the functor $i_{\mathcal{B}_{[2,k]}^{\circ\circ}[x_1]}^{\mathcal{B}^{\circ\circ}}$ is exact and $r_{\mathcal{B}_{[2,k]}^{\circ\circ}[x_1]}^{\mathcal{B}^{\circ\circ}} \circ i_{\mathcal{B}_{[2,k]}^{\circ\circ}[x_1]}^{\mathcal{B}^{\circ\circ}} = \text{Id}$. For $I \subset [2, k]$, one has

$$i_{\mathcal{B}_{[2,k]}^{\circ\circ}[x_1]}^{\mathcal{B}^{\circ\circ}}(P_{[2,k]}[x_1]) \simeq P_{[2,k]}.$$

(iii) *The restriction functor $r_{\mathcal{B}_{[1,k-1]}^{\circ\circ}[x_k]}^{\mathcal{B}^{\circ\circ}}$ has a left adjoint functor*

$$i_{\mathcal{B}_{[1,k-1]}^{\circ\circ}[x_k]}^{\mathcal{B}^{\circ\circ}} : \mathcal{B}_{[1,k-1]}^{\circ\circ}[x_k] - \text{mod} \rightarrow \mathcal{B}^{\circ\circ} - \text{mod},$$

which is exact and satisfies $r_{\mathcal{B}_{[1,k-1]}^{\circ\circ}[x_k]}^{\mathcal{B}^{\circ\circ}} \circ i_{\mathcal{B}_{[1,k-1]}^{\circ\circ}[x_k]}^{\mathcal{B}^{\circ\circ}} = \text{Id}$. Furthermore, for $1 \leq i \leq k-1$, we have

$$i_{\mathcal{B}_{[1,k-1]}^{\circ\circ}[x_k]}^{\mathcal{B}^{\circ\circ}}(P_{[1,i]}[x_k]) \simeq P_{[1,i]}.$$

Proof. (i) For any $\mathcal{B}^{\circ\circ}$ -module M , we have

$$\mathrm{Hom}_{\mathcal{B}^{\circ\circ}}(P_{[1,i]}^{\circ\circ}, M) \simeq M_{[1,i]} = r_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ\circ}}(M)_{[1,i]} \simeq \mathrm{Hom}_{\mathcal{B}^{\circ}}(P_{[1,i]}^{\circ}, r_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ\circ}}(M)),$$

which gives an isomorphism $i_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ\circ}}(P_{[1,i]}^{\circ\circ}) \simeq P_{[1,i]}^{\circ\circ}$. The proof of the isomorphism $i_R^{\mathcal{B}^{\circ\circ}}(R) \simeq P$ is similar.

This implies that our two adjunction maps are isomorphisms on all projective modules, hence on the perfect derived subcategories. It is also easy to see that these maps are isomorphisms on the abelian categories (since the induction functors on the abelian categories can be explicitly described).

(ii),(iii). The proofs are straightforward and similar to (i). \square

For a pair of disjoint subintervals $I, J \subset [1, k]$, such that $I \sqcup J$ is again an interval, let us denote by

$$\alpha\{I, J\} \in \mathrm{Hom}_{\mathcal{B}^{\circ\circ}}(P_I, P_{I \sqcup J}) = \mathrm{Hom}_R(R/x_{I \sqcup J}, R/x_I)$$

the generator corresponding to the natural projection $R/x_{I \sqcup J} \rightarrow R/x_I$, and let

$$\beta\{I, J\} \in \mathrm{Hom}_{\mathcal{B}^{\circ\circ}}(P_{I \sqcup J}, P_J) = \mathrm{Hom}_R(R/x_J, R/x_{I \sqcup J})$$

be the generator corresponding to the map $R/x_J \xrightarrow{x_I} R/x_{I \sqcup J}$. In the case when this makes sense we use the same notation for the similar morphisms in the category of \mathcal{B}° -modules.

Proposition 4.1.2. *For every $i = 2, \dots, k$ let us define a \mathcal{B}° -module \bar{P}_i from the exact sequence*

$$0 \rightarrow P_{[1, i-1]} \xrightarrow{\alpha\{[1, i-1], [i]\}} P_{[1, i]} \rightarrow \bar{P}_i \rightarrow 0,$$

and let $\bar{P}_1 = P_{[1]}$. Then

$$\mathrm{Ext}_{\mathcal{B}^{\circ}}^m(\bar{P}_i, \bar{P}_i) \simeq \begin{cases} R/(x_i) & m = 0 \\ 0 & m \neq 0, \end{cases} \quad (4.3)$$

and one has a semiorthogonal decomposition,

$$D^b(\mathcal{B}^{\circ}) = \langle \langle \bar{P}_k \rangle, \dots, \langle \bar{P}_2 \rangle, \langle P_1 \rangle \rangle.$$

In particular, the algebra \mathcal{B}° has finite global dimension.

Proof. The surjectivity of the map $R/(x_{[1, j]}) \rightarrow R/(x_{[1, i-1]})$ implies that the map $\alpha\{[1, i-1], [i]\}$ is an embedding. Furthermore, it is easy to see that

$$(\bar{P}_i)_{[1, j]} = \begin{cases} R/(x_i), & j \geq i, \\ 0, & j < i. \end{cases}$$

In particular, $\mathrm{Ext}^*(P_{[1, j]}, \bar{P}_i) = 0$ for $j < i$ and $\mathrm{Ext}^*(P_i, \bar{P}_i) = R/(x_i)$. This immediately implies the required semiorthogonalities and the equality (4.3).

Now let us consider the restriction and induction functors

$$r_{\mathcal{B}^{\circ}}^{\mathcal{B}^{\circ}} : D^b(\mathcal{B}^{\circ}) \rightarrow D^b(\mathcal{B}_{[1, k-1]}^{\circ}[x_k]), \quad i_{\mathcal{B}^{\circ}}^{\mathcal{B}^{\circ}} : D^b(\mathcal{B}_{[1, k-1]}^{\circ}[x_k]) \rightarrow D^b(\mathcal{B}^{\circ}).$$

We have $r_{\mathcal{B}_{[1,k-1]}^{\circ}[x_k]}^{\mathcal{B}^{\circ}} \circ i_{\mathcal{B}_{[1,k-1]}^{\circ}[x_k]}^{\mathcal{B}^{\circ}} = \text{Id}$, so the functor $i_{\mathcal{B}_{[1,k-1]}^{\circ}[x_k]}^{\mathcal{B}^{\circ}}$ is fully faithful. Since for $i \leq k-1$ one has

$$i_{\mathcal{B}_{[1,k-1]}^{\circ}[x_k]}^{\mathcal{B}^{\circ}}(P_{[1,i]}[x_k]) = P_{[1,i]},$$

we deduce that

$$i_{\mathcal{B}_{[1,k-1]}^{\circ}[x_k]}^{\mathcal{B}^{\circ}}(\overline{P}_i[x_k]) = \overline{P}_i,$$

for $i \leq k-1$. Using the induction on k , we see that it remains to prove that there is a semiorthogonal decomposition

$$D^b(\mathcal{B}^{\circ}) = \langle \langle \overline{P}_k \rangle, i_{\mathcal{B}_{[1,k-1]}^{\circ}[x_k]}^{\mathcal{B}^{\circ}} D^b(\mathcal{B}_{[1,k-1]}^{\circ}[x_k]) \rangle \rangle.$$

In other words, it remains to check that the kernel of the restriction functor $r_{\mathcal{B}_{[1,k-1]}^{\circ}[x_k]}^{\mathcal{B}^{\circ}}$ is generated by \overline{P}_k . But this immediately follows from the fact that any \mathcal{B}° -module M with $M_{[1,j]} = 0$ for $j < k$ has $x_k M_{[1,k]} = 0$, together with the fact that any finitely generated $R/(x_k)$ -module has a finite free resolution.

Now the fact that \mathcal{B}° has finite global dimension follows from the standard results on homologically smooth dg-algebras (see [18, Prop. 3.8, Prop. 3.11]). \square

Remark 4.1.3. It is easy to see that for every i one has an isomorphism of \mathcal{B}° -modules

$$\overline{P}_i \simeq r_{\mathcal{B}^{\circ}}^{B^{\circ\circ}}(P_{[i]}^{\circ\circ}).$$

Now we introduce a certain family of $\mathcal{B}^{\circ\circ}$ -modules, which will play an important role for us.

Lemma 4.1.4. *For a pair of nonempty disjoint subintervals $I, J \subset [1, k]$, such that $I \sqcup J$ is still an interval, the sequence*

$$0 \rightarrow P_I \xrightarrow{\alpha\{I, J\}} P_{I \sqcup J} \xrightarrow{\beta\{I, J\}} P_J$$

is exact. We define the module $M\{I, J\}$ as the cokernel of the last arrow, so that we have an exact sequence

$$0 \rightarrow P_I \rightarrow P_{I \sqcup J} \rightarrow P_J \rightarrow M\{I, J\} \rightarrow 0.$$

Proof. We just need to show that the kernel of the morphism $P_{I \sqcup J} \rightarrow P_J$ is isomorphic to P_I . But this immediately follows from the exact sequence

$$0 \rightarrow R/(x_J) \xrightarrow{x_I} R/(x_{I \sqcup J}) \rightarrow R/(x_I) \rightarrow 0.$$

\square

For every $j \in [2, k]$, we have a natural exact functor of “extending by zero”,

$$i_{\{j\}} : \mathcal{B}_{[j,k]}^{\circ\circ}[x_1, \dots, x_{j-2}] - \text{mod} \rightarrow \mathcal{B}^{\circ\circ} - \text{mod}.$$

Namely, note that every $\mathcal{B}^{\circ\circ}$ -module M , viewed as an R -module, has a decomposition

$$M = \bigoplus_{I \subset [1, k], I \neq \emptyset} M_I$$

corresponding to the natural idempotents in $\mathcal{B}^{\circ\circ}$. Now we have

$$(i_!\{j\}M)_I = \begin{cases} M_I & I \subset [j, k], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that there is a natural structure of a $\mathcal{B}^{\circ\circ}$ -module on $i_!\{j\}M$.

We will also use the notation $i_!\{1\} = \text{Id} : \mathcal{B}^{\circ\circ} - \text{mod} \rightarrow \mathcal{B}^{\circ\circ} - \text{mod}$.

Lemma 4.1.5. *For every $m \geq j \geq 2$ one has an isomorphism of $\mathcal{B}^{\circ\circ}$ -modules,*

$$M\{[j-1], [j, m]\} \simeq i_!\{j\}(P_{[j,m]}[x_1, \dots, x_{j-2}]).$$

Proof. We need to compute the cokernel C of the map $\beta\{[j-1], [j, m]\} : P_{[j-1,m]} \rightarrow P_{[j,m]}$. We have

$$(P_{[j,m]})_I = \text{Hom}(R/(x_{[j,m]}), R/(x_I)) = (x_{I \setminus [j,m]})/(x_I), \quad (P_{[j-1,m]})_I = (x_{I \setminus [j-1,m]})/(x_I),$$

and our map is given by the multiplication by x_{j-1} .

Assume first that $j-1 \notin I$. Then $I \setminus [j, m] = I \setminus [j-1, m]$, so we get

$$C_I = (P_{[j,m]})_I \otimes R/(x_i).$$

Note that in this case we have either $I \subset [1, j-1]$ or $I \subset [j, k]$, and in the former case, $(P_{[j,m]})_I = 0$, so $C_I = 0$.

On the other hand, if $j-1 \in I$ then setting $I' = I \setminus [j-1, m]$ we have

$$(P_{[j,m]})_I = (x_{j-1}x_{I'})/(x_I), \quad (P_{[j-1,m]})_I = x_{I'}/(x_I),$$

so in this case the map $(P_{[j-1,m]})_I \xrightarrow{x_{j-1}} (P_{[j,m]})_I$ is surjective, so $C_I = 0$.

This easily leads to the identification $C \simeq i_!\{j\}(P_{[j,m]}[x_1, \dots, x_{j-2}])$. \square

Theorem 4.1.6. *For every $j = 1, \dots, k$, the composed functor*

$$\Phi_j = i_!\{j\}i_{\mathcal{B}_{[j,k]}^{\circ\circ}} : D^b(\mathcal{B}_{[j,k]}^{\circ}[x_1, \dots, x_{j-2}]) \rightarrow D^b(\mathcal{B}^{\circ\circ}) \quad (4.4)$$

is fully faithful, and we have a semiorthogonal decomposition

$$D^b(\mathcal{B}^{\circ\circ}) = \langle \Phi_k D^b(\mathcal{B}_{[k]}^{\circ}[x_1, \dots, x_{k-2}]), \dots, \Phi_3 D^b(\mathcal{B}_{[3,k]}^{\circ}[x_1]), \Phi_2 D^b(\mathcal{B}_{[2,k]}^{\circ}), \Phi_1 D^b(\mathcal{B}_{[1,k]}^{\circ}) \rangle.$$

Hence, the algebra $\mathcal{B}^{\circ\circ}$ has finite global dimension.

Proof. Recall that by Lemma 4.1.1(i), the functor $i_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ}} : D^b(\mathcal{B}^{\circ}) \rightarrow D^b(\mathcal{B}^{\circ\circ})$ is fully faithful and the right orthogonal to its image is equal to the kernel of the restriction functor $r_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ}}$.

To check that the functor Φ_j is fully faithful it is enough to prove that it is fully faithful on the subcategory of projective modules $P_{[j,m]}[x_1, \dots, x_{j-2}]$, where $m \in [j, k]$. But this follows from Lemma 4.1.5. Indeed, for any $\mathcal{B}_{[j,k]}^{\circ\circ}[x_1, \dots, x_{j-2}]$ -module M one has $\text{Hom}(P_{[j-1,m]}, i_!\{j\}M) = 0$ for any $m \geq j-1$, so Lemma 4.1.5, together with the defining exact sequence for $M\{[j-1], [j, m]\}$, gives an identification

$$\begin{aligned} \text{Hom}(i_!\{j\}P_{[j,m]}[x_1, \dots, x_{j-2}], i_!\{j\}M) &\simeq \text{Hom}(P_{[j,m]}, i_!\{j\}M) \simeq (i_!\{j\}M)_{[j,m]} \\ &= M_{[j,m]} \simeq \text{Hom}(P_{[j,m]}[x_1, \dots, x_{j-2}], M). \end{aligned}$$

By Proposition 4.1.2, each category $D^b(\mathcal{B}_{[i,k]}^\circ[x_1, \dots, x_{i-2}])$ is admissible and is generated by the standard projective modules. Furthermore, using Lemma 4.1.5 we see that for each $j = 1, \dots, k$, the subcategory

$$\langle \Phi_j D^b(\mathcal{B}_{[j,k]}^\circ[x_1, \dots, x_{j-2}]), \dots, \Phi_2 D^b(\mathcal{B}_{[2,k]}^\circ), \Phi_1 D^b(\mathcal{B}_{[1,k]}^\circ) \rangle$$

is generated by the projective modules $P_{[m,l]}$ with $m \leq j$. This immediately implies that the image of $i_! \{j-1\}$ is right orthogonal to the above subcategory. Thus, we get the required semiorthogonalities.

Finally, the subcategory generated by the images of Φ_1, \dots, Φ_n is admissible (see [24, Lem. 1.1.1]) and its right orthogonal is contained in the right orthogonal of all projective modules P_I . Hence, this right orthogonal is zero, and we get the required semiorthogonal decomposition. \square

Remark 4.1.7. Noncommutative resolutions similar to \mathcal{B}° and $\mathcal{B}^{\circ\circ}$ were considered in [6] and in [13].

4.2. Localization on the B-side. We say that a collection of objects $(X_i)_{i \in I}$ in an abelian category \mathcal{C} *generates* \mathcal{C} if the minimal abelian subcategory of \mathcal{C} closed under extensions and containing all X_i is the entire \mathcal{C} .

We want to describe the restriction functors $r_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}}$ and $r_R^{\mathcal{B}^{\circ\circ}}$ (see (4.1) and (4.2)) as localizations with respect to some subcategories.

Theorem 4.2.1. (i) *We have a semiorthogonal decomposition*

$$D^b(\mathcal{B}^{\circ\circ}) = \langle \ker(r_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}}), D^b(\mathcal{B}^\circ) \rangle$$

and an equivalence of enhanced triangulated categories

$$D^b(\mathcal{B}^{\circ\circ}) / \ker(r_R^{\mathcal{B}^{\circ\circ}}) \simeq D^b(R),$$

where the embedding of $D^b(\mathcal{B}^\circ)$ into $D^b(\mathcal{B}^{\circ\circ})$ is given by the induction functor $i_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}}$.

(ii) *For every $1 \leq i \leq j < m \leq k$, the module $M\{[i, j], [j+1, m]\}$ belongs to the subcategory $\ker(r_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}})$. Furthermore, $\ker(r_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}})$ is generated by the modules $(M\{[i], [i+1, j]\})_{i < j}$ as a triangulated category.*

(iii) *For every pair of disjoint intervals $I, J \subset [1, k]$, such that $I \sqcup J$ is an interval, the modules $M\{I, J\}$ belong to the subcategory $\ker(r_R^{\mathcal{B}^{\circ\circ}})$. Furthermore, $\ker(r_R^{\mathcal{B}^{\circ\circ}})$ is generated by the modules $(M\{[i], [i+1, j]\}, M\{[j], [i, j-1]\})_{i < j}$ as a triangulated category.*

Proof. (i) The semiorthogonal decomposition follows immediately from adjunction of the induction and restriction functors $(i_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}}, r_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}})$ and from the fact that $r_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}} \circ i_{\mathcal{B}^\circ}^{\mathcal{B}^{\circ\circ}} = \text{Id}$ (see Lemma 4.1.1(i)).

The restriction functor between abelian categories

$$r_R^{\mathcal{B}^{\circ\circ}} : \mathcal{B}^\circ - \text{mod} \rightarrow R - \text{mod}$$

is exact. Furthermore, by Lemma 4.1.1(i) the induction functor $i_R^{\mathcal{B}^{\circ\circ}}$ provides a cosection, so the category $R - \text{mod}$ is a colocalization of $\mathcal{B}^{\circ\circ} - \text{mod}$, which gives an equivalence

$$R - \text{mod} \simeq \mathcal{B}^{\circ\circ} - \text{mod} / (\ker(r_R^{\mathcal{B}^{\circ\circ}}) \cap \mathcal{B}^{\circ\circ} - \text{mod}).$$

The similar equivalence with derived categories follows from the work [20].

(ii) The first assertion is equivalent to the surjectivity of the map

$$(P_{[i,m]})_{[1,r]} \rightarrow (P_{[j+1,m]})_{[1,r]}$$

induced by $x_{[i,j]}$. In other words, the claim is that the map

$$(x_{[1,r] \setminus [i,m]}) / (x_{[1,r]}) \xrightarrow{x_{[i,j]}} (x_{[1,r] \setminus [j+1,m]}) / (x_{[1,r]})$$

is surjective for every m . Indeed, if $j \geq r$ then the target is zero. Otherwise $j < r$ and the claim follows from the equality $x_{[1,r] \setminus [j+1,m]} = x_{[i,j]} x_{[1,r] \setminus [i,m]}$.

The kernel of $r_{\mathcal{B}^\circ}^{\mathcal{B}^\circ}$ is the right orthogonal to the image of $i_{\mathcal{B}^\circ}^{\mathcal{B}^\circ} = \Phi_1$. Thus, by Theorem 4.1.6, it is generated by the images of Φ_2, \dots, Φ_n . Since each algebra $\mathcal{B}_{[j,k]}^\circ[x_1, \dots, x_{j-2}]$ has finite global dimension, we deduce that the kernel of $r_{\mathcal{B}^\circ}^{\mathcal{B}^\circ}$ is generated by the images of projective modules under Φ_2, \dots, Φ_n . But $\Phi_{j,n}(P_{[j,m]}[x_1, \dots, x_{j-2}])$ is precisely $M\{[j-1], [j, m]\}$, by Lemma 4.1.5.

(iii) As in (ii), the claim amounts to the surjectivity of the map

$$(x_{[1,k] \setminus (I \sqcup J)}) \xrightarrow{x_I} (x_{[1,k] \setminus J})$$

which follows from the equality $x_{[1,k] \setminus J} = x_I x_{[1,k] \setminus (I \sqcup J)}$.

Let $\mathcal{C} \subset \mathcal{B}^\circ - \text{mod}$ denote the kernel of $r_{\mathcal{B}^\circ}^{\mathcal{B}^\circ}$, i.e., the full subcategory consisting of \mathcal{B}° -modules M such that $M_{[1,k]} = 0$. By part (ii), it is enough to prove that the modules $r_{\mathcal{B}^\circ}^{\mathcal{B}^\circ}(M\{[j], [i, j-1]\})$ generate \mathcal{C} as an abelian category. We will use the induction on k . In the case $k = 1$, we have $\mathcal{C} = 0$ so there is nothing to prove.

Let $\mathcal{D} \subset \mathcal{B}^\circ - \text{mod}$ denote the full subcategory of \mathcal{B}° -modules M such that $M_1 = 0$. Let us set $\overline{\mathcal{B}}^\circ = \mathcal{B}_{[2,k]}^\circ[x_1]$, $\overline{\mathcal{B}} = \mathcal{B}_{[2,k]}^\circ[x_1]$. Note that the restriction functor $r_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ} : \mathcal{B}^\circ - \text{mod} \rightarrow \overline{\mathcal{B}}^\circ - \text{mod}$ induces an equivalence

$$\Phi : \mathcal{D} \simeq \overline{\mathcal{B}}^\circ - \text{mod}.$$

Furthermore, as we have seen in Lemma 4.1.1(ii), we have $r_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ} i_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ}(M) \in \mathcal{D}$ and

$$\Phi(r_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ} i_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ}(M)) = r_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ}(M).$$

Let $\mathcal{C}' \subset \overline{\mathcal{B}}^\circ - \text{mod}$ denote the kernel of $r_{R[x_1]}^{\overline{\mathcal{B}}^\circ}$. By induction assumption, \mathcal{C}' is generated by the modules $r_{\overline{\mathcal{B}}^\circ}^{\overline{\mathcal{B}}^\circ}(\overline{M}\{[j], [i, j-1]\})$, where $2 \leq i < j \leq k$, and $\overline{M}\{[j], [i, j-1]\}$ are $\overline{\mathcal{B}}^\circ$ -modules defined similarly to $M\{[j], [i, j-1]\}$. We have $i_{\overline{\mathcal{B}}^\circ}^{\overline{\mathcal{B}}^\circ} \overline{M}\{[j], [i, j-1]\} = M\{[j], [i, j-1]\}$ for $2 \leq i < j \leq k$, so

$$\Phi^{-1}(r_{\overline{\mathcal{B}}^\circ}^{\overline{\mathcal{B}}^\circ}(\overline{M}\{[j], [i, j-1]\})) = r_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ} i_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ}(\overline{M}\{[j], [i, j-1]\}) = r_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ}(M\{[j], [i, j-1]\}).$$

Thus, the subcategory $\Phi^{-1}(\mathcal{C}') = \mathcal{C} \cap \mathcal{D}$ belongs to the subcategory generated by the modules $r_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ}(M\{[j], [i, j-1]\})$, with $2 \leq i < j \leq k$.

It remains to check that \mathcal{C} is generated by $\mathcal{C} \cap \mathcal{D}$ and by the modules $r_{\overline{\mathcal{B}}^\circ}^{\mathcal{B}^\circ}(M\{[j], [1, j-1]\})$. Note that $\mathcal{C} \cap \mathcal{D}$ is precisely the kernel of the functor

$$r : \mathcal{C} \rightarrow R/(x_1, x_2 \dots x_k) - \text{mod} : M \mapsto M_1.$$

Now for $2 \leq j \leq k$, let define the functor

$$i_*(j) : R/(x_1, x_j) \rightarrow \mathcal{C}$$

sending N to the \mathcal{B}° -module M such that

$$M_{[1,m]} = \begin{cases} N & m < j, \\ 0 & m \geq j, \end{cases}$$

with the maps $M_{1,m-1} \rightarrow M_{1,m}$ (resp., $M_{1,m} \rightarrow M_{1,m-1}$) being the identity maps (resp., multiplication by x_j) for $m < j$.

For $2 \leq j \leq k$, let $\mathcal{C}_j \subset \mathcal{C}$ denote the full subcategory of $M \in \mathcal{C}$ such that $x_j M_1 = 0$. Since for every $M \in \mathcal{C}$ one has $x_2 \dots x_k M_1 = 0$, it is easy to see that the subcategories \mathcal{C}_j generate \mathcal{C} as an abelian category. Now for every $M \in \mathcal{C}_j$, there is a natural morphism

$$i_*(j)(M_1) \rightarrow M$$

inducing the identity map $i_*(j)(M_1)_1 \rightarrow M_1$. Hence, the kernel and the cokernel of the morphism $i_*(j)(M_1) \rightarrow M$ are in $\mathcal{C} \cap \mathcal{D}$. It follows that the images of the functors $i_*(j)$ together with $\mathcal{C} \cap \mathcal{D}$ generate \mathcal{C} .

It remains to observe that for each $j \geq 2$, there is an exact sequence

$$0 \rightarrow i_*(j)(R/(x_1, x_j)) \rightarrow r_{\mathcal{B}^\circ}^{\mathcal{B}^\circ} M\{[j], [1, j-1]\} \rightarrow r_{\mathcal{B}^\circ}^{\mathcal{B}^\circ} M\{[j], [2, j-1]\} \rightarrow 0.$$

This implies that the image of $i_*(j)$ is contained in the subcategory generated by $r_{\mathcal{B}^\circ}^{\mathcal{B}^\circ} M\{[j], [1, j-1]\}$ and $r_{\mathcal{B}^\circ}^{\mathcal{B}^\circ} M\{[j], [2, j-1]\}$. \square

Corollary 4.2.2. *One has an equivalence of enhanced triangulated categories*

$$D^b(\mathcal{B}^\circ) / \ker(r_{\mathcal{B}^\circ}^{\mathcal{B}^\circ}) \simeq D^b(\mathcal{B}^\circ).$$

5. COMPARISON BETWEEN THE A-SIDE AND THE B-SIDE

Now we specialize to the case $n = k - 1$ on the A-side and the \mathbb{Z} -grading on the Fukaya category such that $\deg(x_i) = 0$ (see Theorem 3.2.5). The main observation is that in this case the algebra \mathcal{A}° on the A-side is isomorphic to the algebra $(\mathcal{B}^\circ)^{op}$.

First, we observe that there is the following bijection between the Lagrangians (L_S) in this case and the subsegments $I \subset [1, k]$:

$$L_{[0,k] \setminus \{i,j\}} \leftrightarrow [i+1, j], \tag{5.1}$$

where $0 \leq i < j \leq k$.

Lemma 5.0.1. *For $n = k - 1$ one has an isomorphism of R -algebras*

$$\mathcal{A}^\circ \simeq \mathcal{B}^\circ \simeq (\mathcal{B}^\circ)^{op} = \bigoplus_{I, J \subset [1, k]} \text{Hom}_{\mathcal{B}^\circ}(P_I, P_J),$$

compatible with the correspondence (5.1).

Proof. We have

$$\mathrm{Hom}_{\mathcal{B}^{\circ\circ}}(P_I, P_J) \simeq \mathrm{Hom}_R(R/(x_J), R/(x_I)) \simeq (x_{I \setminus J})/(x_I) = x_{I \setminus J} \cdot R/(x_{I \cap J}).$$

It is easy to see that mapping $\mathrm{Hom}_{\mathcal{B}^{\circ\circ}}(P_I, P_J)$ to $\mathrm{Hom}_{\mathcal{B}^{\circ\circ}}(P_J, P_I)$ using these identifications gives an isomorphism of the algebra $\mathcal{B}^{\circ\circ}$ with its opposite algebra (see below for the computation of compositions).

On the other hand, for $S = [0, k] \setminus \{a, b\}$ and $S' = [0, k] \setminus \{a', b'\}$ we have the following four possibilities, and we can calculate the subintervals I_{\bullet}, J_{\bullet} of Corollary 3.2.4 in each of them.

Case 1. $a < a', b < b'$. If $a' \geq b$ then S and S' are not close, so we can assume that $a' < b$. In this case we have $J_1 = [a, a']$, $J_2 = [b, b']$, so using Theorem 3.2.5(ii) we get

$$\begin{aligned} \mathcal{A}(S, S') &= \mathbf{k}[x_1, \dots, x_a] \otimes \mathbf{k}[x_{a+1}, \dots, x_{a'}] \otimes \mathbf{k}[x_{a'+1}, \dots, x_b]/(x_{[a'+1, b]}) \\ &\otimes \mathbf{k}[x_{b+1}, \dots, x_{b'}] \otimes \mathbf{k}[x_{b'+1}, \dots, x_k] \cdot f_{S, S'} = R/(x_{[a'+1, x_b]}) \cdot f_{S, S'}. \end{aligned}$$

Case 2. $a < a', b \geq b'$. In this case we have $J_1 = [a, a']$, $I_1 = [b', b]$ (I_1 should be omitted if $b = b'$), so by Theorem 3.2.5(ii) we get

$$\begin{aligned} \mathcal{A}(S, S') &= \mathbf{k}[x_1, \dots, x_a] \otimes \mathbf{k}[x_{a+1}, \dots, x_{a'}] \otimes \mathbf{k}[x_{a'+1}, \dots, x_{b'}]/(x_{[a'+1, b']}) \\ &\otimes \mathbf{k}[x_{b'+1}, \dots, x_b] \otimes \mathbf{k}[x_{b+1}, \dots, x_k] \cdot f_{S, S'} = R/(x_{[a'+1, x_{b'}]}) \cdot f_{S, S'}. \end{aligned}$$

Case 3. $a \geq a', b < b'$. In this case we have $I_1 = [a', a]$, $J_1 = [b, b']$ (I_1 should be omitted if $a = a'$), so by Theorem 3.2.5(ii), we get

$$\begin{aligned} \mathcal{A}(S, S') &= \mathbf{k}[x_1, \dots, x_{a'}] \otimes \mathbf{k}[x_{a'+1}, \dots, x_a] \otimes \mathbf{k}[x_{a+1}, \dots, x_b]/(x_{[a+1, b]}) \\ &\otimes \mathbf{k}[x_{b+1}, \dots, x_{b'}] \otimes \mathbf{k}[x_{b'+1}, \dots, x_k] \cdot f_{S, S'} = R/(x_{[a+1, x_b]}) \cdot f_{S, S'}. \end{aligned}$$

Case 4. $a \geq a', b \geq b'$. If $a \geq b'$ then S and S' are not close, so we can assume that $a < b'$. In this case we have $I_1 = [a', a]$, $I_2 = [b', b]$ (where I_1 is omitted if $a = a'$ and I_2 is omitted if $b = b'$). By Theorem 3.2.5(ii), we get

$$\begin{aligned} \mathcal{A}(S, S') &= \mathbf{k}[x_1, \dots, x_{a'}] \otimes \mathbf{k}[x_{a'+1}, \dots, x_a] \otimes \mathbf{k}[x_{a+1}, \dots, x_{b'}]/(x_{[a+1, b']}) \\ &\otimes \mathbf{k}[x_{b'+1}, \dots, x_b] \otimes \mathbf{k}[x_{b+1}, \dots, x_k] \cdot f_{S, S'} = R/(x_{[a+1, x_{b'}]}) \cdot f_{S, S'}. \end{aligned}$$

In all of these cases we deduce that

$$\mathcal{A}(S, S') \simeq R/(x_{I \cap J}) \cdot f_{S, S'},$$

where $I = [a+1, b]$, $J = [a'+1, b']$. So we get an identification of R -modules

$$\mathcal{A}(S, S') \simeq \mathrm{Hom}(P_I, P_J)$$

sending $f_{S, S'}$ to $x_{I \setminus J}$.

To check the compatibility with the composition, we note that for three intervals $I, J, K \subset [1, k]$ one has

$$x_{I \setminus J} \cdot x_{J \setminus K} = x_{I \setminus K} \cdot \prod_{i \in (J \setminus (I \cup K)) \cup (I \cap K \setminus J)} x_i.$$

Thus, the assertion follows from Theorem 3.2.5(iii) and the equality

$$\{i : |[i-1, i] \subset T(S, S', S'')\} = (J \setminus (I \cup K)) \cup (I \cap K \setminus J), \quad (5.2)$$

where L_S , $L_{S'}$ and $L_{S''}$ correspond to I , J and K under (5.1). The proof of (5.2) is a straightforward but tedious check, so we will consider only one of the cases. Let

$$\begin{aligned} S &= [0, k] \setminus \{a, b\}, & S' &= [0, k] \setminus \{a', b'\}, & S'' &= [0, k] \setminus \{a'', b''\}, \\ I &= [a + 1, b], & J &= [a' + 1, b'], & K &= [a'' + 1, b'']. \end{aligned}$$

Assume that $a < a'$, $b < b'$, $a'' < a'$, $b'' < b'$. Then we have $J_1 = [a, a']$, $J_2 = [b, b']$, $I'_1 = [a'', a']$, $I'_2 = [b'', b']$, hence,

$$T(S, S', S'') = [\max(a, a''), a'] \cup [\max(b, b''), b'].$$

On the other hand,

$$I \cap K \setminus J = [\max(a + 1, a'' + 1), a'], \quad J \setminus (I \cup K) = [\max(b, b'') + 1, b'],$$

so the equality (5.2) follows in this case. The other cases are considered similarly. \square

Theorem 5.0.2. *There are equivalences of enhanced triangulated categories over \mathbf{k} ,*

$$\mathcal{W}(M_{k-1,k}, \Lambda_Z) \simeq D^b(\mathcal{B}_{[1,k]}^{\circ\circ}), \quad (5.3)$$

$$\mathcal{W}(M_{k-1,k}, \Lambda_1) \simeq D^b(\mathcal{B}_{[1,k]}^{\circ}), \quad (5.4)$$

$$\mathcal{W}(M_{k-1,k}) \simeq D^b(R_{[1,k]}), \quad (5.5)$$

compatible with localization functors and the restriction functors $r_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ\circ}}$ and $r_{\mathcal{B}^{\circ}}^{\mathcal{B}^{\circ}}$. Furthermore, the first equivalence sends the Lagrangian $L_S = \prod_{a \in S} L_a$ for $S = [0, k] \setminus \{i, j\}$, where $i < j$, to the projective module $P_{[i+1, j]}$. Here the Fukaya categories are equipped with the \mathbb{Z} -grading coming from a unique grading structure on $M_{k-1, k}$ such that $\deg(x_i) = 0$ for $i = 1, \dots, k$ (see Theorem 3.2.5).

Proof. Note that by Theorem 3.2.5(iv), shifting the graded structures on each L_S if needed, we can achieve that the algebra $\mathcal{A}^{\circ\circ}$ is concentrated in degree 0. Since the Lagrangians (L_S) generate the wrapped Fukaya category, the first equivalence is an immediate consequence of the isomorphism of algebras proved in Lemma 5.0.1.

The second and third equivalences are obtained from the first by localization, using Theorem 3.3.1 on the A-side and Theorem 4.2.1 on the B-side. Namely, we observe that under the equivalence (5.3), the resolutions of objects $T_2 \times X$ (resp., of $T_1 \times X$) from Proposition 3.3.3 (in the case $n = k - 1$) correspond to the complexes defining modules $M\{[i, j], [j + 1, m]\}$ (resp., $M\{[j + 1, m], [i, j]\}$). Thus, under (5.3) the subcategory $\mathcal{D}_2 \subset \mathcal{W}(M_{k-1, k})$ (resp., $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$) corresponds to the subcategory $\ker(r_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ\circ}}) \subset D^b(\mathcal{B}^{\circ\circ})$ generated by all the modules $M\{[i, j], [j + 1, m]\}$ (resp., $\ker(r_{\mathcal{B}^{\circ\circ}}^{\mathcal{B}^{\circ\circ}})$ generated by all $M\{I, J\}$). Hence, using Theorem 3.3.1 and Corollary 4.2.2, we derive the equivalences (5.4) and (5.5). \square

Remark 5.0.3. It follows from Remark 4.1.3 that under equivalence (5.4) of Theorem 5.0.2, the object \overline{P}_i defined in Proposition 4.1.2 (involved in a semi-orthogonal decomposition of $D^b(\mathcal{B}_{[1, k]}^{\circ})$) corresponds to $L_S = \prod_{a \in S} L_a$ with $S = [0, k] \setminus \{i - 1, i\}$.

Remark 5.0.4. The semiorthogonal decomposition

$$D^b(\mathcal{B}^{\circ\circ}) = \langle \text{im}(\Phi_k), \dots, \text{im}(\Phi_2), \text{im}(\Phi_1) \rangle$$

constructed in Theorem 4.1.6 has a natural geometric meaning on the A-side. Namely, let $\mathcal{C}_i \subset \mathcal{W}(M_{k-1,k}, \Lambda_Z)$ be the subcategory corresponding to $\text{im}(\Phi_i)$. Then \mathcal{C}_1 is generated by the Lagrangians L_S with $S \subset [1, k]$, and for each $i = 2, \dots, k$, the subcategory \mathcal{C}_i is generated by the Lagrangians of the form $T_2 \times L_{[0,i-3] \sqcup S'}$ with $S' \subset [i, k]$, $|S'| = k - i$. The required semiorthogonalities can be easily seen on the A-side. Also, the natural functor

$$\mathcal{C}_1 \rightarrow \mathcal{W}(M_{k-1,k}, \Lambda_1)$$

is an equivalence, which corresponds to the equivalence of $\text{im}(\Phi_1)$ with $D^b(\mathcal{B}^{\circ})$. On the other hand, for each $i = 2, \dots, k$, there is an equivalence

$$\mathcal{C}'_i \xrightarrow{\sim} \mathcal{C}_i : L \mapsto T_2 \times L$$

from the subcategory $\mathcal{C}'_i \subset \mathcal{W}(M_{k-2,k}, \Lambda_Z)$ generated by all $L_{[0,i-3] \sqcup S'}$ with $S' \subset [i, k]$. Furthermore, since $\text{End}(L_{[0,i-3]}) \simeq k[x_1, \dots, x_{i-2}]$, one can see that there is an equivalence

$$\mathcal{C}''_i \otimes_k k[x_1, \dots, x_{i-2}] \xrightarrow{\sim} \mathcal{C}'_i : L \mapsto L_{[0,i-3]} \times L,$$

where $\mathcal{C}''_i \subset \mathcal{W}(M_{k-i,k}, \Lambda_Z)$ is generated by $L_{S'}$ with $S' \subset [i, k]$. Finally, we have a natural equivalence

$$\mathcal{C}''_i \xrightarrow{\sim} \mathcal{W}(M_{k-i,k-i+1}, \Lambda_1)$$

induced by filling in the first $i - 1$ interior holes and removing one of the stops.

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