HOMOLOGICAL MIRROR SYMMETRY FOR K3 SURFACES VIA MODULI OF A_∞-STRUCTURES

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Abstract. We show that several moduli spaces of lattice polarized K3 surfaces tied with Arnold's strange duality (and its generalization due to Berglund–Hübsch–Krawitz) arise as moduli spaces of A_∞-structures on particular finite-dimensional graded algebras. The same algebras also appear in the Fukaya category of the mirror dual family. Based on these identifications, we discuss applications to homological mirror symmetry for K3 surfaces, and give a proof of homological mirror symmetry for the affine quartic surface. Along the way, we also give a proof of a conjecture of Seidel from [63] which may be of independent interest.

Contents

1. Introduction

Our basic starting point is an algebraic variety with an isolated singularity admitting a $\mathbb{G}_\mathrm{m}$-action. The primordial example is the cusp singularity defined by

$$\left\{ (x, y) \in \mathbb{A}^2 \;\middle|\; y^2 = x^3 \right\}. \tag{1.1}$$

The main construction that we study in this paper originates from [42], where the case of the cusp singularity was studied in detail. We recall this construction in order to ease the reader to our topic before discussing higher-dimensional singularities with a $\mathbb{G}_{\mathrm{m}}$-action.

The cuspidal curve (1.1) has a $\mathbb{G}_{\mathrm{m}}$-action given by $t \cdot (x, y) = (t^2 x, t^3 y)$. Thus the coordinate ring gets a grading with $\deg(x) = 2, \deg(y) = 3$. It can be compactified to the projective cone

$$\left\{ [x : y : z] \in \mathbb{P}(2,3,1) \;\middle|\; y^2 = x^3 \right\} \tag{1.2}$$

by adding one point.

The semiuniversal unfolding $\mathcal{C} \to U \coloneqq \operatorname{Spec} \mathbf{k}[u_4, u_6]$ of the cusp singularity is given by

$$y^2 = x^3 + u_4 x + u_6. \tag{1.3}$$

The $\mathbb{G}_{\mathrm{m}}$-action extends to this family by giving (u_4, u_6) weights $(4, 6)$. By replacing u_4 and u_6 with $u_4 z^4$ and $u_6 z^6$, we consider the projectivized family in the weighted projective space $\mathbb{P}(2, 3, 1)$ given by the homogeneous equation

$$y^2 = x^3 + u_4 x z^4 + u_6 z^6. \tag{1.4}$$

This is the well-known Weierstrass family of cubic curves. Each curve C_u is of arithmetic genus 1, comes equipped with the 'point at infinity' cut out by $z = 0$, call it p, and a holomorphic nowhere-vanishing 1-form ω_u defined on the smooth locus by $dx/F_y = -dy/F_x$ for $F(x, y) = y^2 - x^3 - u_4 x - u_6$. The $\mathbb{G}_{\mathrm{m}}$-action extends to the compactified family, preserving the section $z = 0$, and satisfies

$$t^*(\omega_{t\cdot u}) = t^{-1} \omega_u. \tag{1.5}$$

The curves C_u are elliptic curves outside the discriminant

$$\Delta \coloneqq \left\{ (u_4, u_6) \in U \;\middle|\; 4u_4^3 - 27u_6^2 = 0 \right\}. \tag{1.6}$$

If $u \in \Delta \setminus \mathbf{0}$, then C_u is a rational curve with a single ordinary double point. Note that all curves above a $\mathbb{G}_{\mathrm{m}}$-orbit are isomorphic.

The base space U can be identified with the moduli space of triples (C, p, ω) consisting of a reduced connected curve C of arithmetic genus 1, a smooth marked point p on C such that $h^0(\mathcal{O}_C(p)) = 1$ and $\mathcal{O}_C(p)$ is ample, and a non-zero section ω of the dualizing sheaf of C (see [44, Theorem 1.4.2]). Furthermore, we have an isomorphism

$$\overline{\mathcal{M}}_{1,1} \cong [(U \setminus \mathbf{0})/\mathbb{G}_{\mathrm{m}}] \quad (\cong \mathbb{P}(4,6)) \tag{1.7}$$

with the moduli stack of stable curves of genus one with one marked point.

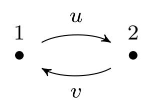
1.1. **Moduli of A_∞-structures.** The condition that $\mathcal{O}_{C_u}(p)$ is ample is equivalent to

$$\mathcal{S}_u \coloneqq \mathcal{O}_{C_u} \oplus \mathcal{O}_p \tag{1.8}$$

being a generator of the perfect derived category $\operatorname{perf} C_u$. On the other hand, the fact that $h^0(\mathcal{O}_C(p)) = 1$ implies that the isomorphism class of the endomorphism Yoneda algebra

$$A \coloneqq \operatorname{End}(\mathcal{S}_u) \tag{1.9}$$

as a graded algebra is independent of $u \in U$. Indeed, it is easy to show that for any u, there is a canonical isomorphism (where we use $H^1(\mathcal{O}_{C_u}) = \mathbf{k} \cdot \omega_u$) between A and the degree one trivial extension algebra of the A_2-quiver. More concretely, this is given by the quiver with relations given in Figure 1.1.



$|u| = 0 \quad |v| = 1 \quad uvu = vuv = 0$

Figure 1.1. Quiver algebra description of A

Thus, considering the algebra A results in a dramatic loss of information hidden in perf $\mathcal{C}_u$, even though $\mathcal{S}_u$ is a generator. This is, of course, no surprise as we have forgotten to derive.

Recall that an A_∞-algebra $\mathcal{A}$ over $\mathbf{k}$ is a graded $\mathbf{k}$-module with a collection $\left(\mu^d\right)_{d=1}^{\infty}$ of $\mathbf{k}$-linear maps $\mu^d \colon \mathcal{A}^{\otimes d} \to \mathcal{A}[2-d]$ satisfying the A_∞-associativity equations

$$\sum_{m,n} (-1)^{|a_1|+\ldots+|a_n|-n} \mu^{d-m+1}(a_d, \ldots, a_{n+m+1}, \mu^m(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots a_1) = 0. \tag{1.10}$$

In particular, $\mu^1 \colon \mathcal{A} \to \mathcal{A}[1]$ is a differential, i.e. $\mu^1 \circ \mu^1 = 0$, and the product

$$a_2 \cdot a_1 = (-1)^{|a_1|} \mu^2(a_2, a_1) \tag{1.11}$$

on $\mathcal{A}$ is associative up to homotopy.

A *minimal A_∞-structure* on a graded associative $\mathbf{k}$-algebra A is an A_∞-structure $(\mu^k)_{k=1}^{\infty}$ on the graded vector space underlying A such that $\mu^1 = 0$ and μ^2 coincides with the given product on A. It is said to be *formal* if $\mu^k = 0$ for $k > 2$.

Recall that the Hochschild cochain complex of a graded algebra A has a bigrading, where $\mathrm{CC}^{r+s}(A)_s$ consists of maps $A^{\otimes r} \to A[s]$. The space of first-order deformations of A as a graded algebra is given by $\mathrm{HH}^2(A)_0$, and deformations to minimal A_∞-structures on A without changing μ^2 is controlled by $\mathrm{HH}^2(A)_{<0} := \bigoplus_{i=1}^{\infty} \mathrm{HH}^2(A)_{-i}$. Moreover, if $\mathrm{HH}^1(A)_{<0}$ vanishes, then [55] shows that the functor sending a $\mathbf{k}$-algebra R to the set of gauge equivalence classes of minimal A_∞-structures on $A \otimes R$ is represented by an affine scheme $\mathcal{U}_\infty(A)$, which is of finite type if $\dim \mathrm{HH}^2(A)_{<0} < \infty$. There is a natural $\mathbb{G}_{\mathrm{m}}$-action on $\mathcal{U}_\infty(A)$ sending $\left(\mu^d\right)_{d=2}^{\infty}$ to $\left(t^{d-2}\mu^d\right)_{d=2}^{\infty}$, and the formal A_∞ structure on A is a fixed point of this action.

Returning back to the Weierstrass family, as explained in [43], the natural dg enhancement $\mathrm{end}(\mathcal{S}_u)$ of $\mathrm{End}(\mathcal{S}_u)$ gives a family of minimal A_∞-structures $\mathcal{A}_u$ on A over U, and hence a morphism

$$U \to \mathcal{U}_\infty(A) \tag{1.12}$$

of affine varieties. We recall the following theorem from [43]. For simplicity, we state it over a field $\mathbf{k}$ with $\mathrm{char}\, \mathbf{k} \neq 2, 3$, see [43] for more general statement.

Theorem 1.1. *If* $\operatorname{char} \mathbf{k} \neq 2, 3$, *then* (1.12) *is a* $\mathbb{G}_{\mathrm{m}}$*-equivariant isomorphism, sending the cuspidal curve* C_0 *to the formal* A_∞*-structure on* A.

There are two main ingredients that enter in the proof of this result:

(i) The formality of the A_∞-algebra $\mathcal{A}_0$ for the cuspidal curve C_0.
(ii) One has $\mathrm{HH}^1(A)_{<0} = 0$, so that $\mathcal{U}_\infty(A)$ can be defined, and
$$\mathrm{HH}^2(A)_{<0} = \mathbf{k}(4) \oplus \mathbf{k}(6), \tag{1.13}$$
so that the tangent spaces of the two moduli spaces agree at the fixed point of the $\mathbb{G}_{\mathrm{m}}$ action.

The Hochschild cohomology computation is done in two different ways in [42] and [43]. We will give yet another way in Section 3.4 below.

To elaborate on (i), first one shows the existence of a chain level $\mathbb{G}_{\mathrm{m}}$-action by taking the Čech complex with respect to a $\mathbb{G}_{\mathrm{m}}$-invariant affine cover. This gives a dg model for $\mathcal{A}_0$. Then, one arranges a $\mathbb{G}_{\mathrm{m}}$-equivariant homotopy to a minimal A_∞-structure, which follows from the fact that one can choose chain level representatives of a basis of $\operatorname{End}(\mathcal{S}_0)$ in such a way that each of them is in a one-dimensional representation of $\mathbb{G}_{\mathrm{m}}$. Finally, to deduce formality, one shows that the weight of the $\mathbb{G}_{\mathrm{m}}$-action on $\operatorname{End}(\mathcal{S}_0)$ agrees with the cohomological grading. But μ^d lowers the cohomological degree by $d-2$, so any $\mathbb{G}_{\mathrm{m}}$-equivariant A_∞-structure must have vanishing μ^d for $d \neq 2$.

Other examples of the above construction were subsequently studied in [55, 44], but all of these work with examples in dimension one. In this paper, we begin to explore higher dimensions.

1.2. **Application to homological mirror symmetry.** Let T_0 be a once-punctured torus viewed as an open symplectic manifold, and $\mathcal{T}_0$ be a rational curve with a single ordinary double point defined by
$$\left\{ [x : y : z] \in \mathbb{P}(2,3,1) \;\middle|\; y^2 + x^3 + xyz = 0 \right\}. \tag{1.14}$$
Theorem 1.1 was obtained in [43] as a tool for proving homological mirror symmetry of T_0 and $\mathcal{T}_0$. Indeed, homological mirror symmetry
$$\mathcal{F}(T_0) \simeq \operatorname{perf}(\mathcal{T}_0) \tag{1.15}$$
proved in [43] gives a quasi-equivalence of pretriangulated A_∞-categories over $\mathbb{Z}$ of the split-closed derived Fukaya category of compact exact Lagrangians in T_0 and the perfect derived category of $\mathcal{T}_0$. The strategy is first to identify generators on both sides, and then match their endomorphism algebras as A_∞-algebras. It is often difficult to explicitly compute such A_∞-algebras, but even if one does, finding a quasi-isomorphism between two different chain models is usually a hard task. The computation of cohomology level structures (and matching them) is much easier, and knowing the moduli of A_∞-structures allows one to appeal to indirect methods to conclude the proof of the existence of a

chain level isomorphism. Such a strategy was applied also for proving homological mirror symmetry in a number of other cases in dimension one. Namely, in [44] a class of curve singularities $C_{1,n}$ for $n \geq 1$ were considered, where $C_{1,1}$ is the cuspidal curve, $C_{1,2}$ is tacnodal curve given by the equation $y^2 = yx^2$, and $C_{1,n}$ is the elliptic n-fold singularity given by n lines in $\mathbb{A}^{n-1}$. These are all the Gorenstein singularities of arithmetic genus one [69, Appendix A]. Carrying out the above strategy has led to a proof of homological mirror symmetry for n-punctured tori [45].

The equivalence (1.15) is an instance of homological mirror symmetry at the large volume limit. The equivalence is known to extend to a formal neighborhood of this limit to give an equivalence

$$\mathcal{F}(T^2) \simeq \operatorname{perf}(\mathcal{T}) \tag{1.16}$$

over $\mathbb{Z}\llbracket q \rrbracket$ where $\mathcal{T}$ is the Tate elliptic curve, a formal neighborhood of the nodal curve $\mathcal{T}_0$ (see [43] for a proof). A general strategy for proving homological mirror symmetry as in (1.16) is to view the categories in (1.16) as deformations of the categories given in (1.15). Hence, in this context deducing homological mirror symmetry for the compact T^2 from homological mirror symmetry for the T_0 is ultimately a question of deformation theory.

1.3. **New results.** In this paper, we lay out a programme that aims to extend the above results to higher dimensions, leading to new homological mirror symmetry conjectures for higher-dimensional Calabi–Yau manifolds at the large volume limit and in its formal neighborhood. It is based on the relation between homological mirror symmetry for Calabi–Yau manifolds and homological mirror symmetry for singularities, which goes back to [39, 50, 75]. We refer the reader to Section 6 for more general conjectures in arbitrary dimension, and we shall only state our results in dimension two here.

Let $\mathbf{w} \in \mathbb{C}[x, y, z]$ be a weighted homogeneous polynomial defined by a 3-by-3 matrix $(a_{ij})_{i,j=1}^3$ with non-negative integer entries and non-zero determinant as

$$\mathbf{w}(x, y, z) = x^{a_{11}} y^{a_{12}} z^{a_{13}} + x^{a_{21}} y^{a_{22}} z^{a_{23}} + x^{a_{31}} y^{a_{32}} z^{a_{33}}. \tag{1.17}$$

We assume that $\mathbf{w}$ has an isolated critical point at the origin. The non-vanishing of the determinant implies that there is a sequence $(d_1, d_2, d_3; h)$ of positive integers, determined uniquely by the condition that $\gcd(d_1, d_2, d_3) = 1$, such that

$$\mathbf{w}\left(t^{d_1} x, t^{d_2} y, t^{d_3} z\right) = t^h \mathbf{w}(x, y, z). \tag{1.18}$$

This sequence is called the *weight system* associated with $\mathbf{w}$. We assume that $d_0 := h - d_1 - d_2 - d_3 > 0$. The *transpose* of $\mathbf{w}$ is defined in [10] as

$$\check{\mathbf{w}}(x, y, z) = x^{a_{11}} y^{a_{21}} z^{a_{31}} + x^{a_{12}} y^{a_{22}} z^{a_{32}} + x^{a_{13}} y^{a_{23}} z^{a_{33}}. \tag{1.19}$$

Let $\left(\check{d}_1, \check{d}_2, \check{d}_3; \check{h}\right)$ be the weight system associated with $\check{\mathbf{w}}$. We assume that $\check{d}_0 := \check{h} - \check{d}_1 - \check{d}_2 - \check{d}_3 = 1$. We will be interested in the symplectic topology of the Milnor fiber of $\check{\mathbf{w}}$ defined by

$$\check{V} := \left\{(x, y, z) \in \mathbb{C}^3 \;\middle|\; \check{\mathbf{w}}(x, y, z) = 1\right\}. \tag{1.20}$$

Introduce a weighted projective hypersurface by

$$Z \coloneqq \{[w:x:y:z] \in \mathbb{P}(d_0,d_1,d_2,d_3) \mid \mathbf{w}(x,y,z) + xyzw = 0\}. \tag{1.21}$$

In general, instead of Z, one considers the quotient stack $[Z/K]$ where $K \subset \mathrm{SL}_4(\mathbb{C})$ is a finite group of diagonal symmetries of $\mathbf{w}$. See Section 6 for details on K. The main conjecture that we introduce in this setting is the following:

Conjecture 1.2. There is a quasi-equivalence

$$\mathcal{F}\left(\check{V}\right) \simeq \operatorname{perf}[Z/K] \tag{1.22}$$

of pretriangulated A_∞-categories.

Now we turn to specific examples that we would like to highlight here for the sake of concreteness. Our first result is the analogue of Theorem 1.1 in dimension two. Let $\mathbf{w}$ be one of weighted homogeneous polynomials in Table 3.1 defining Arnold's 14 exceptional unimodal singularities, and U_+ be the positive part of the base space of the semiuniversal unfolding of $\mathbf{w}$. We obtain a family $\pi\colon \mathcal{Y} \to U_+$ of stacks such that Z is isomorphic to $Y_u \coloneqq \pi^{-1}(u)$ for some non-zero $u \in U_+$, which is the point corresponding to the large complex structure limit. Using [34, 46], we construct a generator $\mathcal{S}_u$ of $\operatorname{perf} Y_u$ for each $u \in U_+$ such that the cohomology of the endomorphism dg algebra $\mathcal{A}_u \coloneqq \operatorname{end} \mathcal{S}_u$ is isomorphic to the degree 2 trivial extension algebra A of the path algebra $A^{\to}$ of the quiver with relations in Figure 1.2, where the lengths of the three paths from the second vertex from the left to the rightmost vertex are given by the Dolgachev numbers δ_1, δ_2 and δ_3 of the singularity, with the relation that the sum of these three paths are zero. Recall that the *degree n trivial extension algebra* (also known as the *Frobenius completion of degree n*) of a finite-dimensional $\mathbf{k}$-algebra $A^{\to}$ has $A^{\to} \oplus \operatorname{Hom}_{\mathbf{k}}(A^{\to}, \mathbf{k})[-n]$ as the underlying vector space, and the multiplication is given by

$$(a,f)\cdot(b,g) = (ab, ag + fb). \tag{1.23}$$

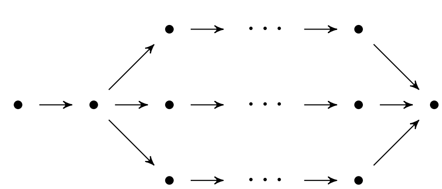


FIGURE 1.2. The quiver associated with exceptional unimodal singularities

Theorem 1.3. *Let $\mathbf{w}$ be a weighted homogeneous polynomial defining an exceptional unimodal singularity, and A be the finite-dimensional associative graded algebra defined above. Then there is a $\mathbb{G}_\mathrm{m}$-equivariant isomorphism $U_+ \xrightarrow{\sim} \mathcal{U}_\infty(A)$ of affine varieties sending the origin $0 \in U_+$ to the formal A_∞-structure on A.*

If $\mathbf{w}$ is a Sebastiani–Thom sum of polynomials of type A or D, i.e., a decoupled sum of polynomials of the form x^{n+1} or $x^2y + y^{n-1}$, then we have an alternative choice of a generator $\mathcal{S}_u$ of $\operatorname{perf} Y_u$, and an alternative algebra A such that Theorem 1.3 still holds. The singularities $Q_{10}, Q_{12}, W_{12}, E_{12}, E_{14}, U_{12}$ from Table 3.1 are of this type, as well as $x^4 + y^4 + z^4$, $x^2 + y^6 + z^6$, and many more. The algebra A is the trivial extension of the tensor product of the path algebras of the Dynkin quivers. As examples the quivers for the E_{12}-singularity and $x^4 + y^4 + z^4$ are shown in Figures 1.3 and 1.4, with the relations that the composition of arrows along the sides of each small square commutes. Homological mirror symmetry for singularities [26, 27] gives a collection $(S_i)_{i=1}^{\mu}$ of Lagrangian spheres in $\check{V}$ such that the cohomology of the total morphism A_∞-algebra $\mathcal{A} \coloneqq \bigoplus_{i,j=1}^{\mu} \hom(S_i, S_j)$ in the compact Fukaya category $\mathcal{F}(\check{V})$ is isomorphic to A.

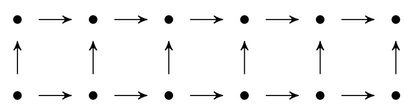


FIGURE 1.3. An alternative quiver for the E_{12}-singularity

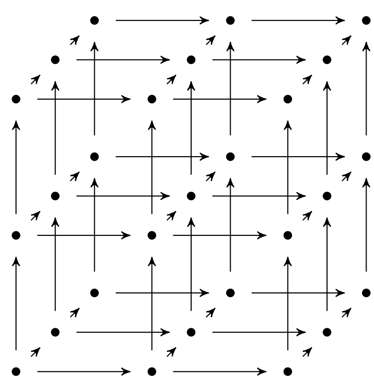


FIGURE 1.4. A quiver for $\mathbf{w} = x^4 + y^4 + z^4$

By combining the proof of a special case of [63, Conjecture 4] which states, under assumptions satisfied for $\check{V}$, an isomorphism

$$\mathrm{SH}^*\left(\check{V}\right) \simeq \mathrm{HH}^*\left(\mathcal{F}\left(\check{V}\right)\right) \tag{1.24}$$

of the symplectic cohomology and the Hochschild cohomology of the compact Fukaya category, with the computation of the symplectic cohomology $\mathrm{SH}^*\left(\check{V}\right)$ using a spectral sequence due to McLean [51], we show that the A_∞-algebra $\mathcal{A}$ is not formal. Hence $\mathcal{A}$ can be identified with a point in the moduli space

$$\mathcal{M}_\infty(A) \coloneqq [(\mathcal{U}_\infty(A) \setminus \mathbf{0})/\mathbb{G}_{\mathrm{m}}] \tag{1.25}$$

of non-formal A_∞-structures. Conjecture 1.2 identifies exactly which point this is, and in order to prove it, one has to distinguish points on $\mathcal{M}_\infty(A)$ by computable invariants of $\mathcal{F}\left(\check{V}\right)$. For 16 examples in dimension 2 that we discuss in this paper, $\mathcal{M}_\infty(A)$ is birational to the coarse moduli space of lattice polarized K3 surfaces. For $\mathbf{w} = x^4 + y^4 + z^4$ and $\mathbf{w} = x^2 + y^6 + z^6$, this space is one-dimensional, and we can prove Conjecture 1.2 in this case:

Theorem 1.4. *(i) Let* $\check{V} := \{(x, y, z) \in \mathbb{C}^3 \mid x^4 + y^4 + z^4 = 1\}$ *be an affine quartic surface considered as an exact symplectic manifold, and* $K := \{[\operatorname{diag}(t_1, t_2, t_3, t_4)] \in \mathrm{PGL}_4(\mathbb{C}) \mid t_1^4 = t_2^4 = t_3^4 = t_1 t_2 t_3 t_4 = 1\}$ *be a finite group acting on* $Z := \{[w, x, y, z] \in \mathbb{P}^3 \mid x^4 + y^4 + z^4 + xyzw = 0\}$. *Then we have a quasi-equivalence*

$$\mathcal{F}\left(\check{V}\right) \simeq \operatorname{perf}\left[Z/K\right] \tag{1.26}$$

of pretriangulated A_∞*-categories over* $\mathbb{C}$.

(ii) Let $\check{V} := \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^6 + z^6 = 1\}$ *be an affine surface considered as an exact symplectic manifold, and* $K := \{[\operatorname{diag}(t_1, t_2, t_3, t_4)] \in \operatorname{Aut}\mathbb{P}(3, 1, 1, 1) \mid t_1^2 = t_2^6 = t_3^6 = t_1 t_2 t_3 t_4 = 1\}$ *be a finite group acting on the weighted projective hypersurface* $Z := \{[x : y : z : w] \in \mathbb{P}(3, 1, 1, 1) \mid x^2 + y^6 + z^6 + xyzw = 0\}$. *Then we have a quasi-equivalence*

$$\mathcal{F}\left(\check{V}\right) \simeq \operatorname{perf}\left[Z/K\right]. \tag{1.27}$$

of pretriangulated A_∞*-categories over* $\mathbb{C}$.

The large complex structure limits in Theorem 1.4 are different from those appearing in [59] and its generalizations [65, 67]. In his construction, Seidel removes the divisor $\{xyz = 0\}$ from the Milnor fiber $\check{V}$ on the A-side and considers the reducible singular variety $\{xyzw = 0\}$ instead of Z on the B-side.

This paper is organized as follows: In Section 2, we set up basic notations for weighted homogeneous polynomials and their semiuniversal unfoldings. In Section 3, we compute Hochschild cohomologies of (not necessarily smooth) proper algebraic stacks associated with weighted homogeneous polynomials using matrix factorizations. In Section 4, we give a generator $\mathcal{S}_u$ of $\operatorname{perf} Y_u$, and prove the formality of $\operatorname{end} \mathcal{S}_0$. We prove Theorem 1.3 in Section 5. In Section 6, we state old and new conjectures in homological mirror symmetry that are relevant to our set-up. In Section 7, we prove that $\mathrm{HH}^*(\mathcal{F}(\check{V}))$ is isomorphic to the symplectic cohomology of $\check{V}$. In Section 8, we give computations of symplectic cohomology of $\check{V}$ and deduce the non-formality result in $\mathcal{F}(\check{V})$. Theorem 1.4 is proved in Section 9.

Through the rest of the paper, we will work over an algebraically closed field $\mathbf{k}$ of characteristic 0. The bounded derived category of coherent sheaves, its full subcategory consisting of perfect complexes, and the unbounded derived category of quasi-coherent sheaves on an algebraic stack X, considered as pretriangulated dg categories, will be denoted by $\operatorname{coh} X$, $\operatorname{perf} X$, and $\operatorname{Qcoh} X$ respectively. All Fukaya categories are completed with respect to cones and direct summands.

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2. Weighted homogeneous singularities

A *weight system* is a sequence $(d_1, \ldots, d_n; h)$ of positive integers satisfying

$$h > \max\{d_1, \ldots, d_n\}. \tag{2.1}$$

We will always assume

$$\gcd(d_1, \ldots, d_n, h) = 1 \tag{2.2}$$

and

$$d_0 := h - d_1 - \cdots - d_n > 0 \tag{2.3}$$

in this paper. Let $\mathbf{w}(x_1, \ldots, x_n) \in \mathbf{k}[x_1, \ldots, x_n]$ be a polynomial in n variables, which is weighted homogeneous of weight $(d_1, \ldots, d_n; h)$;

$$\mathbf{w}\left(t^{d_1}x_1, \ldots, t^{d_n}x_n\right) = t^h \mathbf{w}(x_1, \ldots, x_n), \quad t \in \mathbb{G}_\mathrm{m}. \tag{2.4}$$

It is written as the sum of monomials

$$\mathbf{w}(x_1, \ldots, x_n) = \sum_{\boldsymbol{i}=(i_1,\ldots,i_n)\in I_\mathbf{w}} c_{\boldsymbol{i}} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}, \quad c_{\boldsymbol{i}} \in \mathbb{G}_\mathrm{m}, \tag{2.5}$$

where the index set $I_\mathbf{w}$ is a subset of the set of non-negative integers satisfying

$$d_1 i_1 + d_2 i_2 + \ldots + d_n i_n = h. \tag{2.6}$$

Let $\Gamma_\mathbf{w}$ be the commutative algebraic group defined by

$$\Gamma_\mathbf{w} := \left\{(t_1, \ldots, t_{n+1}) \in \mathbb{G}_\mathrm{m}^{n+1} \;\middle|\; t_1^{i_1} t_2^{i_2} \ldots t_n^{i_n} = t_{n+1} \text{ for all } (i_1, \ldots, i_n) \in I_\mathbf{w}\right\}. \tag{2.7}$$

The group $\widehat{\Gamma}_\mathbf{w} := \operatorname{Hom}(\Gamma_\mathbf{w}, \mathbb{G}_\mathrm{m})$ of characters of $\Gamma_\mathbf{w}$ is written as

$$\widehat{\Gamma}_\mathbf{w} = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_{n+1} \big/ (i_1\chi_1 + \cdots + i_n\chi_n - \chi_{n+1})_{\boldsymbol{i}\in I_\mathbf{w}}, \tag{2.8}$$

where $\chi_i \in \widehat{\Gamma}_\mathbf{w}$ for $1 \le i \le n+1$ is defined by $(t_1, \ldots, t_{n+1}) \mapsto t_i$. Since the composition $\Gamma_\mathbf{w} \hookrightarrow \mathbb{G}_\mathrm{m}^n \times \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}^n$ with the first projection is injective, we will think of $\Gamma_\mathbf{w}$ as a subgroup of $\mathbb{G}_\mathrm{m}^n$, and set $\chi_\mathbf{w} := \chi_{n+1}$. The group $\Gamma_\mathbf{w}$ consists of diagonal transformations of $\mathbb{A}^n$ which keeps $\mathbf{w}$ semi-invariant;

$$\mathbf{w}(t \cdot (x_1, \ldots x_n)) = \chi_\mathbf{w}(t)\mathbf{w}(x_1, \ldots, x_n), \quad t \in \Gamma_\mathbf{w}. \tag{2.9}$$

The injective homomorphism

$$\phi\colon \mathbb{G}_m \to \Gamma_\mathbf{w}, \quad t \mapsto \left(t^{d_1}, \ldots, t^{d_n}\right) \tag{2.10}$$

fits into the exact sequence

$$1 \to \mathbb{G}_\mathrm{m} \xrightarrow{\phi} \Gamma_\mathbf{w} \to \ker\chi_\mathbf{w}/\langle j_\mathbf{w}\rangle \to 1, \tag{2.11}$$

where $j_{\mathbf{w}} := \left(e^{2\pi\sqrt{-1}q_1}, \dots, e^{2\pi\sqrt{-1}q_n}\right)$ is the *grading element* generating the cyclic group $\ker \chi_{\mathbf{w}} \cap \phi(\mathbb{G}_{\mathrm{m}})$ of order h.

Let Γ be a subgroup of $\Gamma_{\mathbf{w}}$ containing $\phi(\mathbb{G}_m)$ as a subgroup of finite index. For such Γ, the kernel of $\chi := \chi_{\mathbf{w}}|_{\Gamma}$ is a finite group, and such subgroups Γ are in bijection with finite subgroups of $\ker \chi_{\mathbf{w}}$ containing the grading element $j_{\mathbf{w}}$.

We set $\overline{R}_0 := \mathbf{k}[x_1, \dots, x_n]/(\mathbf{w})$, where the subscript "0" is placed in anticipation of smoothings that we will study later on. The group Γ acts naturally on $\operatorname{Spec} \overline{R}_0$, and we write the quotient stack of the complement of the origin $\mathbf{0}$ as

$$X := \left[\left(\operatorname{Spec} \overline{R}_0 \setminus \mathbf{0}\right)/\Gamma\right]. \tag{2.12}$$

We extend the Γ-action on $\mathbb{A}^n$ to $\mathbb{A}^{n+1} := \operatorname{Spec} \mathbf{k}[x_0, \dots, x_n]$ by

$$(t_1, \dots, t_n) \cdot (x_0, x_1, \dots, x_n) = (\chi_0 x_0, t_1 x_1, \dots, t_n x_n) \tag{2.13}$$

where

$$\chi_0(t_1, \dots, t_n) := \chi(t_1, \dots, t_n) t_1^{-1} \cdots t_n^{-1}. \tag{2.14}$$

By abuse of notation, we write the pull-back of $\mathbf{w}$ to $\mathbb{A}^{n+1}$ by the same symbol, and set $R_0 := \mathbf{k}[x_0, \dots, x_n]/(\mathbf{w})$. The condition (2.3) ensures that $[(\mathbb{A}^{n+1} \setminus \mathbf{0})/\Gamma]$ is proper, and hence so is its closed substack

$$Y_0 := [(\operatorname{Spec} R_0 \setminus \mathbf{0})/\Gamma]. \tag{2.15}$$

It is a projective cone over X, which is obtained from $V_0 := \left[\operatorname{Spec} \overline{R}_0 / \ker \chi_0\right]$ by adding X at infinity. The weight of the Γ-action on the x_0 variable in (2.13) is chosen so that the dualizing sheaf of Y_0 is trivial.

Assume that $\mathbf{w} \colon \mathbb{A}^n \to \mathbb{A}$ has an isolated critical point at the origin. This is the case if and only if the Jacobi algebra

$$\operatorname{Jac}_{\mathbf{w}} := \mathbf{k}[x_1, \dots, x_n]/(\partial_1 \mathbf{w}, \dots, \partial_n \mathbf{w}) \tag{2.16}$$

is finite-dimensional. The dimension μ of $\operatorname{Jac}_{\mathbf{w}}$ is called the *Milnor number* of $\mathbf{w}$. Let $J_{\mathbf{w}}$ be the set of exponents of monomials representing a basis of $\operatorname{Jac}_{\mathbf{w}}$, and

$$\widetilde{\mathbf{w}} := \mathbf{w}(x_1, \dots, x_n) + \sum_{\boldsymbol{j}=(j_1,\dots,j_n) \in J_{\mathbf{w}}} u_{\boldsymbol{j}} x_1^{j_1} \dots x_n^{j_n} \colon \mathbb{A}^n \times U \to \mathbb{A}^1 \tag{2.17}$$

be a semiuniversal unfolding of $\mathbf{w}$. The base space $U := \operatorname{Spec} \mathbf{k}[u_1, \dots, u_\mu]$ is an affine space of dimension μ. Let $\mathbf{w}_u$ be the restriction of $\widetilde{\mathbf{w}}$ to $\mathbb{A}^n \times \{u\}$ for $u \in U$ and set $\overline{R}_u := \mathbf{k}[x_1, \dots, x_n]/(\mathbf{w}_u)$. We consider the affine subspace U_+ of U where $u_{\boldsymbol{j}}$ can be non-zero only if there exists a positive integer $w_{\boldsymbol{j}}$ satisfying

$$\chi^{w_{\boldsymbol{j}}-1} = t_1^{w_{\boldsymbol{j}}-j_1} t_2^{w_{\boldsymbol{j}}-j_2} \dots t_n^{w_{\boldsymbol{j}}-j_n}. \tag{2.18}$$

Let J_+ be the set of $\boldsymbol{j} \in J_{\mathbf{w}}$ satisfying this condition. Then we have the family

$$\pi_{\mathcal{Y}} \colon \mathcal{Y} := \left[\left(\mathbf{W}_+^{-1}(0) \setminus (\mathbf{0} \times U_+)\right)/\Gamma\right] \to U_+ \tag{2.19}$$

of stacks over U_+ defined by

$$\mathbf{W}_+ \coloneqq \mathbf{w}(x_1, \ldots, x_n) + \sum_{\boldsymbol{j} \in J_+} u_{\boldsymbol{j}} x_0^{w_{\boldsymbol{j}}} x_1^{j_1} \ldots x_n^{j_n} \colon \mathbb{A}^{n+1} \times U_+ \to \mathbb{A}^1, \tag{2.20}$$

whose fiber $Y_u \coloneqq \pi^{-1}(u)$ over $u \in U_+$ is a compactification of $V_u \coloneqq \left[\left(\operatorname{Spec} \overline{R}_u \setminus \mathbf{0} \right) / \ker \chi_0 \right]$. The divisor $Y_u \setminus V_u$ at infinity of $Y_u \coloneqq \pi_{\mathcal{Y}}^{-1}(u)$ is isomorphic to X for all $u \in U_+$. The relative dualizing sheaf $\omega_{\mathcal{Y}/U_+}$ is identified with $\omega_{(\mathbf{W}_+^{-1}(0) \setminus (\mathbf{0} \times U_+))/U_+}$ considered as a Γ-equivariant coherent sheaf, which in turn is isomorphic to the restriction of $\omega_{(\mathbb{A}^{n+1} \times U_+)/U_+} \otimes \chi$ to $\mathbf{W}_+^{-1}(0) \setminus (\mathbf{0} \times U_+)$ since $\mathbf{W}_+$ is a section of $\mathcal{O}_{\mathbb{A}^{n+1} \times U_+}$ of degree χ. This sheaf is Γ-equivariantly trivial, and we fix its trivialization, which is unique up to scaling.

Example 2.1 (tacnode)**.** When $n = 2$, $(d_1, d_2; h) = (2, 1; 4)$, and $\mathbf{w} = x^2 + y^4$, one has

$$\Gamma_{\mathbf{w}} \coloneqq \left\{ (t_1, t_2) \in \mathbb{G}_{\mathrm{m}}^2 \;\middle|\; t_1^2 = t_2^4 \right\} \xrightarrow{\sim} \mathbb{G}_{\mathrm{m}} \times \boldsymbol{\mu}_2, \quad (t_1, t_2) \mapsto (t_2, t_1 t_2^{-2}). \tag{2.21}$$

The image of the injective homomorphism

$$\phi \colon \mathbb{G}_{\mathrm{m}} \to \Gamma_{\mathbf{w}}, \quad t \mapsto (t^2, t) \tag{2.22}$$

is an index 2 subgroup isomorphic to $\mathbb{G}_{\mathrm{m}}$, so that there are two choices of Γ. By construction, we have the semi-invariance property

$$\mathbf{w}(t_1 x, t_2 y) = \chi(t_1, t_2) \mathbf{w}(x, y), \tag{2.23}$$

where $\chi \colon \Gamma \to \mathbb{G}_{\mathrm{m}}$ is the character sending (t_1, t_2) to $t_1^2 = t_2^4$. A semiuniversal unfolding of $\mathbf{w}$ is given by

$$\widetilde{\mathbf{w}}(x, y; u_2, u_3, u_4) = x^2 + y^4 + u_2 y^2 + u_3 y + u_4, \tag{2.24}$$

and one has

$$\mathbf{W}_+(x, y, z; u_2, u_3, u_4) = x^2 + y^4 + u_2 y^2 z^2 + u_3 y z^3 + u_4 z^4 \tag{2.25}$$

if $\Gamma = \phi(\mathbb{G}_{\mathrm{m}})$, and

$$\mathbf{W}_+(x, y, z; u_2, u_4) = x^2 + y^4 + u_2 y^2 z^2 + u_4 z^4. \tag{2.26}$$

if $\Gamma = \Gamma_w$.

Example 2.2 (E_{12}-singularity)**.** When $n = 3$, $(d_1, d_2, d_3; h) = (21, 14, 6; 42)$, and $\mathbf{w}(x, y, z) = x^2 + y^3 + z^7$, one has $\Gamma_{\mathbf{w}} \cong \mathbb{G}_{\mathrm{m}}$, $\operatorname{Jac}_{\mathbf{w}} = \mathbb{A}[x, y, z]/(2x, 3y^2, 7y^6)$, and $\mu = 12$. One can take

$$J_{\mathbf{w}} = \left\{ (i, j, k) \in \mathbb{N}^3 \;\middle|\; i = 0,\ j \le 1,\ k \le 5 \right\}, \tag{2.27}$$

so that a semiuniversal unfolding $\widetilde{\mathbf{w}} \colon \mathbb{A}^3 \times U \to \mathbb{A}^1$ of $\mathbf{w}$ is given by

$$\widetilde{\mathbf{w}} = x^2 + y^3 + z^7 + \sum_{\substack{j=0,1, \\ k=0,1,2,3,4,5}} u_{jk} y^j z^k. \tag{2.28}$$

Since $\phi(\mathbb{G}_{\mathrm{m}}) = \Gamma_{\mathbf{w}}$, the choice of Γ is unique in this case. The exponent

$$m_{jk} = 42 - 14j - 6k \tag{2.29}$$

is positive unless $(j,k)=(1,5)$, so that $U_+ \subset U$ is the 11-dimensional subspace defined by $u_{15}=0$, and $\mathbf{W}_+ \colon \mathbb{A}^4 \times U_+ \to \mathbb{A}^1$ is given by

$$\mathbf{W}_+ = x^2 + y^3 + z^7 + \sum_{(j,k)\neq(1,5)} u_{jk} y^j z^k w^{m_{jk}}. \tag{2.30}$$

3. Hochschild cohomology via matrix factorizations

The Hochschild cohomology of a quasi-projective scheme Y is defined as

$$\mathrm{HH}^*(Y) \coloneqq \mathrm{Ext}^*_{Y\times Y}\left(\mathcal{O}_\Delta, \mathcal{O}_\Delta\right), \tag{3.1}$$

where $\mathcal{O}_\Delta \coloneqq \Delta_* \mathcal{O}_Y$ and $\Delta \colon Y \to Y \times Y$ is the diagonal embedding. The same definition works for perfect derived stacks, where the fiber product is taken in the category of derived stacks [8]. The right hand side of (3.1) is isomorphic to the endomorphism

$$\mathrm{HH}^*(\mathrm{Qcoh}\, Y) \coloneqq \mathrm{Hom}^*_{\mathrm{Fun}^{\mathrm{L}}(\mathrm{Qcoh}\, Y, \mathrm{Qcoh}\, Y)}\left(\mathrm{id}_{\mathrm{Qcoh}\, Y}, \mathrm{id}_{\mathrm{Qcoh}\, Y}\right) \tag{3.2}$$

of the identity in the ∞-category of colimit-preserving endofunctors of $\mathrm{Qcoh}\, Y$ [72, 8].

When Y is a smooth variety over $\mathbf{k}$ (see [3] for a partial extension to positive characteristics), one can compute Hochschild cohomology by appealing to Hochschild–Kostant–Rosenberg isomorphism

$$\mathrm{HH}^n(Y) \cong \bigoplus_{p+q=n} H^p\left(Y, \Lambda^q \mathcal{T}_Y\right). \tag{3.3}$$

However, our main interest is in the case when Y is a singular stack. A generalization of the above decomposition to singular varieties is given by Buchweitz–Flenner [13] which states

$$\mathrm{HH}^n(Y) \cong \bigoplus_{p+q=n} \mathrm{Ext}^p\left(\wedge^q \mathbb{L}_Y, \mathcal{O}_Y\right) \tag{3.4}$$

where $\mathbb{L}_Y$ is the cotangent complex over $\mathbf{k}$ and $\wedge^q$ is the derived exterior product. However, it is not always straightforward to compute with this, even when Y is a variety. We will instead use another strategy which uses the function $\mathbf{w}$ more directly.

Let $S \coloneqq \mathrm{Sym}\, V$ be the symmetric algebra over the vector space $V \coloneqq \mathrm{span}\,\{x_0, x_1, \ldots, x_n\}$ of dimension $n+1$, and $\mathbb{A}^{n+1} = \mathrm{Spec}\, S$ be the affine space. Let further Γ be a finite extension of $\mathbb{G}_\mathrm{m}$ acting linearly on V, $\chi \in \widehat{\Gamma} \coloneqq \mathrm{Hom}(\Gamma, \mathbb{G}_\mathrm{m})$ be a character of Γ, and $\mathbf{W} \in H^0\left(\mathcal{O}_{[\mathbb{A}^{n+1}/\Gamma]}(\chi)\right) \cong S(\chi)^\Gamma$ be a non-zero element of weight χ. The quotient ring $R \coloneqq S/(\mathbf{W})$ inherits a Γ-action.

When χ is isomorphic to the top exterior power of $V^\vee$ as a Γ-module, the bounded derived category $\mathrm{coh}\, Y$ of coherent sheaves on the quotient stack $Y \coloneqq [(\mathrm{Spec}\, R \setminus \mathbf{0})/\Gamma]$ is quasi-equivalent to the pretriangulated dg category $\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{W})$ of Γ-equivariant matrix factorizations;

$$\mathrm{coh}\, Y \cong \mathrm{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{W}). \tag{3.5}$$

This is first proved by Orlov [53, Theorem 3.11] when $\Gamma \cong \mathbb{G}_\mathrm{m}$ in the context of triangulated categories. The generalization to a finite extension of $\mathbb{G}_\mathrm{m}$ is straightforward. The quasi-equivalence of dg categories can be found in [6, 14, 33, 68]. Note also that by [53, Theorem 3.10], $\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{W})$ is equivalent to the bounded stable derived category of the graded ring R, denoted by $D^b_{\mathrm{sing}}(\mathrm{gr}\, R)$. The equivalence (3.5) implies the isomorphism

$$\mathrm{HH}^*(Y) \cong \mathrm{HH}^*(\mathbb{A}^{n+1}, \Gamma, \mathbf{W}), \tag{3.6}$$

where the right hand side is the Hochschild cohomology of the dg category $\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{W})$, which can be computed as follows:

Theorem 3.1 ([18, 14, 58, 6])**.** *Let Γ be an abelian finite extension of $\mathbb{G}_\mathrm{m}$ acting linearly on $\mathbb{A}^{n+1} = \operatorname{Spec} S$, and $\mathbf{W} \in S$ be a non-zero element of degree $\chi \in \widehat{\Gamma} := \operatorname{Hom}(\Gamma, \mathbb{G}_\mathrm{m})$. Assume that the singular locus of the zero set $Z_{(-\mathbf{W}) \boxplus \mathbf{W}}$ of the Sebastiani–Thom sum $(-\mathbf{W}) \boxplus \mathbf{W}$ is contained in the product of the zero sets $Z_\mathbf{W} \times Z_\mathbf{W}$. Then $\mathrm{HH}^t(\mathbb{A}^{n+1}, \Gamma, \mathbf{W})$ is isomorphic to*

$$\left(\bigoplus_{\substack{\gamma \in \ker \chi,\ l \geq 0 \\ t - \dim N_\gamma = 2u}} H^{-2l}(d\mathbf{W}_\gamma) \otimes \chi^{\otimes(u+l)} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right.$$
$$\left. \oplus \bigoplus_{\substack{\gamma \in \ker \chi,\ l \geq 0 \\ t - \dim N_\gamma = 2u+1}} H^{-2l-1}(d\mathbf{W}_\gamma) \otimes \chi^{\otimes(u+l+1)} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma. \tag{3.7}$$

Here $H^i(d\mathbf{W}_\gamma)$ is the i-th cohomology of the Koszul complex

$$C^*(d\mathbf{W}_\gamma) := \left\{ \cdots \to \Lambda^2 V_\gamma^\vee \otimes \chi^{\otimes(-2)} \otimes S_\gamma \to V_\gamma^\vee \otimes \chi^\vee \otimes S_\gamma \to S_\gamma \right\}, \tag{3.8}$$

where the rightmost term S_γ sits in cohomological degree 0, and the differential is the contraction with

$$d\mathbf{W}_\gamma \in (V_\gamma \otimes \chi \otimes S_\gamma)^\Gamma. \tag{3.9}$$

The vector space V_γ is the subspace of γ-invariant elements in V, S_γ is the symmetric algebra of V_γ, $\mathbf{W}_\gamma$ is the restriction of $\mathbf{W}$ to $\operatorname{Spec} S_\gamma$, and N_γ is the complement of V_γ in V so that $V \cong V_\gamma \oplus N_\gamma$ as a Γ-module. The zero-th cohomology of the Koszul complex (3.8) is isomorphic to the Jacobi algebra $\mathrm{Jac}_{\mathbf{W}_\gamma}$. If $\mathbf{W}_\gamma$ has an isolated critical point at the origin, then the cohomology of (3.8) is concentrated in degree 0, so that only the summand

$$\left(\mathrm{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)^\Gamma \tag{3.10}$$

with $l = 0$ contributes in (3.7).

The formula (3.7) is an adaptation of [6, Theorem 1.2], to which we refer the reader for a proof. The slight difference between [6, Theorem 1.2] and (3.7) comes from the convention for the Koszul complex; the latter is convenient in that when V has an additional $\mathbb{G}_\mathrm{m}$-action, (3.7) is equivariant with respect to it.

If the Γ-action on V satisfies $\dim (S \otimes \rho)^\Gamma < \infty$ for any $\rho \in \widehat{\Gamma}$, then one has

$$\dim \mathrm{HH}^t(\mathbb{A}^{n+1}, \mathbf{W}, \Gamma) < \infty \tag{3.11}$$

for any $t \in \mathbb{Z}$, since the Koszul complex (3.8) is bounded, the group $\ker \chi$ is finite, each direct summand in (3.7) is finite-dimensional, and there are only finitely many u contributing to a fixed t.

3.1. **Cones over isolated hypersurface singularities.** Let $\mathbf{w} \in \mathbf{k}[x_1, \dots, x_n]$ be a weighted homogeneous polynomial of weight $(d_1, \dots, d_n; h)$ satisfying (2.3), and Γ be a subgroup of $\Gamma_\mathbf{w}$ containing $\phi(\mathbb{G}_\mathrm{m})$ as a subgroup of finite index as in Section 2. Assume that $\mathbf{w}$ has an isolated critical point at the origin and let $\mathbf{W}$ be the image of $\mathbf{w}$ by the inclusion $\mathbf{k}[x_1, \dots, x_n] \hookrightarrow \mathbf{k}[x_0, \dots, x_n]$. Then $Y_0 \coloneqq [(\mathbf{W}^{-1}(0) \setminus \mathbf{0})/\Gamma]$ has a $\mathbb{G}_\mathrm{m}$-action given by $t \cdot [x_0 : x_1 : \cdots : x_n] = [tx_0 : x_1 : \cdots : x_n]$, which induces a $\mathbb{G}_\mathrm{m}$-action on $\mathrm{HH}^*(Y_0)$. Let $\mathrm{HH}^*(Y_0)_{<0}$ be the negative weight part of this $\mathbb{G}_\mathrm{m}$-action.

Since $\mathbf{W}$ does not contain the variable x_0, the Koszul complex $C^*(d\mathbf{W}_\gamma)$ is isomorphic to the tensor product of $C^*(d\mathbf{w}_\gamma)$ and the complex $\{\mathbf{k}x_0^\vee \otimes \chi^\vee \otimes \mathbf{k}[x_0] \to \mathbf{k}[x_0]\}$ concentrated in cohomological degree $[-1, 0]$ with the zero differential if V_γ contains $\mathbf{k}x_0 \subset V$, and to $C^*(d\mathbf{w}_\gamma)$ otherwise. Only direct summands with $l = 0, -1$ contribute to (3.7) in the former case, and those with $l = 0$ in the latter case. Summands with $l = 0$ contribute

$$\left(\mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)^\Gamma \tag{3.12}$$

to $\mathrm{HH}^{2u + \dim N_\gamma}(Y_0)$, and those with $l = -1$ contribute

$$\left(\mathbf{k}x_0^\vee \otimes \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)^\Gamma \tag{3.13}$$

to $\mathrm{HH}^{2u + \dim N_\gamma + 1}(Y_0)$ since

$$H^{-1}(d\mathbf{W}_\gamma) \cong \mathbf{k}x_0^\vee \otimes \chi^\vee \otimes \mathrm{Jac}_{\mathbf{w}_\gamma} \otimes \mathbf{k}[x_0]. \tag{3.14}$$

Corollary 3.2. *Under the above assumptions, one has* $\mathrm{HH}^0(Y_0) \cong \mathbf{k}$, $\mathrm{HH}^1(Y_0)_0 \ncong 0$, *and* $\mathrm{HH}^1(Y_0)_{<0} \cong 0$.

Proof. If $u \le -1$, then (3.12) vanishes, and if $u = 0$, then (3.12) contribute to $\mathrm{HH}^0(Y_0)$ only if $N_\gamma = 0$, where it is $\mathbf{k}$. (3.12) cannot contribute to $\mathrm{HH}^1(Y_0)$, since $\dim N_\gamma = 1$ is impossible for $\gamma = (t_0, t_1, \dots, t_n) \in \Gamma$ because of the condition $t_0 \cdots t_n = 1$. One always has $u \ge -1$ in (3.13), and one can have $u = -1$ only if $N_\gamma = \operatorname{span}\{x_1, \dots, x_n\}$. Each such γ contribute $\mathbf{k}(-1)$ to $\mathrm{HH}^{n-1}(Y_0)$. The summand with $u = 0$ and $\gamma = 0$ contributes $(\mathbf{k}x_0^\vee \otimes \mathrm{Jac}_\mathbf{w} \otimes \mathbf{k}[x_0])^\Gamma$ to $\mathrm{HH}^1(Y_0)$, which has non-negative $\mathbb{G}_\mathrm{m}$-weights. In particular, the element $x_0^\vee \otimes x_0$ gives a non-zero contribution to $\mathrm{HH}^1(Y_0)_0$. Summands with $u = 0$ and $\gamma \neq 0$ or $u \ge 1$ contribute to $\mathrm{HH}^{\ge 2}(Y_0)$. ∎

3.2. **Projective hypersurfaces.** Consider the case

$$\mathbf{w}(x_1, \dots, x_n) = x_1^{n+1} + \cdots + x_n^{n+1} \tag{3.15}$$

with

$$(d_1, \dots, d_n; h) = (1, \dots, 1; n+1) \tag{3.16}$$

and

$$\Gamma = \left\{ (t_0, \dots, t_n) \in (\mathbb{G}_\mathrm{m})^{n+1} \;\middle|\; t_1^{n+1} = \cdots = t_n^{n+1} = t_0 \cdots t_n \right\}. \tag{3.17}$$

This case appears in mirror symmetry for the Calabi–Yau hypersurface of degree $n+1$ in $\mathbb{P}^n$, and gives the D_4-singularity $x^3 + y^3$ for $n = 2$. The group $\widehat{\Gamma}$ of characters of Γ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$, and we write the character $(t_0, \dots, t_n) \mapsto t_1^{i_1 + \cdots + i_n} t_2^{-i_2} \cdots t_n^{-i_n}$ for $(i_1, \dots, i_n) \in \mathbb{Z} \times (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$ as $\rho_{i_1, \dots, i_n}$. One has $\mathbf{k} x_0^\vee \cong \rho_{1,\dots,1}, \mathbf{k} x_1^\vee \cong \rho_{1,0,\dots,0}, \mathbf{k} x_2^\vee \cong \rho_{1,n,0,\dots,0}, \dots, \mathbf{k} x_n^\vee \cong \rho_{1,0,\dots,0,n}, \chi \cong \rho_{n+1,0,\dots,0}$, and $\ker \chi \cong (\mathbb{Z}/(n+1)\mathbb{Z})^n$.

When γ is the identity element, one has $V_\gamma = V$, $N_\gamma = 0$, $\mathbf{W}_\gamma = \mathbf{w}$ and

$$\mathrm{Jac}_\mathbf{w} \cong \mathbf{k}[x_1, \dots, x_n] / ((n+1) x_1^n, \dots, (n+1) x_n^n). \tag{3.18}$$

The element

$$x_0^{(n+1)(u-i)+1} x_1^i \cdots x_n^i \in \left(\mathrm{Jac}_\mathbf{w} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \right)^\Gamma \tag{3.19}$$

for $i = 0, \dots, \min\{u, n-1\}$ contributes $\mathbf{k}((n+1)(u-i)+1)$ to HH^{2u}, and the element

$$x_0^\vee \otimes x_0^{(n+1)(u-i)+2} x_1^i \cdots x_n^i \in \left(x_0^\vee \otimes \mathrm{Jac}_\mathbf{w} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \right)^\Gamma \tag{3.20}$$

for $i = 0, \dots, \min\{u, n-1\}$ contributes $\mathbf{k}((n+1)(u-i)+1)$ to HH^{2u+1}.

When $V_\gamma = 0$ and $N_\gamma = V$, one has $\mathbf{W}_\gamma = 0$ and the summand

$$\left(\chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \cong \mathbf{k} x_0^\vee \wedge \cdots \wedge x_n^\vee, \tag{3.21}$$

contributes $\mathbf{k}(-1)$ to $\mathrm{HH}^{2u + \dim N_\gamma} = \mathrm{HH}^{-2+n+1} = \mathrm{HH}^{n-1}$. The number of such γ is $2, 21, 204, \dots$ for $n = 2, 3, 4, \dots$ respectively.

When $V_\gamma = \mathbf{k} x_0$ and $N_\gamma = \mathbf{k} x_1 \oplus \cdots \oplus \mathbf{k} x_n$, one has $\mathbf{W}_\gamma = 0$ and the summand

$$\left(\mathrm{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \cong \mathbf{k} x_0^{(n+1)u+n} \otimes x_1^\vee \wedge \cdots \wedge x_n^\vee \tag{3.22}$$

in $\mathrm{HH}^{2u + \dim N_\gamma}$ contributes $\mathbf{k}((n+1)u+n)$ to HH^{2u+n} for $u \geq 0$, and the summand

$$\left(x_0^\vee \otimes \mathrm{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \cong \mathbf{k} x_0^\vee \otimes x_0^{(n+1)u+n+1} \otimes x_1^\vee \wedge \cdots \wedge x_n^\vee \tag{3.23}$$

in $\mathrm{HH}^{2u + \dim N_\gamma + 1}$ contributes $\mathbf{k}((n+1)u+n)$ to HH^{2u+n+1} for $u \geq -1$. The number of such γ is $2, 6, 52, \dots$ for $n = 2, 3, 4, \dots$ respectively.

When $V_\gamma = \mathbf{k} x_0 \oplus \cdots \oplus \mathbf{k} x_i$ and $\Lambda^{\dim N_\gamma} N_\gamma^\vee = \mathbf{k} x_{i+1}^\vee \wedge \cdots \wedge x_n^\vee$ for $0 < i < n$, one has $\mathbf{W}_\gamma = x_1^{n+1} + \cdots + x_i^{n+1}$ and

$$\mathrm{Jac}_{\mathbf{W}_\gamma} = \mathbf{k}[x_0] \otimes \operatorname{span}\left\{ 1, x_1, \dots, x_1^{n-1} \right\} \otimes \cdots \otimes \operatorname{span}\left\{ 1, x_i, \dots, x_i^{n-1} \right\}. \tag{3.24}$$

Since the weight of

$$x_0^{k_0} \cdots x_i^{k_i} \otimes x_{i+1}^\vee \wedge \cdots \wedge x_n^\vee \in \operatorname{Jac}_{\mathbf{W}_\gamma} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \tag{3.25}$$

for $(k_0, \ldots, k_i) \in \mathbb{N} \times \{0, \ldots, n-1\}^i$ can never be proportional to χ, one has

$$\left(\operatorname{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma \cong 0 \tag{3.26}$$

for any $u \in \mathbb{Z}$ and similarly for $\left(x_0^\vee \otimes \operatorname{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee \right)^\Gamma$, so that such γ does not contribute to HH^*. In total, one has

$$\begin{aligned} \mathrm{HH}^0(Y_0) &\cong \mathbf{k}, \\ \mathrm{HH}^1(Y_0) &\cong \mathbf{k} \oplus \mathbf{k}(-1)^{\oplus 4}, \\ \mathrm{HH}^{2i+2}(Y_0) &\cong \mathrm{HH}^{2i+3}(Y_0) \cong \mathbf{k}(3i+1) \oplus \mathbf{k}(3i+2)^{\oplus 2} \oplus \mathbf{k}(3i+3) \quad \text{for } i \geq 0 \end{aligned}$$

for $n = 2$, and

$$\begin{aligned} \mathrm{HH}^0(Y_0) &\cong \mathrm{HH}^1(Y_0) \cong \mathbf{k}, \\ \mathrm{HH}^2(Y_0) &\cong \mathbf{k}(-1)^{\oplus 27} \oplus \mathbf{k}(1) \oplus \mathbf{k}(4), \\ \mathrm{HH}^3(Y_0) &\cong \mathbf{k}(1) \oplus \mathbf{k}(3)^{\oplus 6} \oplus \mathbf{k}(4), \\ \mathrm{HH}^{2i+4}(Y_0) &\cong \mathbf{k}(4i+2) \oplus \mathbf{k}(4i+3)^{\oplus 6} \oplus \mathbf{k}(4i+5) \oplus \mathbf{k}(4i+8) \quad \text{for } i \geq 0, \\ \mathrm{HH}^{2i+5}(Y_0) &\cong \mathbf{k}(4i+2) \oplus \mathbf{k}(4i+5) \oplus \mathbf{k}(4i+7)^{\oplus 6} \oplus \mathbf{k}(4i+8) \quad \text{for } i \geq 0 \end{aligned}$$

for $n = 3$.

3.3. **Double covers of projective spaces.**

Consider the case

$$\mathbf{w}(x_1, \ldots, x_n) = x_1^2 + x_2^{2n} + \cdots + x_n^{2n} \tag{3.27}$$

with

$$(d_1, \ldots, d_n; h) = (n, 1, \ldots, 1; 2n) \tag{3.28}$$

and

$$\Gamma = \left\{ (t_0, \ldots, t_n) \in (\mathbb{G}_\mathrm{m})^{n+1} \;\middle|\; t_1^2 = t_2^{2n} = \cdots = t_n^{2n} = t_0 \cdots t_n \right\}. \tag{3.29}$$

This case appears in mirror symmetry for the double cover of $\mathbb{P}^{n-1}$ branched over a hypersurface of degree $2n$, and gives the tacnode singularity $x^2 + y^4$ for $n = 2$. One has $\widehat{\Gamma} \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2n\mathbb{Z})^{n-2}$ and $\ker \chi \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2n\mathbb{Z})^{n-1}$.

When γ is the identity element, one has $V_\gamma = V$, $N_\gamma = 0$, $\mathbf{W}_\gamma = \mathbf{w}$ and

$$\operatorname{Jac}_\mathbf{w} \cong \mathbf{k}[x_1, \ldots, x_n]/(2x_1, 2nx_2^{2n-1}, \ldots, 2nx_n^{2n-1}) \tag{3.30}$$

The element

$$x_0^{2(u-i)n+2i} x_2^{2i} \cdots x_n^{2i} \in \left(\operatorname{Jac}_\mathbf{w} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \right)^\Gamma \tag{3.31}$$

for $i = 0, \ldots, \min\{u, n-1\}$ contributes $\mathbf{k}(2(u-i)n + 2i)$ to HH^{2u}, and the element

$$x_0^\vee \otimes x_0^{2(u-i)n+2i+1} x_2^{2i} \cdots x_n^{2i} \in \left(x_0^\vee \otimes \operatorname{Jac}_\mathbf{w} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u} \right)^\Gamma \tag{3.32}$$

for $i = 0, \dots, \min\{u, n-1\}$ contributes $\mathbf{k}(2(u-i)n + 2i)$ to HH^{2u+1}.

When $V_\gamma = 0$ and $N_\gamma = V$, one has $\mathbf{W}_\gamma = 0$ and the summand

$$\left(\chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)^\Gamma \cong \mathbf{k} x_0^\vee \wedge \cdots \wedge x_n^\vee, \tag{3.33}$$

contributes $\mathbf{k}(-1)$ to $\mathrm{HH}^{2u+\dim N_\gamma} = \mathrm{HH}^{-2+n+1} = \mathrm{HH}^{n-1}$. The set of such γ is bijective with the set of $(i_0, i_2, \dots, i_{n-1}) \in \{0, \dots, 2n-1\}^{n-1}$ satisfying $i_0 + n + i_2 + \cdots + i_n \equiv 0$ modulo $2n$. The number of such γ is $2, 21, 300, \dots$ for $n = 2, 3, 4, \dots$ respectively.

When $V_\gamma = \mathbf{k}x_0$ and $N_\gamma = \mathbf{k}x_1 \oplus \cdots \oplus \mathbf{k}x_n$, one has $\mathbf{W}_\gamma = 0$ and the summand

$$\left(\mathrm{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)^\Gamma \cong \mathbf{k} x_0^{2nu+2n-1} \otimes x_1^\vee \wedge \cdots \wedge x_n^\vee \tag{3.34}$$

in $\mathrm{HH}^{2u+\dim N_\gamma}$ contributes $\mathbf{k}(2nu + 2n - 1)$ to HH^{2u+n} for $u \geq 0$, and the summand

$$\left(x_0^\vee \otimes \mathrm{Jac}_{\mathbf{W}_\gamma} \otimes \chi^{\otimes u} \otimes \Lambda^{\dim N_\gamma} N_\gamma^\vee\right)^\Gamma \cong \mathbf{k} x_0^\vee \otimes x_0^{2nu+2n} \otimes x_1^\vee \wedge \cdots \wedge x_n^\vee \tag{3.35}$$

in $\mathrm{HH}^{2u+\dim N_\gamma+1}$ contributes $\mathbf{k}(2nu + 2n - 1)$ to HH^{2u+n+1} for $u \geq -1$. The number of such γ is $1, 4, 43, \dots$ for $n = 2, 3, 4, \dots$ respectively.

Other γ does not contribute, and the result is summarized as

$$\begin{aligned}
\mathrm{HH}^0(Y_0) &\cong \mathbf{k}, \\
\mathrm{HH}^1(Y_0) &\cong \mathbf{k} \oplus \mathbf{k}(-1)^{\oplus 3}, \\
\mathrm{HH}^{2i+2}(Y_0) \cong \mathrm{HH}^{2i+3}(Y_0) &\cong \mathbf{k}(4i+2) \oplus \mathbf{k}(4i+3) \oplus \mathbf{k}(4i+4) \quad \text{for } i \geq 0
\end{aligned}$$

for $n = 2$, and

$$\begin{aligned}
\mathrm{HH}^0(Y_0) \cong \mathrm{HH}^1(Y_0) &\cong \mathbf{k}, \\
\mathrm{HH}^2(Y_0) &\cong \mathbf{k}(-1)^{\oplus 25} \oplus \mathbf{k}(2) \oplus \mathbf{k}(6), \\
\mathrm{HH}^3(Y_0) &\cong \mathbf{k}(2) \oplus \mathbf{k}(5)^{\oplus 4} \oplus \mathbf{k}(6), \\
\mathrm{HH}^{2i+4}(Y_0) &\cong \mathbf{k}(6i+4) \oplus \mathbf{k}(6i+5)^{\oplus 4} \oplus \mathbf{k}(6i+8) \oplus \mathbf{k}(6i+12) \quad \text{for } i \geq 0, \\
\mathrm{HH}^{2i+5}(Y_0) &\cong \mathbf{k}(6i+4) \oplus \mathbf{k}(6i+8) \oplus \mathbf{k}(6i+11)^{\oplus 4} \oplus \mathbf{k}(6i+12) \quad \text{for } i \geq 0
\end{aligned}$$

for $n = 3$.

3.4. **Sylvester's sequence.** Consider the case $\mathbf{w}(x_1, \dots, x_n) = x_1^{s_1} + \cdots + x_n^{s_n}$ where $(s_i)_{i=1}^\infty = (2, 3, 7, 43, 1807, \dots)$ is the Sylvester's sequence defined by $s_i = 1 + s_1 \cdots s_{i-1}$. This case appears in mirror symmetry for the Calabi–Yau hypersurface in $\mathbb{P}(1, s_1, \dots, s_n)$, and gives the cusp singularity $x^2 + y^3$ for $n = 2$. One has

$$(d_0, d_1, \dots, d_n; h) = (1, h/s_1, \dots, h/s_n; s_{n+1} - 1) \tag{3.36}$$

and $\phi \colon \mathbb{G}_\mathrm{m} \to \Gamma$ is an isomorphism.

When γ is the identity element, one has $V_\gamma = V$, $N_\gamma = 0$, $\mathbf{W}_\gamma = \mathbf{w}$ and

$$\mathrm{Jac}_\mathbf{w} \cong \mathbf{k}[x_1, \dots, x_n]/(s_1 x_1^{s_1-1}, \dots, s_n x_n^{s_n-1}). \tag{3.37}$$

The monomial $x_0^{w_{\boldsymbol{j}}+(u-1)h} x_1^{j_1} \cdots x_n^{j_n}$ from the summand

$$\left(\operatorname{Jac}_{\mathbf{w}} \otimes \mathbf{k}[x_0] \otimes \chi^{\otimes u}\right)^\Gamma \tag{3.38}$$

contributes $\mathbf{k}(w_{\boldsymbol{j}} + (u-1)h)$ to HH^{2u} for each $\boldsymbol{j} = (j_1, \ldots, j_n)$ satisfying $0 \le j_i \le s_i - 1$ for $i = 1, \ldots, n$ and $w_{\boldsymbol{j}} := h - d_1 j_1 - \cdots - d_n j_n \ge -(u-1)h$. Such $\boldsymbol{j}$ also contributes $\mathbf{k}(w_{\boldsymbol{j}} + (u-1)h)$ to HH^{2u+1} just as in Section 3.2.

Each γ with $V_\gamma = 0$ contributes $\mathbf{k}(-1)$ to HH^{n-1}. The set of such γ can be identified with the set of integers from 0 to $h-1$ prime to all s_i for $i = 1, \ldots, n$. The cardinality of this set is given by $2, 12, 504, \ldots$ for $n = 2, 3, 4, \ldots$ respectively.

One never has $V_\gamma = \mathbf{k}x_0$ in this case. Any γ with $V_\gamma \ne 0, V$ does not contribute to HH^* just as in Section 3.2.

The result is summarized as

$$\mathrm{HH}^0(Y_0) \cong \mathbf{k}, \tag{3.39}$$

$$\mathrm{HH}^1(Y_0) \cong \mathbf{k} \oplus \mathbf{k}(-1)^{\oplus 2}, \tag{3.40}$$

$$\mathrm{HH}^{2i+2}(Y_0) \cong \mathrm{HH}^{2i+3}(Y_0) \cong \mathbf{k}(6i+4) \oplus \mathbf{k}(6i+6) \quad \text{for } i \ge 0 \tag{3.41}$$

for $n = 2$, and

$$\mathrm{HH}^0(Y_0) \cong \mathbf{k}, \tag{3.42}$$

$$\mathrm{HH}^1(Y_0) \cong \mathbf{k}, \tag{3.43}$$

$$\mathrm{HH}^2(Y_0) \cong \mathbf{k}(-1)^{\oplus 12} \oplus \mathbf{k}(\boldsymbol{w}), \tag{3.44}$$

$$\mathrm{HH}^3(Y_0) \cong \mathbf{k}(\boldsymbol{w}), \tag{3.45}$$

$$\mathrm{HH}^{2i+4}(Y_0) \cong \mathrm{HH}^{2i+5} \cong \mathbf{k}(\widetilde{\boldsymbol{w}} + 42(i+1)) \quad \text{for } i \ge 0 \tag{3.46}$$

where $\boldsymbol{w} = (4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42)$ and $\widetilde{\boldsymbol{w}} = (-2, \boldsymbol{w})$ for $n = 3$.

3.5. **Exceptional unimodal singularities.** Consider the weighted homogeneous polynomials given in Table 3.1, which define Arnold's 14 exceptional unimodal singularities [4, Table 14]. We take $\Gamma = \phi(\mathbb{G}_\mathrm{m})$. The Hilbert polynomial for the Jacobi ring

$$\operatorname{Jac}_{\mathbf{w}} := \mathbf{k}[x_1, x_2, x_3]/(\partial_1 \mathbf{w}, \partial_2 \mathbf{w}, \partial_3 \mathbf{w}) \tag{3.47}$$

is given by

$$\frac{(1 - T^{h-d_1})(1 - T^{h-d_2})(1 - T^{h-d_3})}{(1 - T^{d_1})(1 - T^{d_2})(1 - T^{d_3})}. \tag{3.48}$$

We define a non-decreasing sequence $\widetilde{\boldsymbol{w}} = (w_0 \le \cdots \le w_{\mu-1})$ of integers in such a way that (3.48) is equal to $\sum_{i=0}^{\mu-1} T^{h-w_i}$. Then one always has $w_0 = -2$, and $\boldsymbol{w} := (w_i)_{i=1}^{\mu-1}$ is as in Table 3.1. The identity element $\gamma = \mathrm{id}_V$ contributes $\mathbf{k}$ to HH^0 and HH^1, $\mathbf{k}(\boldsymbol{w})$ to HH^2 and HH^3, and $\mathbf{k}(\widetilde{\boldsymbol{w}} + (i+1)h)$ to HH^{2i+4} and HH^{2i+5} for $i \ge 0$. By adding the term x_0^h, one obtains a smooth Deligne–Mumford stack Y_1 derived-equivalent to a K3 surface. Since V^γ for $\gamma \ne \mathrm{id}_V$ does not contain the x_0-axis, contributions from $\gamma \ne \mathrm{id}_V$ is the same

Name	Normal form	$(d_1, d_2, d_3; h)$	μ	$\boldsymbol{w}$
Q_{10}	$x^2z + y^3 + z^4$	$(9, 8, 6; 24)$	10	$(4, 6, 7, 10, 12, 15, 16, 18, 24)$
Q_{11}	$x^2z + y^3 + yz^3$	$(7, 6, 4; 18)$	11	$(2, 4, 5, 6, 8, 10, 11, 12, 14, 18)$
Q_{12}	$x^2z + y^3 + z^5$	$(6, 5, 3; 15)$	12	$(1, 3, 4, 4, 6, 7, 9, 9, 10, 12, 15)$
Z_{11}	$x^2 + y^3z + z^5$	$(15, 8, 6; 30)$	11	$(4, 6, 10, 12, 14, 16, 18, 22, 24, 30)$
Z_{12}	$x^2 + y^3z + yz^4$	$(11, 6, 4; 22)$	12	$(2, 4, 6, 8, 10, 10, 12, 14, 16, 18, 22)$
Z_{13}	$x^2 + y^3z + z^6$	$(9, 5, 3; 18)$	13	$(1, 3, 4, 6, 7, 8, 9, 10, 12, 13, 15, 18)$
S_{11}	$x^2z + xy^2 + z^4$	$(6, 5, 4; 16)$	11	$(2, 3, 4, 6, 7, 8, 10, 11, 12, 16)$
S_{12}	$x^2z + xy^2 + yz^3$	$(5, 4, 3; 13)$	12	$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13)$
W_{12}	$x^2 + y^4 + z^5$	$(10, 5, 4; 20)$	12	$(2, 3, 6, 7, 8, 10, 11, 12, 15, 16, 20)$
W_{13}	$x^2 + y^4 + yz^4$	$(8, 4, 3; 16)$	13	$(1, 2, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16)$
E_{12}	$x^2 + y^3 + z^7$	$(21, 14, 6; 42)$	12	$(4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42)$
E_{13}	$x^2 + y^3 + yz^5$	$(15, 10, 4; 30)$	13	$(2, 6, 8, 10, 12, 14, 16, 18, 20, 22, 26, 30)$
E_{14}	$x^2 + y^3 + z^8$	$(12, 8, 3; 24)$	14	$(1, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 21, 24)$
U_{12}	$x^3 + y^3 + z^4$	$(4, 4, 3; 12)$	12	$(1, 2, 2, 4, 5, 5, 6, 8, 8, 9, 12)$

(3.54)

TABLE 3.1. 14 exceptional unimodal singularities

for Y_0 and Y_1. On the other hand, the rank of the total Hochschild cohomology of Y_1 is 24, and $\gamma = \mathrm{id}_V$ contributes $\mathbf{k}$ to $\mathrm{HH}^0(Y_1)$ via the element $1 \in \mathrm{Jac}_{\mathbf{w}}$ of degree 0, $\mathbf{k}^{\oplus(\mu-2)}$ to $\mathrm{HH}^2(Y_1)$ via elements of degrees between 1 and $h+1$, and $\mathbf{k}$ to HH^4 via the element of degree $h+2$. It follows that $\gamma \neq \mathrm{id}_V$ contribute $\mathbf{k}^{\oplus(24-\mu)}$ to $\mathrm{HH}^2(Y_1)$. Since V^γ does not contain the x_0-axis, each of these contributions contains $x_0^\vee$ from $\Lambda^{\dim N_\gamma} N_\gamma$, and hence the $\mathbb{G}_\mathrm{m}$-weight for the contribution to $\mathrm{HH}^2(Y_0)$ is 1. This shows

$$\mathrm{HH}^0(Y_0) \cong \mathbf{k}, \tag{3.49}$$

$$\mathrm{HH}^1(Y_0) \cong \mathbf{k}, \tag{3.50}$$

$$\mathrm{HH}^2(Y_0) \cong \mathbf{k}(-1)^{\oplus(24-\mu)} \oplus \mathbf{k}(\boldsymbol{w}), \tag{3.51}$$

$$\mathrm{HH}^3(Y_0) \cong \mathbf{k}(\boldsymbol{w}), \tag{3.52}$$

$$\mathrm{HH}^{2i+4}(Y_0) \cong \mathrm{HH}^{2i+5}(Y_0) \cong \mathbf{k}(\widetilde{\boldsymbol{w}} + (i+1)h) \quad \text{for } i \geq 0. \tag{3.53}$$

3.6. **Cusp singularities.** Consider the case

$$\mathbf{W}(x_0, \ldots, x_n) = x_1^{n+1} + \cdots + x_n^{n+1} + x_0 \cdots x_n \tag{3.55}$$

with the same weight (3.16) and the group (3.17) as in Section 3.2.

When γ is the identity element, one has $V_\gamma = V$, $N_\gamma = 0$, and $\mathbf{W}_\gamma = \mathbf{W}$. The subring of S consisting of semi-invariants with respect to χ is equal to the invariant ring with respect to $\ker \chi \cong (\boldsymbol{\mu}_{n+1})^n$. This ring is generated by $n+2$ monomials $x_0^{n+1}, \ldots, x_n^{n+1}, x_0 \cdots x_n$ with one relation $x_0^{n+1} \cdots x_n^{n+1} = (x_0 \cdots x_n)^{n+1}$. The $n+1$ monomials $x_0^{n+1}, \ldots, x_n^{n+1}$ belong to the same class in $\operatorname{Jac}_{\mathbf{W}}$, and the monomial $x_0 \cdots x_n$ is zero in $\operatorname{Jac}_{\mathbf{W}}$, so that

$$\dim \left(\operatorname{Jac}_{\mathbf{W}} \otimes \chi^{\otimes u} \right)^\Gamma = \begin{cases} 0 & u \le -1, \\ 1 & u \ge 0. \end{cases} \tag{3.56}$$

The Grothendieck ring $\operatorname{rep}_\Gamma$ of finite-dimensional Γ-vector spaces can be identified with the group ring of $\widehat{\Gamma}$, generated by $[x_0], \ldots, [x_n]$ and their inverses with relations $[x_0]^{n+1} = \cdots = [x_n]^{n+1} = [x_0] \cdots [x_n]$. The ring S is a $\widehat{\Gamma}$-graded ring, and the class $[C^*(d\mathbf{W})]$ of the Koszul complex is an element of a suitable completion of $\operatorname{rep}_\Gamma$ given by

$$[C^*(d\mathbf{W})] = (1 + [x_0] + \cdots + [x_0]^{n-1}) \cdots (1 + [x_n] + \cdots + [x_n]^{n-1}). \tag{3.57}$$

Among n^{n+1} monomials in (3.57), only $[x_0]^i \cdots [x_n]^i$ for $i = 0, \ldots, n-1$ are proportional to a power of $[\chi]$. By projecting to the subring generated by $T \coloneqq [x_0] \cdots [x_n]$, one obtains

$$\left[(C^*(d\mathbf{W}))^\Gamma \right] = 1 + T + \cdots + T^{n-1}. \tag{3.58}$$

Since $(\partial_i \mathbf{W})_{i=0}^{n-1}$ is a regular sequence in S, the cohomology of the Koszul complex is concentrated in degree -1 and 0. It follows that

$$[\operatorname{Jac}_{\mathbf{W}}] - \left[H^{-1}(d\mathbf{W}) \right] = 1 + T + \cdots + T^{n-1}, \tag{3.59}$$

so that

$$\dim \left(H^{-1}(d\mathbf{W}) \otimes \chi^{\otimes (u+1)} \right)^\Gamma = \begin{cases} 0 & u \le n-2, \\ 1 & u \ge n-1. \end{cases} \tag{3.60}$$

Hence $\gamma = 0$ contributes $\mathbf{k}$ to HH^{2u} for $u \ge 0$ and HH^{2u+1} for $u \ge n-1$.

Contributions from non-trivial γ is the same as in Section 3.2, and the result is summarized as

$$\begin{aligned} \operatorname{HH}^0(Y_0) &\cong \mathbf{k}, \\ \operatorname{HH}^{i+1}(Y_0) &\cong \mathbf{k}^{\oplus 3} \quad \text{for } i \ge 0 \end{aligned}$$

for $n = 2$, and

$$\begin{aligned} \operatorname{HH}^0(Y_0) &\cong \mathbf{k}, \\ \operatorname{HH}^1(Y_0) &\cong 0, \\ \operatorname{HH}^2(Y_0) &\cong \mathbf{k}^{\oplus 28}, \\ \operatorname{HH}^3(Y_0) &\cong \mathbf{k}^{\oplus 6}, \\ \operatorname{HH}^{4+i}(Y_0) &\cong \mathbf{k}^{\oplus 7} \quad \text{for } i \ge 0 \end{aligned}$$

for $n = 3$.

3.7. **Ordinary double points.** Consider the case $\mathbf{W}(x, y, z, w) = x_0^{n+1} + \cdots + x_n^{n+1} - (n+1)x_0 \cdots x_n$ with the same weight (3.16) and the group (3.17) as in Section 3.2.

When γ is the identity element, one has $V_\gamma = V$, $N_\gamma = 0$, and $\mathbf{W}_\gamma = \mathbf{W}$. The generators $x_0^{n+1}, \ldots, x_n^{n+1}, x_0 \cdots x_n$ of the invariant ring $S^{\ker \chi}$ belongs to the same class in $\mathrm{Jac}_{\mathbf{W}}$, so that

$$\dim \left(H^0(d\mathbf{W}) \otimes \chi^{\otimes k} \right)^\Gamma = \begin{cases} 0 & k \le -1, \\ 1 & k \ge 0. \end{cases} \tag{3.61}$$

The same reasoning as in Section 3.6 shows that $\gamma = 0$ contributes $\mathbf{k}$ to HH^{2i} for $i \ge 0$ and HH^{2i+1} for $i \ge 2$.

Contributions from non-trivial γ is the same as in Section 3.6, except that the coordinate x_0 behaves exactly the same way as other coordinates. The result is summarized as

$$\begin{aligned} \mathrm{HH}^0(Y_0) &\cong \mathbf{k}, \\ \mathrm{HH}^1(Y_0) &\cong \mathbf{k}^{\oplus 2}, \\ \mathrm{HH}^{i+2}(Y_0) &\cong \mathbf{k} \quad \text{for } i \ge 0 \end{aligned}$$

for $n = 2$, and

$$\begin{aligned} \mathrm{HH}^0(Y_0) &\cong \mathbf{k}, \\ \mathrm{HH}^1(Y_0) &\cong 0, \\ \mathrm{HH}^2(Y_0) &\cong \mathbf{k}^{\oplus 22}, \\ \mathrm{HH}^3(Y_0) &\cong 0, \\ \mathrm{HH}^{4+i}(Y_0) &\cong \mathbf{k} \quad \text{for } i \ge 0 \end{aligned}$$

for $n = 3$.

4. Generators and formality

We use the same notation as in Section 2, and assume that $\mathbf{w} \colon \mathbb{A}^n \to \mathbb{A}^1$ has an isolated critical point at the origin. In order to relate U_+ to the moduli space of minimal A_∞-structures on a fixed graded algebra, we need generators $\mathcal{S}_u$ of $\operatorname{perf} Y_u$ for $u \in U_+$ such that

(i) the isomorphism class of the endomorphism algebra $\operatorname{End}(\mathcal{S}_u)$ as a graded algebra does not depend on $u \in U_+$,
(ii) the generator $\mathcal{S}_0$ at $u = 0$ admits a $\mathbb{G}_{\mathrm{m}}$-equivariant structure such that the cohomological grading on $\operatorname{End}(\mathcal{S}_u)$ is proportional to the weight of the $\mathbb{G}_{\mathrm{m}}$-action.

Since the ambient space $[(\mathbb{A}^{n+1} \setminus \mathbf{0})/\Gamma_{\mathbf{w}}]$ is a toric stack of Picard number 1, it has a full strong exceptional collection of line bundles [35], whose restriction gives a generator of $\operatorname{perf} Y_u$ satisfying (i). Although this generator is good for the comparison of homological mirror symmetry for the ambient Fano stack with that for the anti-canonical hypersurface, it does not satisfy (ii), and hence is not good for the present purpose. Instead, we make the following assumption:

Assumption 4.1. The singularity category $\operatorname{mf}(\mathbb{A}^n, \mathbf{w}, \Gamma)$ has a tilting object $\mathcal{E}$.

Here, an object $\mathcal{E}$ of $\operatorname{mf}(\mathbb{A}^n, \mathbf{w}, \Gamma)$ is a *tilting object* if $\operatorname{End}^i(\mathcal{E}) \cong 0$ for $i \neq 0$ and $\hom(\mathcal{E}, X) \simeq 0$ implies $X \cong 0$. When $\mathbf{w}$ comes from one of Arnold's 14 exceptional unimodal singularities and $\Gamma = \phi(\mathbb{G}_m)$, a tilting object of $\operatorname{mf}(\mathbb{A}^n, \mathbf{w}, \Gamma)$ is given in [34, 46], so that Assumption 4.1 is satisfied in this case. The endomorphism ring of this tilting object is described by the quiver in Figure 1.2. When $\mathbf{w}$ is the Sebastiani–Thom sum of polynomials of types A or D, another tilting object in the singularity category is given in [26, 27], whose endomorphism ring is the tensor product of the path algebras of Dynkin quivers of the corresponding types.

Let $\mathcal{S}_u$ be the image of $\mathcal{E}$ under composition of the push-forward functor

$$\operatorname{mf}(\mathbb{A}^n, \Gamma, \mathbf{w}) \to \operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{W}_u) \tag{4.1}$$

of matrix factorizations and the equivalence (3.5).

Theorem 4.2. *Under Assumption 4.1, the object $\mathcal{S}_u$ split-generates* $\operatorname{perf} Y_u$.

Proof. For the simplicity of notation, we assume $\Gamma \cong \mathbb{G}_m$, so that Y is an anti-canonical hypersurface in $\mathbb{P} := \mathbb{P}(d_0, \dots, d_n)$; the extension to the general case is straightforward (cf. e.g., [73, Section 3]). We will work with $D^b_{\mathrm{sing}}(\operatorname{gr} \overline{R})$ and $D^b_{\mathrm{sing}}(\operatorname{gr} R)$ instead of $\operatorname{mf}(\mathbb{A}^n, \Gamma, \mathbf{w})$ and $\operatorname{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{W}_u)$, which are equivalent by [53, Theorem 39]. Note that $D^b_{\mathrm{sing}}(\operatorname{gr} \overline{R})$ is split-generated by the structure sheaf of the origin since $\mathbf{w}$ has an isolated critical point at the origin. Since the structure sheaf of the origin in $D^b_{\mathrm{sing}}(\operatorname{gr} \overline{R})$ is described as a cone constructed out of $\mathcal{E}$, and its push-forward in $D^b_{\mathrm{sing}}(\operatorname{gr} R)$ is the structure sheaf $R/\mathfrak{m}$ of the origin, it suffices to show that the image of $R/\mathfrak{m}(i)$ for $i \in \mathbb{Z}$ under Orlov equivalence

$$D^b_{\mathrm{sing}}(\operatorname{gr} R) \cong \operatorname{coh} Y_u \tag{4.2}$$

split-generates $\operatorname{perf} Y_u$. The proof of [53, Theorem 16] shows the existence of semiorthogonal decompositions

$$D^b(\operatorname{gr} R_{\geq 0}) = \langle \mathcal{D}_0, \mathcal{S}_{\geq 0} \rangle = \langle \mathcal{P}_{\geq 0}, \mathcal{T}_0 \rangle \,, \tag{4.3}$$

equivalences

$$\mathcal{D}_0 \cong \operatorname{coh} Y_u, \quad \mathcal{T}_0 \cong D^b_{\mathrm{sing}}(\operatorname{gr} R), \tag{4.4}$$

and an equality

$$\mathcal{D}_0 = \mathcal{T}_0. \tag{4.5}$$

Therefore, in order to send an object $\overline{Z} \in D^b_{\mathrm{sing}}(\operatorname{gr} R)$ by the equivalence

$$D^b_{\mathrm{sing}}(\operatorname{gr} R) \cong \mathcal{T}_0 = \mathcal{D}_0 \cong \operatorname{coh} Y_u, \tag{4.6}$$

we

(1) find an object $Z \in D^b(\operatorname{gr} R_{\geq 0})$ which goes to $\overline{Z}$ by the localization functor $D^b(\operatorname{gr} R_{\geq 0}) \to D^b_{\mathrm{sing}}(\operatorname{gr} R)$,
(2) take the semiorthogonal component M of Z, i.e., find a distinguished triangle

$$M \to Z \to N \to M[1] \tag{4.7}$$

such that $M \in \mathcal{T}_0 = {}^\perp \mathcal{P}_{\geq 0}$ and $N \in \mathcal{P}_{\geq 0}$, and
(3) take the image $\mathcal{M}$ of M by the localization functor $\pi \colon D^b(\operatorname{gr} R_{\geq 0}) \to \operatorname{coh} Y_u$.

Since R is Gorenstein with parameter zero and dimension n, one has

$$\hom_R(R/\mathfrak{m}(-i), R(j)) = \begin{cases} \boldsymbol{k}[-n] & i = -j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

If we start with $Z_i = (R/\mathfrak{m})(-i)[-n+1]$ for $0 \leq i < h$, then

$$\operatorname{Cone}\left((R/\mathfrak{m})(-i)[-n] \to R(-i)\right) \tag{4.9}$$

belongs to $\mathcal{S}^\perp_{\geq i+1}$, which is equal to ${}^\perp \mathcal{P}_{\geq i+1}$ in the semiorthogonal decomposition

$$D^b(\operatorname{gr} R_{\geq 0}) = \langle \mathcal{P}_{\geq 0}, \mathcal{T}_0 \rangle = \langle \mathcal{P}_{\geq i+1}, R(-i), R(-i+1), \ldots, R, \mathcal{T}_0 \rangle. \tag{4.10}$$

Since $(R/\mathfrak{m})(-i)$ is orthogonal to $R(-i+1), \ldots, R$ and its image in $D^b \operatorname{coh} Y$ is zero, the image $\mathcal{M}_i \in D^b \operatorname{coh} Y$ of the semiorthogonal component $M_i \in \mathcal{T}_0 = \mathcal{D}_0$ of Z_i is isomorphic to the image of the semiorthogonal component of $R(-i)$.

Since $i < h$, the operation of taking the semiorthogonal component of $R(-i)$ is the same as that for the polynomial ring T, and the resulting object $\mathcal{M}_i$ is the restriction to Y of the object $\mathcal{E}_i$ in $D^b \operatorname{coh} \mathbb{P}$ obtained by mutating $\mathcal{O}_{\mathbb{P}}(-i)$ across $\mathcal{O}_{\mathbb{P}}(-i+1), \ldots, \mathcal{O}_{\mathbb{P}}$. The collection $(\mathcal{E}_i)_{i=0}^{h-1}$ is left dual to the full exceptional collection $(\mathcal{O}_{\mathbb{P}}(-i))_{i=0}^{h-1}$ by construction, and hence is full again. Now [59, Lemma 5.4] shows $\bigoplus_{i=0}^{h-1} \mathcal{M}_i$ generates $\operatorname{perf} Y$. □

Under Assumption 4.1, let $\mathcal{A}_u$ be the minimal model of the dg endomorphism algebra $\operatorname{end}(\mathcal{S}_u)$, so that one has a quasi-equivalence

$$\operatorname{Qcoh} Y_u \cong D(\mathcal{A}_u) \tag{4.11}$$

and an isomorphism

$$\mathrm{HH}^*(Y_u) \cong \mathrm{HH}^*(\mathcal{A}_u). \tag{4.12}$$

The cohomology algebra $A := H^*(\mathcal{A}_u)$ is the degree n trivial extension of the endomorphism algebra of the tilting object $\mathcal{E}$ by [75], and hence independent of u.

By using the additional $\mathbb{G}_{\mathrm{m}}$-action, we can prove the following:

Theorem 4.3. $\mathcal{A}_0$ *is formal.*

Proof. We fix an equivariant structure on $\mathcal{S}_0$ with respect to the $\mathbb{G}_\mathrm{m}$-action $(x_0, x_1, \dots, x_n) \mapsto (\alpha x_0, x_1, \dots, x_n)$ on $\mathbb{A}^{n+1}$ in such a way that $\operatorname{End}^0(\mathcal{S}_0) \cong \operatorname{End}^0(\mathcal{E})$ is $\mathbb{G}_\mathrm{m}$-invariant. Note that the dualizing sheaf of Y_0 is trivial as a $\mathcal{O}_{Y_0}$-module, but has weight 1 with respect to the $\mathbb{G}_\mathrm{m}$-action. It follows that the weight for the $\mathbb{G}_\mathrm{m}$-action on $\operatorname{End}^{n-1}(\mathcal{S}_0) \cong \left(\operatorname{End}^0(\mathcal{E})\right)^\vee$ is one. This shows that the cohomological degree on the $\mathbb{N}$-graded algebra $\operatorname{End}^*(\mathcal{S}_0)$ is $(n-1)$ times the $\mathbb{G}_\mathrm{m}$-weight. Since the group $\mathbb{G}_\mathrm{m}$ is reductive, the chain homotopy to transfer the dg structure on $\operatorname{end}(\mathcal{S}_0)$ to the minimal model $\mathcal{A}_0$ can be chosen to be $\mathbb{G}_\mathrm{m}$-equivariant, so that the resulting A_∞-operations are $\mathbb{G}_\mathrm{m}$-equivariant. Since the A_∞-operation μ^d has the cohomological degree $2-d$ and the cohomological degree is proportional to the $\mathbb{G}_\mathrm{m}$-weight, one obtains $\mu^d = 0$ for $d \neq 2$. □

As a result, we have

$$\operatorname{HH}^*(A) \cong \operatorname{HH}^*(Y_0). \tag{4.13}$$

5. Moduli of K3 surfaces as moduli of A_∞-structures

We prove Theorem 1.3 in this section. Let $\mathbf{w}$ be one of the defining polynomials of exceptional unimodal singularities in Table 3.1 and $\pi_\mathcal{Y} \colon \mathcal{Y} \to U_+$ be the family of proper Deligne–Mumford stacks obtained as a fiberwise compactification of the positive part of the semiuniversal unfolding as in Section 2. As recalled in Section 4, Assumption 4.1 is satisfied in this case, so that one has an object $\mathcal{S}$ of $\operatorname{perf} \mathcal{Y}$, which restricts to a split-generator $\mathcal{S}_u$ of $\operatorname{perf} Y_u$ for each $u \in U_+$. We fix an isomorphism $\operatorname{End}^0(\mathcal{S}) \xrightarrow{\sim} \operatorname{End}^0(\mathcal{E}) \otimes \mathcal{O}_{U_+}$ as an $\mathcal{O}_{U_+}$-algebra, which gives an isomorphism $\operatorname{End}^0(\mathcal{S}_u) \xrightarrow{\sim} \operatorname{End}^0(\mathcal{E})$ for each $u \in U_+$. We also fix a section of the relative dualizing sheaf $\omega_{\mathcal{Y}/U_+}$, which gives a basis of $H^0(\omega_{Y_u})$ for each $u \in U_+$. This fixes an isomorphism of A, defined as the degree 2 trivial extension of $\operatorname{End}^0(\mathcal{E})$, with $\operatorname{End}(\mathcal{S}_u)$. By taking the minimal model of the dg endomorphism ring $\operatorname{end}(\mathcal{S})$, one obtains a family of minimal A_∞-structures on A over U_+, and hence a morphism $\varphi \colon U_+ \to \mathcal{U}_\infty(A)$ of schemes, which is $\mathbb{G}_\mathrm{m}$-equivariant since the $\mathbb{G}_\mathrm{m}$-action on u can be offset by the $\mathbb{G}_\mathrm{m}$-action on x_0, which rescales the section of $\omega_{\mathcal{Y}/U_+}$.

Recall from [48, Section (A.5)] that an $\overline{R}$-polarized scheme is a triple (W, W_∞, ϕ) consisting of a projective scheme W, an ample reduced Weil divisor W_∞ on W, and an isomorphism $\phi \colon R/x_0R \xrightarrow{\sim} \overline{R}$ of graded rings, where $R := \bigoplus_{l=0}^\infty H^0(\mathcal{O}_W(lW_\infty))$ and $x_0 \in H^0(\mathcal{O}_W(W_\infty))$ is the element corresponding to 1. It is shown in [48, Section A.5] that U_+ is a fine moduli scheme of $\overline{R}$-polarized schemes.

The universal family over U_+ is given by the coarse moduli space $\mathcal{W}$ of $\mathcal{Y}$, and one has $H^0(\mathcal{O}_\mathcal{W}(l\mathcal{W}_\infty)) \cong H^0(\mathcal{O}_\mathcal{Y}(l))$. Since $\mathcal{O}_\mathcal{Y}(l)$ can be described as a particular direct summand of some complex constructed from $\mathcal{S}$, a family of $\overline{R}$-polarized schemes can be reconstructed from a family of A_∞-algebras. This implies that φ induces an injection on tangent spaces, and hence an isomorphism since $\dim U_+ = \dim \operatorname{HH}^2(A)_{<0} \geq \dim \mathcal{U}_\infty(A)$. Since φ is a $\mathbb{G}_\mathrm{m}$-equivariant morphism from an affine space to an affine scheme with good

$\mathbb{G}_{\mathrm{m}}$-actions inducing an isomorphism on tangent spaces, it is an isomorphism of schemes, and Theorem 1.3 is proved.

For an even lattice M of signature $(1, \rho - 1)$, an M-polarized K3 surface is defined by Nikulin [52] as a pair (Y, j) of a K3 surface Y and a primitive embedding $j\colon M \to \operatorname{Pic} Y$ whose image contains a pseudo-ample line bundle. Over the field $\mathbb{C}$ of complex numbers, the coarse moduli space $\mathbf{M}_{\mathrm{K3}}$ of M-polarized K3 surfaces is the quotient of the symmetric domain of type IV associated with the orthogonal lattice $N := M^\perp$ inside the K3 lattice $E_8 \perp E_8 \perp U \perp U \perp U$. It follows from [48, Theorems 1.8 and 6.4] that when N is the Milnor lattice of an exceptional unimodal singularity, the positive part U_+ of the base space of the semiuniversal unfolding is birational to the $\mathbb{G}_{\mathrm{m}}$-bundle over $\mathbf{M}_{\mathrm{K3}}$ parametrizing the pair of a lattice polarized K3 surface and a holomorphic volume form on it. For E_{12}, Z_{11}, and Q_{10}-singularity, it is also known [12, Theorem 5] that $(U_+ \setminus \mathbf{0})/\mathbb{C}^\times$ is isomorphic to the Satake–Baily–Borel compactification of $\mathbf{M}_{\mathrm{K3}}$.

6. Homological mirror symmetry for invertible polynomials

A polynomial $\mathbf{w} \in \mathbb{C}[x_1, \ldots, x_n]$ is *invertible* if there is an integer matrix $A = (a_{ij})_{i,j=1}^n$ with non-zero determinant such that

$$\mathbf{w} = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ij}}, \tag{6.1}$$

and $\mathbf{w}$ has an isolated critical point at the origin. An invertible polynomial is weighted homogeneous, and the corresponding weight system $(d_1, \ldots, d_n; h)$ satisfying (2.2) is determined uniquely.

We conjecture that Assumption 4.1 holds when $\mathbf{w}$ is an invertible polynomial and $\Gamma = \Gamma_{\mathbf{w}}$:

Conjecture 6.1. For any invertible polynomial $\mathbf{w}$, the singularity category $\operatorname{mf}(\mathbb{A}^n, \mathbf{w}, \Gamma_{\mathbf{w}})$ has a tilting object.

Conjecture 6.1 is related to Conjecture 6.2 below by Orlov's theorem [53, Thoerem 16]:

Conjecture 6.2. For any invertible polynomial $\mathbf{w}$, the bounded derived category of coherent sheaves on the stack

$$X_{\mathbf{w}} := [((\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/(\mathbf{w})) \setminus \mathbf{0})/\Gamma_{\mathbf{w}}] \tag{6.2}$$

has a tilting object, which is a direct sum of line bundles.

Note that $X_{\mathbf{w}}$ is a smooth proper rational stack of Picard number one. Conjecture 6.2 is an analogue of a conjecture of King [38, Conjecture 9.3], which states that a smooth complete toric variety has a tilting object, which is a direct sum of line bundles. King's conjecture is proved for smooth proper toric Deligne–Mumford stacks of Picard number at most two [35, 11], and disproved in general [31, 20]. Besides toric cases, [32, 30] give evidences for Conjecture 6.2.

The *transpose* of $\mathbf{w}$ is defined in [10] as

$$\check{\mathbf{w}} = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ji}}, \tag{6.3}$$

whose exponent matrix $\check{A}$ is the transpose matrix of A.

Homological mirror symmetry for invertible polynomials is the following:

Conjecture 6.3. For any invertible polynomial $\mathbf{w}$, one has a quasi-equivalence

$$\operatorname{mf}(\mathbb{A}^n, \Gamma_{\mathbf{w}}, \mathbf{w}) \simeq \mathcal{F}(\check{\mathbf{w}}). \tag{6.4}$$

Conjecture 6.3 is stated for Brieskorn–Pham singularities in 3 variables in [73], for polynomials in 3 variables associated with a regular system of weights of dual type in the sense of Saito in [70], for invertible polynomials in 3 variables in [19], and in general in [25]. It is proved for Sebastiani–Thom sums of polynomials of type A and D in [26, 27].

We assume that the weight system $\left(\check{d}_1, \ldots, \check{d}_n; \check{h}\right)$ of the transpose $\check{\mathbf{w}}$ satisfies

$$\check{d}_0 := \check{h} - \check{d}_1 - \cdots - \check{d}_n = 1, \tag{6.5}$$

so that the Milnor fiber $\check{V}_{\check{\mathbf{w}}} := \check{\mathbf{w}}^{-1}(1)$ can be compactified to a Calabi–Yau hypersurface $\check{Y}$ in $\mathbb{P}\left(\check{d}_0, \ldots, \check{d}_n\right)$. The group $\Gamma_{\mathbf{w}}$ can be identified with the group

$$\left\{ (t_0, \ldots, t_n) \in (\mathbb{G}_\mathrm{m})^{n+1} \;\middle|\; t_1^{a_{11}} \cdots t_n^{a_{1n}} = \cdots = t_1^{a_{n1}} \cdots t_n^{a_{nn}} = t_0 \cdots t_n \right\} \tag{6.6}$$

acting naturally on $\mathbb{A}^{n+1}$, and we set

$$Z_{\mathbf{w}} := \left[(\operatorname{Spec} \mathbb{C}[x_0, \ldots, x_n]/(\mathbf{w} + x_0 x_1 \cdots x_n) \setminus \mathbf{0})/\Gamma_{\mathbf{w}} \right]. \tag{6.7}$$

Conjecture 6.4. One has quasi-equivalences

$$\operatorname{perf} Z_{\mathbf{w}} \simeq \mathcal{F}\left(\check{V}_{\check{\mathbf{w}}}\right) \tag{6.8}$$

and

$$\operatorname{coh} Z_{\mathbf{w}} \simeq \mathcal{W}\left(\check{V}_{\check{\mathbf{w}}}\right). \tag{6.9}$$

Let

$$\mathcal{Y}_{\mathbf{w}} := \left[\left(\operatorname{Spec} \Lambda_{\mathbb{N}}[x_0, \ldots, x_n]/(\mathbf{w} + q x_0^h + x_0 x_1 \cdots x_n) \setminus \mathbf{0} \right) / \Gamma_{\check{\mathbf{w}}} \right] \tag{6.10}$$

be a stack over $\Lambda_{\mathbb{N}} := \mathbb{C}\llbracket q \rrbracket$, and

$$\check{Y}_{\check{\mathbf{w}}} := \left\{ [x_0 : \cdots : x_n] \in \mathbb{P}\left(\check{d}_0, \ldots, \check{d}_n\right) \;\middle|\; \check{\mathbf{w}}(x_1, \ldots, x_n) + x_0^h = 0 \right\} \tag{6.11}$$

be the compactification of $\check{V}_{\check{\mathbf{w}}}$ in $\mathbb{P}\left(\check{d}_0, \ldots, \check{d}_n\right)$ considered as an orbifold. Let further $\check{D}_{\check{\mathbf{w}}} := \check{Y}_{\check{\mathbf{w}}} \setminus \check{V}_{\check{\mathbf{w}}}$ be the divisor at infinity and $\mathcal{F}\left(\check{Y}_{\check{\mathbf{w}}}, \check{D}_{\check{\mathbf{w}}}\right)$ be the relative Fukaya category.

Conjecture 6.5. There exists a q-adically continuous automorphism $\psi \in \operatorname{End}(\Lambda_{\mathbb{N}})^\times$ of $\Lambda_{\mathbb{N}}$ as a $\mathbb{C}$-algebra and a quasi-equivalence

$$\psi^* \operatorname{coh} \mathcal{Y}_{\mathbf{w}} \simeq \mathcal{F}\left(\check{Y}_{\check{\mathbf{w}}}, \check{D}_{\check{\mathbf{w}}}\right). \tag{6.12}$$

Strictly speaking, Conjecture 6.5 is not a conjecture in this generality since the foundation of relative Fukaya categories is not worked out yet for orbifolds. However, we expect this generalization to be rather mild since $\check{V}_{\check{\mathbf{w}}} = \check{Y}_{\check{\mathbf{w}}} \setminus \check{D}_{\check{\mathbf{w}}}$ is a manifold. One can also restrict to the case where $\check{V}_{\check{\mathbf{w}}}$ admits a compactification to a Calabi–Yau manifold, and replace $\check{Y}_{\check{\mathbf{w}}}$ with it. The resulting divisor at infinity can contain several components, and we expect the relation with the crepant resolution conjecture for orbifold Gromov–Witten invariants. The absolute Fukaya category and the relative Fukaya category should be related by a base change to the Novikov field, as in [43] or more recent [66, Section 5.5].

If we write

$$A^{-1} = \begin{pmatrix} \varphi_1^{(1)} & \varphi_1^{(2)} & \cdots & \varphi_1^{(n)} \\ \varphi_2^{(1)} & \varphi_2^{(2)} & \cdots & \varphi_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_n^{(1)} & \varphi_n^{(2)} & \cdots & \varphi_n^{(n)} \end{pmatrix}, \tag{6.13}$$

then $\ker \chi_{\mathbf{w}}$ is generated by

$$\rho_k = \left(\exp\left(2\pi\sqrt{-1}\varphi_1^{(k)}\right), \dots, \exp\left(2\pi\sqrt{-1}\varphi_n^{(k)}\right)\right), \qquad k = 1, \dots, n. \tag{6.14}$$

The *transpose* of a subgroup $G \subset \ker \chi_{\mathbf{w}}$ is defined in [9, 40] as

$$\check{G} := \left\{ \prod_{j=1}^n (\check{\rho}_j)^{r_j} \,\middle|\, \prod_{j=1}^n x_j^{r_j} \in \mathbb{C}[x_1, \dots, x_n]^G \right\}. \tag{6.15}$$

In particular, the transpose of the trivial group is $\ker \chi_{\check{\mathbf{w}}}$, and the transpose of $\langle j_{\mathbf{w}} \rangle$ is $\ker \chi_{\check{\mathbf{w}}} \cap \mathrm{SL}_n(\mathbb{C})$.

Assume that G and $\check{G}$ contain $\langle j_{\mathbf{w}} \rangle$ and $\langle j_{\check{\mathbf{w}}} \rangle$, so that one can define Γ as the pull-back of $G/\langle j_{\mathbf{w}} \rangle$ by the map $\Gamma_{\mathbf{w}} \to \Gamma_{\mathbf{w}}/\phi_{\mathbf{w}}(\mathbb{G}_{\mathrm{m}}) \cong G/\langle j_{\mathbf{w}} \rangle$, and similarly for $\check{\Gamma}$. Set

$$Z := [(\operatorname{Spec} \mathbb{C}[x_0, \dots, x_n]/(\mathbf{w} + x_0 x_1 \cdots x_n) \setminus \mathbf{0})/\Gamma], \tag{6.16}$$

$$\mathcal{Y} := \left[\left(\operatorname{Spec} \Lambda_{\mathbb{N}}[x_0, \dots, x_n]/(\mathbf{w} + q x_0^h + x_0 x_1 \cdots x_n) \setminus \mathbf{0}\right)/\Gamma\right], \tag{6.17}$$

and

$$\check{Y} := \left[\left(\operatorname{Spec} \mathbb{C}[x_0, \dots, x_n]/(\check{\mathbf{w}} + x_0^{\check{h}}) \setminus \mathbf{0}\right)\Big/\check{\Gamma}\right]. \tag{6.18}$$

It is natural to expect the existence of the 'orbifold Fukaya–Seidel category' $\mathcal{F}(\check{\mathbf{w}}, \check{G})$, the 'orbifold Fukaya category' $\mathcal{F}\left(\left[\check{\mathbf{w}}^{-1}(1)/\check{G}\right]\right)$, the 'wrapped orbifold Fukaya category' $\mathcal{W}\left(\left[\check{\mathbf{w}}^{-1}(1)/\check{G}\right]\right)$, and the 'orbifold relative Fukaya category' $\mathcal{F}\left(\check{Y}, \check{D}\right)$ satisfying

$$\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w}) \simeq \mathcal{F}(\check{\mathbf{w}}, \check{G}), \tag{6.19}$$

$$\operatorname{perf} Z \simeq \mathcal{F}\left(\left[\check{\mathbf{w}}^{-1}(1)/\check{G}\right]\right), \tag{6.20}$$

$$\operatorname{coh} Z \simeq \mathcal{W}\left(\left[\check{\mathbf{w}}^{-1}(1)/\check{G}\right]\right), \tag{6.21}$$

and

$$\psi^* \operatorname{coh} \mathcal{Y} \cong \mathcal{F}\left(\check{Y}, \check{D}\right) \tag{6.22}$$

for some $\psi \in \operatorname{End}(\Lambda_{\mathbb{N}})^\times$.

Instead of the restrictive assumption (6.5), one can start with an invertible polynomial $\mathbf{W}$ in $n+1$ variables, together with a group G satisfying $\langle J_{\mathbf{W}} \rangle \subset G \subset \mathrm{SL}_{n+1}(\mathbb{C}) \cap \ker \chi_{\mathbf{W}}$, and generalize (6.20) in such a way that removing a monomial from $\mathbf{W}$ corresponds to removing a divisor from $\check{Y}$. By removing all the monomials from $\mathbf{W}$, one will be left with homological mirror symmetry for an unramified cover of a higher-dimensional pair of pants proved in [64].

7. Hochschild cohomology of the Fukaya category of the Milnor fiber

For an object X of an A_∞-category $\mathscr{A}$, the *left Yoneda module* $\mathscr{Y}_X^{\mathrm{l}}$ is defined on objects by

$$\mathscr{Y}_X^{\mathrm{l}}(Y) = \hom_{\mathscr{A}}(X, Y). \tag{7.1}$$

The *right Yoneda module* $\mathscr{Y}_X^{\mathrm{r}}$ is defined similarly by

$$\mathscr{Y}_X^{\mathrm{r}}(Y) = \hom_{\mathscr{A}}(Y, X). \tag{7.2}$$

There exists a full and faithful functor $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \operatorname{Bimod} \mathscr{A}$ sending $X \otimes Y$ to $\mathscr{Y}_X^{\mathrm{l}} \otimes \mathscr{Y}_Y^{\mathrm{r}}$.

For a functor $F \colon \mathscr{A} \to \mathscr{B}$, the *graph bimodule* Γ_F is the $\mathscr{B}$-$\mathscr{A}$-bimodule defined on objects by

$$\Gamma_F(b, a) = \hom_{\mathscr{B}}(F(a), b) \tag{7.3}$$

for $a \in \mathscr{A}$ and $b \in \mathscr{B}$. The composition of A_∞-functors is compatible with the tensor product of bimodules. For $F \colon \mathscr{A} \to \mathscr{B}$ and $G \colon \mathscr{B} \to \mathscr{C}$, one has

$$\Gamma_G \otimes_{\mathscr{B}} \Gamma_F \coloneqq \Gamma_G \otimes T\mathscr{B} \otimes \Gamma_F \simeq \Gamma_{G \circ F}, \tag{7.4}$$

where $T\mathscr{B}$ is the bar complex of $\mathscr{B}$.

The Hochschild cohomology of an A_∞-category $\mathscr{A}$ is defined as the endomorphism of the *diagonal bimodule*, which in turn is defined as the graph bimodule $\Delta_{\mathscr{A}} \coloneqq \Gamma_{\mathrm{id}_{\mathscr{A}}}$ of the identity functor $\mathrm{id}_{\mathscr{A}}$.

Although Hochschild cohomology is less functorial than Hochschild homology, it has the restriction morphism $F^* \colon \mathrm{HH}^*(\mathscr{B}) \to \mathrm{HH}^*(\mathscr{A})$ with respect to a full and faithful functor $F \colon \mathscr{A} \to \mathscr{B}$.

Theorem 7.1 ([72, Corollary 8.2], cf. also [49] and references therein)**.** *The restriction morphism with respect to the Yoneda embedding $\mathscr{A} \to \operatorname{Mod}(\mathscr{A})$ is an isomorphism.*

Corollary 7.2. *If $\mathscr{A}$ is a full subcategory of $\mathscr{B}$ and $\mathscr{B}$ is a full subcategory of $\operatorname{Mod} \mathscr{A}$, then $\mathrm{HH}^*(\mathscr{A})$ is isomorphic to $\mathrm{HH}^*(\mathscr{B})$.*

Proof. The sequence

$$\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \operatorname{Mod}\mathscr{A} \xrightarrow{H} \operatorname{Mod}\mathscr{B} \tag{7.5}$$

of full and faithful functors gives a sequence

$$\mathrm{HH}^*(\operatorname{Mod}\mathscr{B}) \xrightarrow{H^*} \mathrm{HH}^*(\operatorname{Mod}\mathscr{A}) \xrightarrow{G^*} \mathrm{HH}^*(\mathscr{B}) \xrightarrow{F^*} \mathrm{HH}^*(\mathscr{A}) \tag{7.6}$$

of restriction morphisms. Then G^* is surjective since $G^* \circ H^*$ is an isomorphism, and G^* is injective since $F^* \circ G^*$ is an isomorphism. □

The diagonal argument [7] shows the following:

Lemma 7.3. *If $\mathscr{A}$ is a full subcategory of $\mathscr{B}$ and the diagonal bimodule $\Delta_{\mathscr{B}}$ is a colimit of images of $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ by the Yoneda embedding, then $\mathscr{B}$ is a full subcategory of* $\operatorname{Mod}\mathscr{A}$*.*

Let $\check{V}$ be the Milnor fiber of a weighted homogeneous polynomial $\check{\mathbf{w}} \colon \mathbb{C}^n \to \mathbb{C}$ with an isolated critical point at the origin. The split-closed derived Fukaya category $\mathcal{F}(\check{V})$ is a full subcategory of the split-closed derived wrapped Fukaya category $\mathcal{W}(\check{V})$. Let $(S_i)_{i=1}^{\mu}$ be a distinguished basis of vanishing cycles, and $\mathcal{S}$ be the full subcategory of $\mathcal{F}(\check{V})$ consisting of $(S_i)_{i=1}^{\mu}$. The total morphism A_∞-algebra of $\mathcal{S}$ will be denoted by

$$\mathcal{A} := \bigoplus_{i,j=1}^{\mu} \hom_{\mathcal{F}(\check{V})}(S_i, S_j). \tag{7.7}$$

We assume

$$\check{d}_0 := \check{h} - \check{d}_1 - \cdots - \check{d}_n > 0. \tag{7.8}$$

It is shown in [62, 4.c] that

$$\left(T_{S_1} \circ \cdots \circ T_{S_\mu}\right)^h = [2\check{d}_0]. \tag{7.9}$$

It follows by [59, Lemma 5.4] that $\mathcal{S}$ split-generates $\mathcal{F}(\check{V})$, so that

$$\mathcal{F}(\check{V}) \cong \operatorname{perf}\mathcal{S} \tag{7.10}$$

and hence

$$\mathrm{HH}^*(\mathcal{F}(\check{V})) \cong \mathrm{HH}^*(\mathcal{S}). \tag{7.11}$$

Theorem 7.4. *Under the assumption* (7.8)*, one has an isomorphism*

$$\mathrm{HH}^*(\mathcal{W}(\check{V})) \cong \mathrm{HH}^*(\mathcal{S}). \tag{7.12}$$

Theorem 7.4 fails without (7.8); one can take $\check{\mathbf{w}} = x^2 + y^2$ as a counter-example. Theorem 7.4 should be understood as a consequence of Koszul duality between $\mathcal{F}(\check{V})$ and $\mathcal{W}(\check{V})$ (see [23, 22, 47]), although we do not give a proof of this here.

Recall that a Liouville manifold is said to be *non-degenerate* if there is a finite collection of Lagrangians such that the open-closed map from the Hochschild homology of the full subcategory of the wrapped Fukaya category consisting of them to the symplectic cohomology hits the identity element [1]. Any Weinstein manifold is known to be non-degenerate [15].

Theorem 7.5 ([28])**.** *There is an A_∞-category $\mathcal{W}^2(\check{V})$, containing product Lagrangians $L \times L'$ and the diagonal $\Delta_{\check{V}}$ in $\check{V}^- \times \check{V}$, and an A_∞-functor*

$$\mathbf{M}\colon \mathcal{W}^2(\check{V}) \to \operatorname{Bimod} \mathcal{W}(\check{V}) \tag{7.13}$$

which is full on the full subcategory of $\mathcal{W}(\check{V})^2$ consisting of product Lagrangians, and $\Delta_{\check{V}}$ is sent to the diagonal bimodule $\Delta_{\mathcal{W}(\check{V})}$. One has

$$\operatorname{Hom}^*_{\mathcal{W}^2(\check{V})}(\Delta_{\check{V}}, \Delta_{\check{V}}) \cong \operatorname{SH}^*(\check{V}). \tag{7.14}$$

If $\check{V}$ is non-degenerate, then $\Delta_{\check{V}}$ is split-generated by product Lagrangians, and $\mathbf{M}$ induces an isomorphism

$$\operatorname{Hom}^*_{\mathcal{W}^2(\check{V})}(\Delta_{\check{V}}, \Delta_{\check{V}}) \cong \operatorname{HH}^*(\mathcal{W}(\check{V})) \coloneqq \operatorname{Hom}_{\operatorname{Bimod} \mathcal{W}(\check{V})}(\Delta_{\mathcal{W}(\check{V})}, \Delta_{\mathcal{W}(\check{V})}). \tag{7.15}$$

Theorem 7.4 combined with Theorem 7.5 gives a proof of [63, Conjecture 4] in our case:

Corollary 7.6. *Under the assumption* (7.8)*, one has an isomorphism*

$$\operatorname{SH}^*(\check{V}) \cong \operatorname{HH}^*(\mathcal{F}(\check{V})). \tag{7.16}$$

To prove Theorem 7.4, it suffices to show the following:

Proposition 7.7. *$\Delta_{\mathcal{W}(\check{V})}$ is a colimit of objects of the image of $\mathcal{S}^{\mathrm{op}} \otimes \mathcal{S}$ in* $\operatorname{Bimod} \mathcal{W}(\check{V})$.

Proof. We write $\mathcal{W} = \mathcal{W}(\check{V})$ and $\Delta = \Delta_{\mathcal{W}}$. It suffices to find a sequence

$$\begin{array}{ccccccc} \Delta & \longrightarrow & \Delta[a] & \longrightarrow & \Delta[2a] & \to & \cdots \\ & \nwarrow \; \swarrow & & \nwarrow \; \swarrow & & & \\ & U_1' & & U_2' & & & \end{array} \tag{7.17}$$

with $a \neq 0$ and $U_m' \in \mathcal{S}^{\mathrm{op}} \otimes \mathcal{S}$. Then the octahedral axiom gives distinguished triangles

$$U_m \to U_{m+1} \to U_{m+1}' \xrightarrow{[1]} \tag{7.18}$$

and

$$\Delta \to \Delta[ma] \to U_m \xrightarrow{[1]} \tag{7.19}$$

with $U_m \in \mathcal{S}^{\mathrm{op}} \otimes \mathcal{S}$. Then for any $X, Y \in \mathcal{W}$, one has $\operatorname{colim}_m U_m[-1](X,Y) \cong \Delta(X,Y)$ in $D(\mathbb{C})$, and hence $\operatorname{colim}_m U_m[-1] \cong \Delta$ in $\operatorname{Bimod} \mathcal{W}$.

For $S \in \mathcal{S}$, the *dual twist functor* is defined on objects by the distinguished triangle

$$T_S^\vee(X) \to X \xrightarrow{\mathrm{ev}^\vee} \hom(X,S)^\vee \otimes S \xrightarrow{[1]} . \tag{7.20}$$

This gives

$$\hom(S,Y) \otimes \hom(X,S) \to \hom(X,Y) \to \hom(T_S^\vee(X), Y) \xrightarrow{[1]}, \tag{7.21}$$

so that

$$\Delta_S \to \Delta \to \Gamma_{T_S^\vee} \xrightarrow{[1]} \tag{7.22}$$

where $\Delta_S \coloneqq \mathscr{Y}_S^{\mathrm{l}} \otimes \mathscr{Y}_S^{\mathrm{r}}$.

For a pair (S_1, S_2) of spherical objects, one has the distinguished triangle

$$\hom(S_2, Y) \otimes \hom(T_{S_1}^\vee(X), S_2) \to \hom(T_{S_1}^\vee(X), Y) \to \hom(T_{S_2}^\vee \circ T_{S_1}^\vee(X), Y) \xrightarrow{[1]} . \tag{7.23}$$

Since the dual twist functor is inverse to the *twist functor* defined on objects by

$$X \mapsto T_S(X) \coloneqq \{\hom(S, X) \otimes S \to X\}, \tag{7.24}$$

the right $\mathcal{W}$-module

$$X \mapsto \hom(T_{S_1}^\vee(X), S_2) \tag{7.25}$$

is isomorphic to the right $\mathcal{W}$-module

$$X \mapsto \hom(X, T_{S_1}(S_2)), \tag{7.26}$$

so that the above distinguished triangle gives a distinguished triangle

$$\mathscr{Y}_{S_2}^{\mathrm{l}} \otimes \mathscr{Y}_{T_{S_1}(S_2)}^{\mathrm{r}} \to \Gamma_{T_{S_1}^\vee} \to \Gamma_{T_{S_2}^\vee \circ T_{S_1}^\vee} \xrightarrow{[1]} \tag{7.27}$$

of $\mathcal{W}$-bimodules. Similarly, one obtains

$$\mathscr{Y}_{S_3}^{\mathrm{l}} \otimes \mathscr{Y}_{T_{S_1} \circ T_{S_2}(S_3)}^{\mathrm{r}} \to \Gamma_{T_{S_2}^\vee \circ T_{S_1}^\vee} \to \Gamma_{T_{S_3}^\vee \circ T_{S_2}^\vee \circ T_{S_1}^\vee} \xrightarrow{[1]} \tag{7.28}$$

and so on. This gives a diagram of the form

$$\begin{array}{ccccccc} \Delta & \longrightarrow & \Gamma_{T_{S_1}^\vee} & \longrightarrow & \Gamma_{T_{S_2}^\vee \circ T_{S_1}^\vee} & \longrightarrow & \cdots \\ \nwarrow & \swarrow & \nwarrow & \swarrow & \nwarrow & & \\ & \mathscr{Y}_{S_1}^{\mathrm{l}} \otimes \mathscr{Y}_{S_1}^{\mathrm{r}}[1] & & \mathscr{Y}_{S_2}^{\mathrm{l}} \otimes \mathscr{Y}_{T_{S_1}(S_2)}^{\mathrm{r}}[1] & & \cdots & \end{array} . \tag{7.29}$$

Together with (7.9), this concludes the proof of Proposition 7.7. □

8. Symplectic cohomology of the Milnor fiber

In thi section, we recall a spectral sequence converging to $\mathrm{SH}^*\left(\check{V}\right)$ associated to a normal crossings compactification of $\check{V}$ due to McLean [51]. It is based on a standard model of Reeb flow in a neighborhood of compactification divisor and can be perceived as an elaborate version of the standard Morse-Bott model discussed in [60] when the compactification divisor is smooth. See also [29] and [16] for related results.

Let $\tilde{Y}$ be a smooth projective variety containing an affine variety with $c_1\left(\check{V}\right) = 0$ in such a way that $\check{D} \coloneqq \tilde{Y} \setminus \check{V}$ is a normal crossing divisor;

$$\check{D} = \bigcup_{i \in I} \check{D}_i. \tag{8.1}$$

For $J \subset I$, we set $\check{D}_J = \bigcap_{i \in J} \check{D}_i$, and also set $\check{D}_\emptyset = \check{V}$.

Choose a sequence $\kappa = (\kappa_i)_{i \in I}$ of positive integers such that the divisor $\sum_{i \in I} \kappa_i \check{D}_i$ on $\tilde{Y}$ is ample. Let $(c_i)_{i \in I}$ be another sequence of integers such that $\sum_{i \in I} c_i \check{D}_i$ is linearly equivalent to the canonical divisor of $\tilde{Y}$. When $\tilde{Y}$ is a Calabi–Yau manifold, one can set $c_i = 0$ for all $i \in I$.

Still following [51], for each $J \subset I$, we let $N\check{D}_J$ be a small tubular neighborhood of $\check{D}_J$ such that $N\check{D}_J \cap \check{D}_{J'}$ is a tubular neighborhood of $\check{D}_{J \cup J'}$ for all $J' \cap S$. Moreover, we require that the boundary $\partial N\check{D}_J$ intersects $\check{D}_{J'}$ for all $J' \subset I$. Next, we let

$$\overset{\circ}{N}\check{D}_J = N\check{D}_J \setminus \cup_{i \in I} \check{D}_i \tag{8.2}$$

be the punctured tubular neighborhood.

Theorem 8.1 ([51] (see also [29, Remark 3.9])). *There is a cohomological spectral sequence converging to* $\mathrm{SH}^*(\check{V})$ *with* E_1*-page given by*

$$E_1^{p,q} = \bigoplus_{\left\{(k_i) \in \mathbb{Z}_{\geq 0}^I \middle| \sum k_i \kappa_i = -p\right\}} H^{p+q-2\sum_i k_i(c_i+1)}\left(\overset{\circ}{N}\check{D}_{J_{(k_i)}}\right) \tag{8.3}$$

where $J_{(k_i)} = \{i \in I \mid k_i \neq 0\}$.

Since κ_i is positive for all i, for each p, we have $E_1^{p,q} \neq 0$ only for finitely many q, and is a finite sum of finite-dimensional vector spaces. Moreover, if $c_i > -1$ for all i, then the spectral sequence is regular.

We will apply this spectral sequence to deduce $\mathrm{SH}^1\left(\check{V}\right) = 0$, where $\check{V}$ is the Milnor fiber of a weighted homogeneous singularity.

Corollary 8.2. *If the Milnor fiber $\check{V}$ of a weighted homogeneous polynomial satisfying* (2.3) *and* $\dim \check{V} \geq 2$ *admits a compactification to a Calabi–Yau manifold by adding a normal crossing divisor, then one has* $\mathrm{SH}^i\left(\check{V}\right) = 0$ *for* $i < 0$*,* $\mathrm{SH}^0\left(\check{V}\right) = \mathbb{C}$*, and* $\mathrm{SH}^1\left(\check{V}\right) = 0$.

Proof. Since $\check{V}$ is simply connected, we do not get any contribution from $H^1(\check{V}) = 0$. The vanishing of c_i and the positivity of κ_i imply that the orbits coming from the normal crossing divisor contribute to $\mathrm{SH}^i\left(\check{V}\right)$ for $i \geq 2$. □

Now we can prove a generalization of the non-formality result in [42], which corresponds to the case $\mathbf{w} = x^2 + y^3$.

Theorem 8.3. *Under the same assumption as Corollary 8.2, $\mathcal{A}$ is not formal.*

Proof. By Corollary 3.2, we have $\mathrm{HH}^1(A) \neq 0$. On the other hand, we know by Corollary 7.6 that $\mathrm{HH}^1(\mathcal{A},\mathcal{A})$ is isomorphic to $\mathrm{SH}^1(\check{V})$, which is zero by Corollary 8.2. Hence we conclude that $\mathcal{A}$ is not formal. □

A non-zero element of $\mathrm{HH}^1(A)$ is given by the Euler derivation defined by

$$\mathrm{eu}(x) = \deg(x)x. \tag{8.4}$$

Recall that for any A_∞-algebra $\mathscr{A}$ with $H^*(\mathscr{A}) = A$, there exists a length spectral sequence converging to $\mathrm{HH}^*(\mathscr{A})$ with E_2-page given by $E_2^{p,q} = \mathrm{HH}^{p+q}(A)_q$. It is shown in [59, Equation 3.14] that the class of the Euler vector field is killed by the differential on E_2 if $\mathscr{A}$ is non-formal.

In dimension 2, Theorem 8.3 can also be proved as follows: If $\mathcal{A}$ is formal, then $\mathrm{HH}^*(\mathcal{A}) \cong \mathrm{HH}^*(Y_0)$ has a dilation since the BV operator on $\mathrm{HH}^*(Y_0)$ induced by the holomorphic volume form sends $\mathrm{eu}/2 \in \mathrm{HH}^1$ to $1 \in \mathrm{HH}^0$. On the other hand, $\mathrm{SH}^*\left(\check{V}\right)$ cannot have a dilation due to the existence of an exact Lagrangian torus in $\check{V}$ proved in [36]. Note that this argument uses that the BV operator on $\mathrm{SH}^*\left(\check{V}\right)$ agrees with BV operator on $\mathrm{HH}^*(\mathcal{A})$, which holds since any two BV operators differ by an invertible element in HH^0, which is of rank 1 in our case.

We give computations of the spectral sequence in a few examples.

8.1. **The affine quartic surface.** Let $\check{V} = \mathbf{w}^{-1}(1)$ be the Milnor fiber of the quartic polynomial $\mathbf{w}(x,y,z) = x^4+y^4+z^4$, which can be compactified to a quartic K3 surface $\tilde{Y}$ in $\mathbb{P}^3$ by adding a smooth curve $\check{D}$ of genus 3. We can take $\kappa = 1$ and $c = 0$, so that the E_1-page of the resulting spectral sequence is given in Table 8.1. We immediately conclude

					q
	$\mathbb{C}^6$	0	0	0	⋮
	$\mathbb{C}$	$\mathbb{C}$	0	0	9
	0	$\mathbb{C}^6$	0	0	8
	0	$\mathbb{C}^6$	0	0	7
	0	$\mathbb{C}$	$\mathbb{C}$	0	6
	0	0	$\mathbb{C}^6$	0	5
	0	0	$\mathbb{C}^6$	0	4
	0	0	$\mathbb{C}$	0	3
	0	0	0	$\mathbb{C}^{27}$	2
	0	0	0	0	1
	0	0	0	$\mathbb{C}$	0
p	…	-2	-1	0	

TABLE 8.1. E_1 page of the spectral sequence for $x^4+y^4+z^4$.

that $\mathrm{SH}^0(\check{V}) = \mathbb{C}$, $\mathrm{SH}^1(\check{V}) = 0$, $\mathrm{SH}^2(\check{V}) = \mathbb{C}^{28}$, $\mathrm{SH}^3(\check{V}) = \mathbb{C}^6$, and $\mathrm{SH}^i(\check{V}) = \mathbb{C}^6$ or $\mathbb{C}^7$ for $i > 3$.

8.2. **The double cover of the plane branched along a sextic.** Let $\check{V} = \mathbf{w}^{-1}(1)$ be the Milnor fiber of the polynomial $\mathbf{w}(x, y, z) = x^2 + y^6 + z^6$, which can be compactified to the double cover $\tilde{Y}$ of $\mathbb{P}^2$ branch along a smooth sextic curve by adding a smooth curve $\check{D}$ of genus 2. We can take $\kappa = 1$ and $c = 0$, so that the E_1-page of the resulting spectral sequence is given in Table 8.2. We immediately conclude that $\mathrm{SH}^0(\check{V}) = \mathbb{C}$, $\mathrm{SH}^1(\check{V}) = 0$,

					q
	$\mathbb{C}^4$	0	0	0	$\vdots$
	$\mathbb{C}$	$\mathbb{C}$	0	0	9
	0	$\mathbb{C}^4$	0	0	8
	0	$\mathbb{C}^4$	0	0	7
	0	$\mathbb{C}$	$\mathbb{C}$	0	6
	0	0	$\mathbb{C}^4$	0	5
	0	0	$\mathbb{C}^4$	0	4
	0	0	$\mathbb{C}$	0	3
	0	0	0	$\mathbb{C}^{25}$	2
	0	0	0	0	1
	0	0	0	$\mathbb{C}$	0
p	$\ldots$	-2	-1	0	

TABLE 8.2. E_1 page of the spectral sequence for $x^2 + y^6 + z^6$.

$\mathrm{SH}^2(\check{V}) = \mathbb{C}^{26}$, $\mathrm{SH}^3(\check{V}) = \mathbb{C}^4$, and $\mathrm{SH}^i(\check{V}) = \mathbb{C}^4$ or $\mathbb{C}^5$ for $i > 3$.

8.3. E_{12}**-singularity.** Let $\check{V} = \mathbf{w}^{-1}(1)$ be the Milnor fiber of the polynomial

$$\mathbf{w}(x, y, z) = x^2 + y^3 + z^7, \tag{8.5}$$

which can be compactified to a K3 surface $\tilde{Y}$. The divisor at infinity is the normal crossing union of 10 smooth rational curves, which generates the lattice $E_{10} \cong E_8 \perp U$ as described in Figure 8.1.

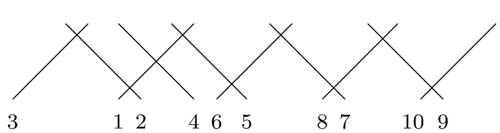

FIGURE 8.1. Divisor at infinity for $x^2 + y^3 + z^7$

The non-empty strata of intersections are $\check{D}_\emptyset$, $\check{D}_1, \check{D}_2, \dots \check{D}_{10}$, and $\check{D}_{1\,2}$, $\check{D}_{1\,4}$, $\check{D}_{1\,5}$, $\check{D}_{2\,3}$, $\check{D}_{5\,6}$, $\check{D}_{6\,7}$, $\check{D}_{7\,8}$,$\check{D}_{8\,9}$,$\check{D}_{9\,10}$. The cohomology computations for these is as follows:

$$H^*(\mathring{N}\check{D}_\emptyset) = \begin{cases} \mathbb{C} & * = 0, \\ \mathbb{C}^{12} & * = 2. \end{cases} \tag{8.6}$$

$$H^*(\mathring{N}\check{D}_1) = \begin{cases} \mathbb{C} & * = 0, \\ \mathbb{C}^{3} & * = 1, \\ \mathbb{C}^{2} & * = 2. \end{cases} \tag{8.7}$$

For $i = 3, 4, 10$

$$H^*(\mathring{N}\check{D}_i) = \begin{cases} \mathbb{C} & * = 0, \\ \mathbb{C} & * = 1. \end{cases} \tag{8.8}$$

For $i = 2, 5, 6, 7, 8, 9$

$$H^*(\mathring{N}\check{D}_i) = \begin{cases} \mathbb{C} & * = 0, \\ \mathbb{C}^{2} & * = 1, \\ \mathbb{C} & * = 2. \end{cases} \tag{8.9}$$

$$H^*(\mathring{N}\check{D}_{i\,j}) = \begin{cases} \mathbb{C} & * = 0, \\ \mathbb{C}^{2} & * = 1, \\ \mathbb{C} & * = 2. \end{cases} \tag{8.10}$$

The Nakai–Moishezon criterion implies that a divisor $D = \sum_{i=1}^{10} \kappa_i \check{D}_i$ is ample if $D.D > 0$ and $D.\check{D}_j > 0$ for all j. An example of a solution to these inequalities is given by $\kappa = (231, 153, 76, 115, 195, 160, 126, 93, 61, 30)$. The E_1-page of the spectral is given by

$$\bigoplus_{p+q=0} E_1^{p,q} = \mathbb{C}, \tag{8.11}$$

$$\bigoplus_{p+q=1} E_1^{p,q} = 0, \tag{8.12}$$

$$\bigoplus_{p+q=2} E_1^{p,q} = E_1^{0,2} \oplus \bigoplus_{i=1}^{10} E_1^{-\kappa_i,\kappa_i+2} = H^2(\check{V}) \oplus \bigoplus_{i=1}^{10} H^0(\mathring{N}\check{D}_i) = \mathbb{C}^{22} \tag{8.13}$$

$$\bigoplus_{p+q=3} E_1^{p,q} = \bigoplus_{i=1}^{10} E_1^{-\kappa_i,\kappa_i+3} = \bigoplus_{i=1}^{10} H^1(\mathring{N}\check{D}_i) = \mathbb{C}^{18} \tag{8.14}$$

$$\bigoplus_{p+q=4} E_1^{p,q} = \bigoplus_i E_1^{-\kappa_i,\kappa_i+4} \oplus \bigoplus_{i=1}^{10} E_1^{-2\kappa_i,2\kappa_i+4} \oplus \bigoplus_{i,j} E_1^{-\kappa_i-\kappa_j,\kappa_i+\kappa_j+4} \tag{8.15}$$

$$= \bigoplus_i H^2(\mathring{N}\check{D}_i) \oplus \bigoplus_{i=1}^{10} H^0(\mathring{N}\check{D}_i) \oplus \bigoplus_{i,j} H^0(\mathring{N}\check{D}_{i\,j}) = \mathbb{C}^{27}, \tag{8.16}$$

so that $\mathrm{SH}^0(\check{V}) = \mathbb{C}$, $\mathrm{SH}^1(\check{V}) = 0$, $\mathrm{SH}^2(\check{V}) = \mathbb{C}^{22}$, $\dim_{\mathbb{C}} \mathrm{SH}^3(\check{V}) \le 18$, and $\dim_{\mathbb{C}} \mathrm{SH}^4(\check{V}) \le 27$.

9. Homological mirror symmetry for affine K3 surfaces

We prove Theorem 1.4 in this section. Let $\check{V} := \{(x,y,z) \in \mathbb{C}^3 \mid x^4+y^4+z^4 = 1\}$ be the Milnor fiber of $\mathbf{w} = x^4+y^4+z^4$. A distinguished basis $(S_i)_{i=1}^{27}$ of vanishing cycles generates the compact Fukaya category of $\check{V}$, and the cohomology A of the total morphism A_∞-algebra $\mathcal{A} := \bigoplus_{i,j=1}^{27} \hom(S_i, S_j)$ is the trivial extension algebra of the tensor product $\mathfrak{A}_3 \otimes \mathfrak{A}_3 \otimes \mathfrak{A}_3$ of the Dynkin quiver $\mathfrak{A}_3$ of type A_3. The A_∞-algebra $\mathcal{A}$ is not formal by Theorem 8.3, and $\mathrm{HH}^*(\mathcal{F}(\check{V}))$ is isomorphic to $\mathrm{SH}^*(\check{V})$ computed in Section 8.1.

The graded algebra A also appears as the cohomology of the endomorphism dg algebra $\mathcal{A}_u$ of a generator $\mathcal{S}_u$ of $\operatorname{perf} Y_u$ where Y_u for $u \in U_+ := \operatorname{Spec} \mathbb{C}[u_1,u_4]$ is the quotient stack $[(\operatorname{Spec} S_u \setminus \mathbf{0})/\Gamma]$ for $S_u := \mathbb{C}[w,x,y,z]/(x^4+y^4+z^4+u_1xyzw+u_4w^4)$ and $\Gamma := \{(t_1,t_2,t_3) \in \mathbb{G}_\mathrm{m}^3 \mid t_1^4 = t_2^4 = t_3^4\}$. The moduli space $\mathcal{U}_\infty(A)$ of minimal A_∞-structures on A is identified with U_+.

In order to identify $u \in U_+$ satisfying $\mathcal{A} \simeq \mathcal{A}_u$, we compare $\mathrm{HH}^*(\mathcal{A}_u)$ and $\mathrm{HH}^*(\mathcal{A}) \cong \mathrm{SH}^*(\check{V})$ as graded vector spaces. Since $\mathrm{SH}^*(\check{V})$ is infinite-dimensional over $\mathbf{k}$, the mirror surface Y_u must be singular. Up to the action of $\mathbb{G}_\mathrm{m}$ on U_+, there are precisely two non-zero $u \in U_+$ such that Y_u is singular, i.e., $(u_1,u_4) = (1,0)$ and $(-4,1)$. The Hochschild cohomologies of these singular surfaces are computed in Sections 3.6 and 3.7. Comparing this with $\mathrm{SH}^*(\check{V})$ computed in Section 8.1, we conclude that the mirror of the $\check{V}$ is the surface associated with $(u_1,u_4) = (1,0)$.

The proof for $\check{V} := \{(x,y,z) \in \mathbb{C}^3 \mid x^2+y^6+z^6 = 1\}$ goes along the same line. The cohomology A of the total morphism A_∞-algebra of a distinguished basis of vanishing cycles is given by the degree 2 trivial extension algebra of $\mathfrak{A}_5 \otimes \mathfrak{A}_5$. The moduli space $\mathcal{U}_\infty(A)$ of minimal A_∞-structures is identified with U_+, and there are precisely two non-zero $u \in U_+$ up to the action of $\mathbb{G}_\mathrm{m}$ such that Y_u is singular. The mirror is identified with Y_u for $u = (u_1,u_6) = (1,0)$ by comparing $\mathrm{HH}^*(Y_u)$ with $\mathrm{SH}^*(\check{V})$ given in Section 8.2.

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