DERIVED EQUIVALENCES OF GENTLE ALGEBRAS VIA FUKAYA CATEGORIES

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Abstract. Following the approach of Haiden-Katzarkov-Kontsevich [10], to any homologically smooth $\mathbb{Z}$-graded gentle algebra A we associate a triple $(\Sigma_A, \Lambda_A; \eta_A)$, where Σ_A is an oriented smooth surface with non-empty boundary, Λ_A is a set of stops on $\partial\Sigma_A$ and η_A is a line field on Σ_A, such that the derived category of perfect dg-modules of A is equivalent to the partially wrapped Fukaya category of $(\Sigma_A, \Lambda_A; \eta_A)$. Modifying arguments of Johnson and Kawazumi, we classify the orbit decomposition of the action of the (symplectic) mapping class group of Σ_A on the homotopy classes of line fields. As a result we obtain a sufficient criterion for homologically smooth graded gentle algebras to be derived equivalent. Our criterion uses numerical invariants generalizing those given by Avella-Alaminos-Geiss in [5], as well as some other numerical invariants. As an application, we find many new cases when the AAG-invariants determine the derived Morita class. As another application, we establish some derived equivalences between the stacky nodal curves considered in [16].

Introduction

Given a Liouville manifold $(M, \omega = d\lambda)$, a rigorous definition of the compact Fukaya category, $\mathcal{F}(M)$, appears in the monograph [20]. This is a triangulated A_∞-category linear over some base ring $\mathbb{K}$. Roughly speaking, the objects of $\mathcal{F}(M)$ are compact, exact, oriented Lagrangian submanifolds in M, equipped with spin structures (if char$\mathbb{K} \neq 2$). The orientations on each Lagrangian determine a $\mathbb{Z}_2$-grading on $\mathcal{F}(M)$, and the spin structures enter in orienting the moduli spaces of holomorphic polygons that enter into the definition of structure constants of the A_∞ operations. It is often convenient to upgrade the $\mathbb{Z}_2$-grading on $\mathcal{F}(M)$ to a $\mathbb{Z}$-grading, which can be done under the additional assumption that $2c_1(M) = 0$ (see [15], [19]). Under this assumption, one defines a notion of a *grading structure* on M, and correspondingly considers only *graded Lagrangians* as objects of $\mathcal{F}(M)$, which now becomes a $\mathbb{Z}$-graded category. We refer to [19] for these general notions. In this paper, we focus our attention to the case where $M = \Sigma$ is punctured (real) 2-dimensional surface, equipped with an area form. A grading structure on Σ can be concretely described as a homotopy class of a section η of the projectivized tangent bundle of $\mathbb{P}(T\Sigma)$. Note that there is an effective $H^1(\Sigma)$'s worth of choices (see Sec. 1). A Lagrangian can be graded if the winding number of η along L vanishes, and in such a situation a grading is a choice of a homotopy from the tangent lift $L \to TL \subset T\Sigma$ to $\eta_{|L}$ along L. These gradings extend in a straightforward manner to the wrapped Fukaya category $\mathcal{W}(\Sigma)$ which contains $\mathcal{F}(\Sigma)$ as a full subcategory, but also allows non-compact Lagrangians in Σ and more generally, partially wrapped category $\mathcal{W}(\Sigma, \Lambda)$, as studied in [10, Sec. 2.1], where Σ is a surface with boundary and Λ is a collection of stops (i.e., marked points) on $\partial\Sigma$.

Given two graded surfaces with stops, $(\Sigma_i, \Lambda_i; \eta_i)$ for $i = 1, 2$, a homeomorphism $\phi : \Sigma_1 \to \Sigma_2$, which restricts to a bijection $\Lambda_1 \to \Lambda_2$, and a homotopy between $\phi_*(\eta_1)$ and η_2, one gets an equivalence between the partially wrapped Fukaya categories $\mathcal{W}(\Sigma_1, \Lambda_1; \eta_1)$ and $\mathcal{W}(\Sigma_2, \Lambda_2; \eta_2)$. Thus, it is important to have a set of explicit computable invariants of a line field η on a surface with boundary that determine the orbit of η under the action of the mapping class group of Σ. Our first result (see Theorem 1.2.5) gives such invariants in terms of winding numbers of η. In the most interesting case when genus is ≥ 2, the invariants consist of the winding numbers along all the boundary components, plus two more invariants, each taking values 0 and 1. The first of them is a $\mathbb{Z}_2$ valued invariant which decides whether the line field η is induced by a non-vanishing vector field, while the second is the Arf-invariant of a certain quadratic form over $\mathbb{Z}_2$. Note that from the numerical invariants of Theorem 1.2.5 one can also recover the genus of the surface and the numbers of stops on the boundary components, so if these invariants match then then the corresponding partially wrapped Fukaya categories are equivalent.

Next, we use this result to construct derived equivalences between *gentle algebras*, introduced by Assem and Skowrónski in [3]. This is a remarkable class algebras with monomial quadratic relations of special kind with a well understood structure of indecomposable modules. Furthermore, their derived categories of modules also enjoy many nice properties (see [7] and references therein). Avella-Alaminos and Geiss [5] gave a combinatorial definition of derived invariants of finite-dimensional gentle algebras, which form a collection of pairs of non-negative integers (m, n) with multiplicities. We refer to these as *AAG-invariants*. It is known that these invariants do not completely determine the derived Morita class of a gentle algebra in general (for example, see [1]).

We consider $\mathbb{Z}$-graded gentle algebras and their perfect derived categories (the classical case corresponds to algebras concentrated in degree 0). For such an algebra A, we denote by $D(A)$ the perfect derived category of dg-modules over A viewed as a dg-algebra with zero differential. The category $D(A)$ has a natural dg-enhancement which we take into account when talking about equivalences involving $D(A)$.

The connection between graded gentle algebras and Fukaya categories was established by Haiden, Katzarkov and Kontsevich in [10] (cf. [6]): they constructed collections of formal generators in some partially wrapped Fukaya categories whose endomorphism algebras are graded gentle algebras. In Theorem 3.2.2 we give an inverse construction[1]: starting from a homologically smooth graded gentle algebra A we construct a graded surface with stops $(\Sigma_A, \Lambda_A; \eta_A)$ together with a set formal generators whose endomorphism algebra is isomorphic to A. This leads to an equivalence of the partially wrapped Fukaya category $\mathcal{W}(\Sigma, \Lambda)$ with the derived category $D(A)$. In addition, we generalize the combinatorial definition of AAG-invariants to possibly infinite-dimensional graded gentle algebras and show that they can be recovered from the winding numbers of η_A along all boundary components.

[1]The existence of such construction is mentioned in [10]

Now recalling our numerical invariants of graded surfaces with stops from Theorem 1.2.5 we obtain a sufficient criterion for derived equivalence between homologically smooth graded gentle algebras. Namely, if we start with two such algebras A and A' and find that the corresponding invariants from Theorem 1.2.5, determined by winding numbers of η_A and $\eta_{A'}$, coincide then we get a derived equivalence between A and A'. Note that this involves checking that A and A' have the same AAG-invariants, and in addition that two more invariants with values in $\{0, 1\}$ match.

As an application, using the above approach we obtain a sufficient criterion for derived equivalence of homologically smooth graded gentle algebras given purely in terms of AAG-invariants (see Corollary 3.2.5). Using Koszul duality, we also get a sufficient criterion for derived equivalence of finite-dimensional gentle algebras with grading in degree 0 (see Corollary 3.2.6).

In a different direction, we construct derived equivalences between stacky nodal curves studied in [16], Namely, these are either chains or rings of weighted projective lines glued to form stacky nodes, locally modelled by quotients $(xy = 0)/(x, y) \sim (\zeta^k x, \zeta y)$, where $\zeta^r = 1$ and $k \in (\mathbb{Z}/r)^*$. In [16, Thm. B] we constructed an equivalence of the derived category of coherent sheaves on such a stacky curve with the partially wrapped Fukaya category of some graded surface with stops (this can be viewed as an instance of homological mirror symmetry). Thus, using Theorem 1.2.5 we get many nontrivial derived equivalences between our stacky curves. In the case of balanced nodes (those with $k = -1$) we recover the equivalences between tcnc curves from [21].

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1. Line fields on surfaces

1.1. **Basics on line fields.** Let Σ be an oriented smooth surface of genus $g(\Sigma)$ with non-empty boundary with connected components $\partial\Sigma = \bigsqcup_{i=1}^{b} \partial_i\Sigma$. The mapping class group of Σ is

$$\mathcal{M}(\Sigma) = \pi_0(Homeo^+(\Sigma, \partial\Sigma)),$$

where $Homeo^+(\Sigma, \partial\Sigma)$ is the space of orientation preserving homeomorphism of Σ which are the identity pointwise on $\partial\Sigma$.

Definition 1.1.1. An (unoriented) line field η on Σ is a section of the projectivized tangent bundle $\mathbb{P}(T\Sigma)$. We denote by

$$G(\Sigma) = \pi_0(\Gamma(\Sigma, \mathbb{P}(T\Sigma)))$$

the set of homotopy classes of unoriented line fields.

A non-vanishing vector field i.e. a section of the unit tangent bundle $\mathbb{S}\Sigma$ induces a line field via the bundle map $\mathbb{S}\Sigma \to \mathbb{P}(T\Sigma)$ which is a fibrewise double covering. However, not all line fields come from non-vanishing vector fields: a section of $\mathbb{P}(T\Sigma)$ may not lift to a section of $\mathbb{S}\Sigma$.

The trivial circle fibration $S^1 \xrightarrow{\iota} \mathbb{P}(T\Sigma) \xrightarrow{p} \Sigma$ induces an exact sequence

$$0 \to H^1(\Sigma) \xrightarrow{p^*} H^1(\mathbb{P}(T\Sigma)) \xrightarrow{\iota^*} H^1(S^1) \to 0$$

A line field η determines a class $[\eta] \in H^1(\mathbb{P}(T\Sigma))$ such that $\iota^*[\eta]([S^1]) = 1$ by taking the Poincaré-Lefschetz dual of the class of the image $[\eta(\Sigma)] \subset H_2(\mathbb{P}(T\Sigma), \partial\mathbb{P}(T\Sigma))$. Via this construction, we get an identification

$$G(\Sigma) = (\iota^*)^{-1}(\zeta) \subset H^1(\mathbb{P}(T\Sigma)),$$

where $\zeta \in H^1(S^1)$ is the generator which integrates to 1 along S^1. Thus, the set $G(\Sigma)$ is a torsor over $H^1(\Sigma)$. We denote the corresponding action of $c \in H^1(\Sigma)$ on $G(\Sigma)$ by $\eta \mapsto \eta + c$, where

$$[\eta + c] = [\eta] + p^* c.$$

The mapping class group $\mathcal{M}(\Sigma)$ acts on $G(\Sigma)$ on the right. Our goal in this section is to understand the orbit decomposition of $G(\Sigma)$ with respect to this action.

Given an immersed curve $\gamma : S^1 \to \Sigma$, one can consider its tangent lift $\tilde{\gamma} : S^1 \to \mathbb{P}(T\Sigma)$ given by $(\gamma, T\gamma)$, where $T\gamma$ is the tangent space to the curve γ.

Definition 1.1.2. Given a line field η and an immersed curve γ, define the winding number of γ with respect to η to be

$$w_\eta(\gamma) := \langle [\eta], [\tilde{\gamma}] \rangle,$$

where $\langle\ ,\ \rangle : H^1(\mathbb{P}(T\Sigma)) \times H_1(\mathbb{P}(T\Sigma)) \to \mathbb{Z}$ is the natural pairing.

The winding number $w_\eta(\gamma)$ with respect to η only depends on the homotopy class of η and the regular homotopy class of γ. From the definition we immediately get the following compatibility with the action of $H^1(\Sigma)$:

$$w_{\eta+c}(\gamma) = w_\eta(\gamma) + \langle c, [\gamma] \rangle.$$

Throughout, $\partial\Sigma$ is oriented with respect to the natural orientation as the boundary of Σ. In particular, $w_\eta(\partial\mathbb{D}^2) = 2$ for the unique homotopy class of line fields on $\mathbb{D}^2$. For a boundary component $B \subset \partial\Sigma$ with the opposite orientation, we write $-B$. Then, we have $w_\eta(-B) = -w_\eta(B)$.

1.2. **Invariants under the action of the mapping class group.** The winding numbers along boundary components of Σ gives the first set of invariants of elements of $G(\Sigma)$.

Definition 1.2.1. Let Σ be a surface with boundary $\partial\Sigma = \sqcup_{i=1}^d \partial_i\Sigma$. The *boundary invariants of a line field* η are the numbers

$$r_i(\eta) := w_\eta(\partial_i\Sigma) + 2 \quad \text{for } i = 1, \dots b.$$

They depend only on the homotopy class of η and are invariant under the action of the mapping class group $\mathcal{M}(\Sigma)$.

To go further, we need to study the winding numbers along non-separating curves on Σ. As is well-known, the winding number invariants do not descend to a map from $H_1(\Sigma)$. Indeed, if $S \subset \Sigma$ is a compact subsurface with boundary $\partial S = \bigsqcup_{i=1}^n \partial_i S$, by Poincaré-Hopf index theorem (see [11, Ch. 3]), we have:

$$\sum_{i=1}^{n} w_\eta(\partial_i S) = 2\chi(S) \tag{1.1}$$

However, considering the reduction modulo 2 we still get a well-defined homomorphism:

$$[w_\eta]^{(2)} : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$$

i.e an element $H^1(\Sigma; \mathbb{Z}_2)$.

Definition 1.2.2. We define the $\mathbb{Z}_2$-valued invariant

$$\sigma : \mathbb{G}(\Sigma) \to \mathbb{Z}_2$$
$$\eta \mapsto \begin{cases} 0 \text{ if } [w_\eta]^{(2)} = 0 \\ 1 \text{ otherwise} \end{cases}$$

We have a natural inclusion induced map

$$i : H_1(\partial\Sigma; \mathbb{Z}_2) \cong \mathbb{Z}_2^b \to H_1(\Sigma; \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g+b-1}.$$

The cokernel of i is isomorphic to $\mathbb{Z}_2^{2g}$ and comes equipped with a non-degenerate intersection pairing.

Note that the numbers $r_i(\eta) \bmod 2$ are precisely the values of $[w_\eta]^{(2)}$ on the boundary cycles. Thus, if $r_i(\eta)$ is odd for at least one i then $\sigma(\eta) = 1$. If all $r_i(\eta)$ are even then we can check whether $\sigma(\eta) = 0$ by looking at the winding numbers of a collection of cycles projecting to a basis of the cokernel of i.

Proposition 1.2.3. *Suppose η is a line field on Σ defined by the class $[\eta] \in H^1(\mathbb{P}(T\Sigma))$. There is well defined map*

$$q_\eta : H_1(\Sigma; \mathbb{Z}_4) \to \mathbb{Z}_4$$

given by

$$q_\eta(\sum_{i=1}^{m} \alpha_i) = \sum_{i=1}^{m} w_\eta(\alpha_i) + 2m \in \mathbb{Z}_4,$$

where α_i are simple closed curves. It satisfies

$$q_\eta(a+b) = q_\eta(a) + q_\eta(b) + 2(a \cdot b) \in \mathbb{Z}_4$$

where $a, b \in H_1(\Sigma; \mathbb{Z}_4)$, and $a \cdot b$ denotes the intersection pairing on $H_1(\Sigma; \mathbb{Z}_4)$.

Proof. In the case when η comes from a vector field v, we have $w_\eta(a) = 2w_v(a)$, where $w_v(\cdot)$ is the winding number of the vector field. Hence, the assertion in this case follows from [12, Thm 1A, Thm 1B]. In general we have $[\eta] = \eta_0 + c$, for some $c \in H^1(\Sigma)$. Thus, the function $q_\eta(a) := q_{\eta_0}(a) + \langle c, a \rangle$ has the claimed properties. □

Lemma 1.2.4. *Suppose that $g(\Sigma) \geq 2$. Assume that line fields η and θ have $r_i(\eta) = r_i(\theta)$ for $i = 1, \ldots, d$, and $q_\eta = q_\theta$. Then their homotopy classes lie in the same $\mathcal{M}(\Sigma)$-orbit.*

Proof. The assumption $q_\eta = q_\theta$ implies that $w_\eta(a) \equiv w_\theta(a) \bmod 4$ for any $a \in H_1(\Sigma)$. Thus, we have $\theta = \eta + 4c$ for some $c \in H^1(\Sigma)$. Furthermore, the condition $r_i(\eta) = r_i(\theta)$ implies that c has zero restriction to $H_1(\partial\Sigma)$. Hence, there exists $\alpha \in H_1(\Sigma)$, such that $\langle c, \gamma\rangle = (\alpha \cdot \gamma)$ for any $\gamma \in H_1(\Sigma)$. Now the fact that η and θ lie in the same $\mathcal{M}(\Sigma)$-orbit is proved in exactly the same way as in the proof of [13, Thm. 2.5]. □

Thus, the study of the $\mathcal{M}(\Sigma)$-orbits on $G(\Sigma)$ reduces to the study of $\mathcal{M}(\Sigma)$-orbits on the set of functions $q : H_1(\Sigma, \mathbb{Z}_4) \to \mathbb{Z}_4$ satisfying

$$q(a + b) = q(a) + q(b) + 2(a \cdot b). \tag{1.2}$$

Let us denote by $\mathrm{Quad}_4 = \mathrm{Quad}_4(\Sigma)$ the set of all such functions (it is an $H^1(\Sigma, \mathbb{Z}_4)$-torsor).

Recall that given a symplectic vector space $V, (- \cdot -)$ over $\mathbb{Z}_2$, one can consider the set $\mathrm{Quad}(V)$ of quadratic forms $\overline{q} : V \to \mathbb{Z}_2$ satisfying

$$\overline{q}(x + y) = \overline{q}(x) + \overline{q}(y) + (x \cdot y). \tag{1.3}$$

For every $\overline{q} \in \mathrm{Quad}(V)$, the Arf-invariant ([2],[8]) is the element of $\mathbb{Z}_2$ given by

$$A(\overline{q}) = \sum_{i=1}^{n} \overline{q}(a_i)\overline{q}(b_i),$$

where (a_i, b_i) is a symplectic basis of V. The Arf invariant is the value that $\overline{q}$ attains on the majority of vectors in V.

In the case when $r_i(\eta) = w_\eta(\partial_i\Sigma) + 2 \in 4\mathbb{Z}$ for every $i = 1, \ldots, d$, and the quadratic function $q = q_\eta$ takes values in $2\mathbb{Z}_4$, we can associate to q a certain quadratic form on a $\mathbb{Z}_2$-vector space, and its Arf-invariant will give us an additional invariant of η modulo the mapping class group action.

Let us set $H := H_1(\Sigma, \mathbb{Z}_4)$, $K = \mathrm{im}(i_* : H_1(\partial\Sigma, \mathbb{Z}_4) \to H_1(\Sigma, \mathbb{Z}_4))$, $\overline{H} = H/2H$, $\overline{K} = K/2K$. Since K lies in the kernel of the intersection pairing, for any $q \in \mathrm{Quad}_4$ the restriction $q|_K$ is a homomorphism $K \to \mathbb{Z}_4$. Note that for $q = q_\eta$ the value of this homomorphism on $[\partial_i\Sigma]$ is $r_i(\eta) \bmod 4$.

Now let $q \in \mathrm{Quad}_4$ be such that $q|_K$ is zero. Then it is easy to see that q descends to a well defined function $q_{H/K}$ on H/K. Assume in addition that $\sigma(\eta) = 0$, i.e., q takes values in $2\mathbb{Z}_4$. In this case we have $q_{H/K} = 2\overline{q}$, where $\overline{q}$ is a function $H/K \to \mathbb{Z}_2$ satisfying (1.3). It is easy to see that $\overline{q}(x+2y) = \overline{q}(x)$, so $\overline{q}$ can be viewed as a $\mathbb{Z}_2$-valued quadratic form on $\overline{H}/\overline{K} \simeq \mathbb{Z}_2^{2g}$. Thus, $\overline{q}$ is an element of $\mathrm{Quad}(\overline{H}/\overline{K})$ and we define $A(\eta)$ as the Arf-invariant of $\overline{q}$.

Theorem 1.2.5. (i) *Suppose $g(\Sigma) = 0$, then the action of $\mathcal{M}(\Sigma)$ on $G(\Sigma)$ is trivial. Moreover, two line fields η and θ are homotopic if and only if*

$$w_\eta(\partial_i\Sigma) = w_\theta(\partial_i\Sigma) \quad \text{for all } i = 1, \ldots d.$$

(ii) *Suppose* $g(\Sigma) = 1$. *Then two line fields* η *and* θ *are in the same* $\mathcal{M}(\Sigma)$*-orbit if and only if*

$$w_\eta(\partial_i\Sigma) = w_\theta(\partial_i\Sigma) \quad \textit{for all } i = 1, \dots d.$$

and

$$\widetilde{A}(\eta) = \widetilde{A}(\theta) \in \mathbb{Z}_{\geq 0},$$

where

$$\widetilde{A}(\eta) := \gcd(\{w_\eta(\gamma) : \gamma \textit{ non-separating}\}) = \gcd(\{w_\eta(\alpha), w_\eta(\beta), w_\eta(\partial_1\Sigma) + 2, \dots, w_\eta(\partial_d\Sigma) + 2\}).$$

Here α, β *are simple curves such that* $[\alpha]$ *and* $[\beta]$ *project to a basis of* $H_1(\Sigma)/\operatorname{im}(i_*)$.

(iii) *Suppose* $g(\Sigma) \geq 2$. *Then two line fields* η *and* θ *are in the same* $\mathcal{M}(\Sigma)$ *orbit if and only if the following conditions are satisfied:*
  (1) $w_\eta(\partial_i\Sigma) = w_\theta(\partial_i\Sigma)$ *for all* $i = 1, \dots d$;
  (2) $\sigma(\eta) = \sigma(\theta)$ *(this only needs to be checked if all* $w_\eta(\partial_i\Sigma)$ *are even);*
  (3) *if* $w_\eta(\partial_i\Sigma) = w_\theta(\partial_i\Sigma) \in 2 + 4\mathbb{Z}$ *and* $\sigma(\eta) = \sigma(\theta) = 0$ *then additionally one must have*

$$A(\eta) = A(\theta),$$

  where A *is an Arf invariant defined above.*

Proof. (i) This follows immediately from the fact that $G(\Sigma)$ is an $H^1(\Sigma)$-torsor and the boundary curves $\partial_i\Sigma$ generate the group $H_1(\Sigma)$.
(ii) This is proved in the same way as Theorem 2.8 in [13].
(iii) We need to prove that if the invariants match then η and θ are in the same $\mathcal{M}(\Sigma)$-orbit. Note that $\sigma(\eta)$ is determined by whether the quadratic function q_η is trivial modulo 2 or not. By Lemma 1.2.4, it is enough to prove that the quadratic functions q_η and q_θ are in the same $\mathcal{M}(\Sigma)$-orbit.

First, let us analyze the result of the action of a transvection

$$T_a(x) = x + (a \cdot x)a$$

on quadratic functions in Quad_4. We have

$$q(T_a(x)) = q(x) + (a \cdot x)q(a) + 2(a \cdot x)(x \cdot a) = q(x) + (q(a) + 2)(a \cdot x). \tag{1.4}$$

In particular, if $q(a) = -1$ then $qT_a = q + (a \cdot ?)$.

If $q, q' \in \mathrm{Quad}_4$ have $q|_K = q'|_K$ then $(q' - q)$ is a homomorphism $H \to \mathbb{Z}_4$, vanishing on K, hence it has form $x \mapsto (a \cdot x)$ for some $a \in H$.

Assume now that $q \in \mathrm{Quad}_4$ is such that $q|_K$ is surjective, i.e., the reduction of $q|_K$ modulo 2 is nonzero. Then we claim that any $q' \in \mathrm{Quad}_4$ with $q'|_K = q|_K$ lies in the $\mathcal{M}(\Sigma)$-orbit of q. Indeed, we have $q' - q = (a \cdot ?)$ for some $a \in H$. By surjectivity of $q|_K$ we can find $k \in K$ such that $q(k) = -1 - q(a)$, i.e., $q(a + k) = -1$. Then from (1.4) we get

$$qT_{a+k} = q'.$$

Next, let us consider $q \in \mathrm{Quad}_4$ such that $q|_K$ takes values in $2\mathbb{Z}_4$. Assume also that $q \bmod 2 \neq 0$. We claim that in this case the $\mathcal{M}(\Sigma)$-orbit of q is determined by $q|_K$. Note that $q \bmod 2$ is a homomorphism $H \to \mathbb{Z}_2$ trivial on K, so it is an element of $\mathrm{Hom}(H/K, \mathbb{Z}_2)$. Since $\mathcal{M}(\Sigma)$ acts transitively on nonzero elements in $\mathrm{Hom}(H/K, \mathbb{Z}_2)$, it is enough to prove that if $q' \equiv q \bmod 2$ and $q'|_K = q|_K$ then q' and q are in the same $\mathcal{M}(\Sigma)$-orbit. As before we deduce that $q' - q = 2(a\cdot ?)$ for some $a \in H$. If $q(a) \equiv 1 \bmod 2$ then this immediately gives $q' = qT_a^2$. On the other hand, if $q'(a) \equiv q(a) \equiv 0 \bmod 2$ then for any element b with $q(b) \equiv 1 \bmod 2$ we have

$$qT_{a+b}^2 = q + 2((a+b)\cdot ?) = q' + 2(b\cdot ?) = q'T_b^2,$$

so q' and q are in the same orbit.

Finally, if q takes values in $2\mathbb{Z}_4$ then we have $q = 2\overline{q}$ for a quadratic form $\overline{q}$ on $\overline{H}$ satisfying (1.3), and we can use the description of $\mathcal{M}(\Sigma)$-orbits on such forms from [13, Thm. 1.3]. □

Remark 1.2.6. 1. It follows from (1.1) that the genus of the surface is determined by the boundary invariants of η via the formula

$$4 - 4g(\Sigma) = \sum_{i=1}^{d} (w_\eta(\partial_i \Sigma) + 2). \tag{1.5}$$

2. In the case $g(\Sigma) = 1$, let α, β be the standard non-separating curves in Σ. Then, it can be shown as in [13, Lemma 2.6] that

$$\gcd(\{w_\eta(\gamma) : \gamma \text{ non-separating }\}) = \gcd(\{w_\eta(\alpha), w_\eta(\beta), w_\eta(\partial_1\Sigma) + 2, \dots, w_\eta(\partial_d\Sigma) + 2\})$$

We also note that in the case $d = 1$, $w_\eta(\partial\Sigma) = -2$, hence this invariant reduces to $\gcd(w_\eta(\alpha), w_\eta(\beta))$ considered in [1].
3. In the case $\sigma(\eta) = 0$, the line field η is induced by a non-vanishing vector field v. This induces a spin structure on the surface Σ (by considering its mod 2 reduction). The condition that $w_\eta(\partial_i\Sigma) \in 2 + 4\mathbb{Z}$ means that this spin structure extends to a spin structure on the compact surface obtained from Σ by capping off the boundaries with a disk. Now, it is a theorem of Atiyah [4] (see also [12]) that the action of the mapping class group on the spin structures on a compact Riemann surface has exactly 2 orbits distinguished by the Arf invariant.

2. Partially wrapped Fukaya categories

The partially wrapped Fukaya category $\mathcal{W}(\Sigma, \Lambda; \eta)$ (with coefficients in a field $\mathbb{K}$) is associated to a graded surface $(\Sigma, \Lambda; \eta)$, where Σ is a connected compact surface with non-empty boundary $\partial\Sigma$, $\Lambda \subset \partial\Sigma$ is a collection of marked points called *stops*, and η is a line field on Σ. There is a combinatorial description of $\mathcal{W}(\Sigma, \Lambda; \eta)$ provided in [10]. A set of pairwise disjoint and non-isotopic Lagrangians $\{L_i\}$ in $\Sigma \backslash \Lambda$ generates the partially wrapped Fukaya category $\mathcal{W}(\Sigma, \Lambda; \eta)$ as a triangulated category if the complement of the Lagrangians

$$\Sigma \setminus \{\bigsqcup_i L_i\} = \bigcup_f D_f$$

is a union of disks D_f each of which has exactly one stop on its boundary. Figure 1 illustrates how each D_f may look like, where the blue arcs are in $\bigsqcup_i L_i$ while the black arcs lie in $\partial\Sigma$.

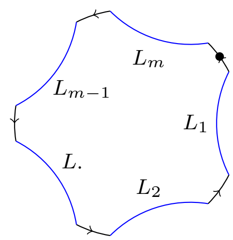


FIGURE 1. An example of a disk D_f

Furthermore, in this case, the associative $\mathbb{K}$-algebra

$$A_{L_\bullet} := \bigoplus_{i,j} \hom(L_i, L_j)$$

is formal, and it can be described by a graded gentle algebra (see Def. 3.1.1). The generators of this quiver can easily be described following the flow lines corresponding to rotation around the boundary components of Σ connecting the Lagrangians. Note that each boundary component of Σ is an oriented circle (where the boundary orientation is induced by the area form on Σ). Specifically, a flowline that goes from L_j to L_i gives a generator for $\hom(L_i, L_j)$ (note the reversal of indices). The data of Λ enters by disallowing flows that pass through a marked point. The algebra structure is given by concatenation of flow lines. Given $\alpha_i \in \hom(L_i, L_{i+1})$ for $i = 1, \ldots, n$, we write

$$\alpha_n \alpha_{n-1} \ldots \alpha_1 \in \hom(L_1, L_n)$$

for their product, read from right-to-left, and if non-zero, this expression corresponds to a flow from L_n to L_1.

Finally, the line field η is used to grade the morphism spaces. A convenient way to determine the line field η is by describing its restrictions along each of the disks D_f. Each such disk is as in Figure 1. Different disks are glued along the curves L_i (the blue parts in their boundary). Changing a line field by homotopy, we can arrange that it is tangent to L_i (as L_i are contractible). Thus, every line field on Σ (up to homotopy) can be glued out of such line fields on the disks D_f.

Note that if we have an embedded segment $c \subset \Sigma$ and a line field η, which is transversal to c at the ends p_1, p_2 of c, then we can define the winding number $w_\eta(c)$ (first, one can trivialize $T\Sigma$ along c in such a way that the tangent line to c is constant, then count the number of times (with sign) η coincides with the tangent line to c along c. An equivalent definition is given in [10, Sec. 3.2]). Now a line field on a disk D_f, tangent to $\{L_i\}$, is determined (up to homotopy) by the integers θ_i, for $i = 1, \ldots, m$, given by its winding numbers along the boundary parts on $\partial\Sigma$ (the black parts in Figure 1). By definition, these numbers are the degrees of the corresponding morphisms in the wrapped Fukaya category.

The numbers θ_i can be chosen arbitrarily subject to the constraint

$$\sum_{i=1}^{m} \theta_i = m - 2. \tag{2.1}$$

This last constraint is the topological condition that needs to be satisfied in order for the line field to extend to the interior of the disk. (Note that the stops do not play a role in this discussion.)

The gentle algebra $A_{L_\bullet}$ is always homologically smooth since so is $\mathcal{W}(\Sigma, \Lambda; \eta)$. The algebra $A_{L_\bullet}$ is proper (i.e., finite-dimensional) if and only if there is at least one marked point on every boundary component. The "if" part is [10, Cor. 3.1]. On the other hand, if there is a boundary component with no stops, then we can compose flows along this boundary indefinitely, so $A_{L_\bullet}$ is not proper.

In what follows, it will be convenient to consider $A^{op}_{L_\bullet}$ as a quiver algebra $\mathbb{K}Q/I$, so that flow lines from L_i to L_j correspond to arrows from the i^{th} vertex to j^{th} vertex. Note that the collection $\{L_i\}$ generates the partially wrapped Fukaya category $\mathcal{W}(\Sigma, \Lambda; \eta)$. Therefore, we have an equivalence

$$D(A^{op}_{L_\bullet}) \cong \mathcal{W}(\Sigma, \Lambda; \eta),$$

where the category on the left denotes the bounded derived category of perfect (left) dg-modules over $A^{op}_{L_\bullet}$.

3. Gentle algebras and Fukaya categories

3.1. **Graded gentle algebras and AAG-invariants.** A quiver is a tuple $Q = (Q_0, Q_1, s, t)$ where Q_0 is the set of vertices, Q_1 is the set of arrows, $s, t : Q_1 \to Q_0$ is the functions that determine the source and target of the arrows. We always assume Q to be finite. A *path* in Q is a sequence of arrows $\alpha_n \dots \alpha_2\alpha_1$ such that $s(\alpha_{i+1}) = t(\alpha_i)$ for $i = 1, \dots, (n-1)$. A *cycle* in Q is a path of length ≥ 1 in which the beginning and the end vertices coincide but otherwise the vertices are distinct. For $\mathbb{K}$ a field, let $\mathbb{K}Q$ be the path algebra, with paths in Q as a basis and multiplication induced by concatenation. Note that the source s and target t maps have obvious extensions to paths in Q.

Definition 3.1.1. A *gentle algebra*[2] $A = \mathbb{K}Q/I$ is given by a quiver Q with relations I such that

(1) Each vertex has at most two incoming and at most two outgoing edges.
(2) The ideal I is generated by composable paths of length 2.
(3) For each arrow α, there is at most one arrow β such that $\alpha\beta \in I$ and there is at most one arrow β such that $\beta\alpha \in I$.
(4) For each arrow α, there is at most one arrow β such that $\alpha\beta \notin I$ and there is at most one arrow β such that $\beta\alpha \notin I$.

In addition, we always assume Q to be connected.

[2]Our terminology is the same as in [18], so we do not impose the condition of finite-dimensionality in the definition of a gentle algebras. What we call "gentle algebra" is sometimes referred to as "locally gentle algebra".

We will consider $\mathbb{Z}$-graded gentle algebras, i.e., every arrow in Q should have a degree assigned to it. For a $\mathbb{Z}$-graded gentle algebra A we denote by $D(A)$ the derived category of perfect dg-modules over A, where A is viewed as a dg-algebra with its natural grading and zero differential.

Remark 3.1.2. Note that $D(A)$ is different from the derived category of graded A-modules. On the other hand, if the grading of A is zero then $D(A)$ is equivalent to the perfect derived category of ungraded A-modules.

Lemma 3.1.3. *(i) A gentle algebra is homologically smooth if and only if there are no* forbidden cycles *i.e. cycles* $\alpha_n \dots \alpha_2\alpha_1$ *in* $\mathbb{K}Q$ *such that* $\alpha_{i+1}\alpha_i \in I$ *for* $i \in \mathbb{Z}/n$.
(ii) A gentle algebra is proper (i.e., finite-dimensional) if and only if there are no permitted cycles *i.e. paths* $\alpha_n \dots \alpha_2\alpha_1$ *in* $\mathbb{K}Q$ *such that* $\alpha_{i+1}\alpha_i \notin I$ *for* $i \in \mathbb{Z}/n$.

Proof. The "if" direction is proved in [10, Prop. 3.4]. The "only if" for properness is well known. It remains to prove that if a gentle algebra A is homologically smooth then there are no forbidden cycles. Since A is homologically smooth, the diagonal bimodule is perfect dg-module over $A^{op} \otimes A$. Thus, for every simple A-module S (corresponding to one of the vertices), we get a quasi-isomorphism of S with a perfect dg-module over A. It follows that the space $\mathrm{Ext}^*_{A-\mathrm{dgmod}}(S,S)$ is finite-dimensional. Equivalently, the space $\mathrm{Ext}^*_A(S,S)$, computed in the category of ungraded A-modules, is finite-dimensional (see [17, Thm. 1.3.3]). But the latter space can be computed using the standard Koszul complex, and the presence of forbidden cycles would mean that for some S the space $\mathrm{Ext}^*_A(S,S)$ is infinite-dimensional. □

We will use the following notions from [5].

Definition 3.1.4. A *forbidden path* is a path in Q of the form

$$f = \alpha_{n-1} \dots \alpha_2\alpha_1 \in \mathbb{K}Q$$

such that all (α_i) are distinct and for all $i = 1, \dots, (n-2)$, $\alpha_{i+1}\alpha_i \in I$. It is a *forbidden thread* if for all $\beta \in Q_1$ neither $\beta\alpha_n \dots \alpha_2\alpha_1$ nor $\alpha_n \dots \alpha_2\alpha_1\beta$ is a forbidden path. In addition, if $v \in Q_0$ with $\#\{\alpha \in Q_1 | s(\alpha) = v\} \le 1, \#\{\alpha \in Q_1 | t(\alpha) = v\} \le 1$, then we consider the idempotent e_v as a (trivial) forbidden thread in the following cases:

- either there are no α with $s(\alpha) = v$ or there are no α with $t(\alpha) = v$;
- we have $\beta, \gamma \in Q_1$ with $s(\gamma) = v = t(\beta)$ and $\gamma\beta \in I$.

The *grading* of a forbidden thread is defined by

$$|f| = \sum_{i=1}^{n-1} |\alpha_i| - (n-2).$$

Definition 3.1.5. A *permitted path* is a path in Q of the form

$$p = \alpha_n \dots \alpha_2\alpha_1 \in \mathbb{K}Q$$

such that all (α_i) are distinct and for all $i = 1, \dots, (n-1)$, $\alpha_{i+1}\alpha_i \notin I$, and it is a *permitted thread* if for all $\beta \in Q_1$ neither $\beta\alpha_n \dots \alpha_2\alpha_1$ nor $\alpha_n \dots \alpha_2\alpha_1\beta$ is a permitted path. In addition, if $v \in Q_0$ with $\#\{\alpha \in Q_1 | s(\alpha) = v\} \le 1, \#\{\alpha \in Q_1 | t(\alpha) = v\} \le 1$, then we consider the idempotent e_v as a (trivial) permitted thread in the following cases:

- either there are no α with $s(\alpha) = v$ or there are no α with $t(\alpha) = v$;
- we have $\beta, \gamma \in Q_1$ with $s(\gamma) = v = t(\beta)$ and $\gamma\beta \notin I$.

The *grading* of a permitted thread is defined by

$$|p| = -\sum_{i=1}^{n} |\alpha_i|.$$

Remark 3.1.6. Inclusion of the idempotents as forbidden and permitted threads ensures that every vertex appears in exactly two forbidden threads/cycles and exactly two permitted threads/cycles.

Definition 3.1.7. For a gentle algebra A, a *combinatorial boundary component of type I* is an alternating cyclic sequence of forbidden and permitted threads:

$$b = p_n f_n \dots p_2 f_2 p_1 f_1$$

such that $s(f_i) = s(p_i)$ for $i \in \mathbb{Z}/n$, and $t(p_i) = t(f_{i+1})$ for $i \in \mathbb{Z}/n$ with the following condition:

($\star$) For each $i \in \mathbb{Z}/n$, if $f_{i+1} = \alpha_k \dots \alpha_1$, $p_i = \beta_m \dots \beta_1$, and $f_i = \gamma_n \dots \gamma_1$ such that $s(f_i) = s(p_i)$ and $t(p_i) = t(f_{i+1})$, we have

$$\gamma_1 \neq \beta_1 \text{ and } \beta_m \neq \alpha_k.$$

The winding number associated to a combinatorial boundary component b of type I is defined to be

$$w(b) := \sum_{i=1}^{r} (|p_i| + |f_i|).$$

We also denote the number n of forbidden threads in b as $n(b)$.

A *combinatorial boundary component of type II* (that can appear only if A is not proper) is simply a permitted cycle

$$pc = \alpha_m \dots \alpha_1.$$

The winding number associated to such a cycle is

$$w(pc) := -\sum_{i=1}^{m} |\alpha_i|.$$

A *combinatorial boundary component of type II'* (that can appear only if A is not homologically smooth) is simply a forbidden cycle

$$fc = \alpha_m \dots \alpha_1.$$

The winding number associated to such a cycle is

$$w(fc) := \sum_{i=1}^{m} |\alpha_i| - m.$$

For combinatorial boundary components of types II and II' we set $n(b) = 0$.

Lemma 3.1.8. *Let A be a proper gentle algebra, with grading in degree zero. Then the collection of pairs $(n(b), n(b) - w(b))$, over all combinatorial boundary components (taken with multiplicities) coincides with AAG-invariants of A.*

Proof. This follows directly from the description of the AAG-invariants in [5, Sec. 3]. Note that the pair $(0, m)$ in Step (3) of the algorithm of [5, Sec. 3] associated to a forbidden cycle $fc = \alpha_m \dots \alpha_1$ match with the pair $(0, w(fc))$ associated with the corresponding combinatorial component of type II'. Indeed, $w(fc) = m$ since the grading of A is in degree 0. □

From now on we will always assume that our gentle algebras are homologically smooth, with the exception of Remark 3.2.10.

Motivated by Lemma 3.1.8 we extend the definition of the AAG-invariants to graded gentle algebras.

Definition 3.1.9. For a graded gentle algebra A we define the *AAG-invariants* to be the collection of pairs $(n(b), n(b) - w(b))$, taken with multiplicities, where b runs over all combinatorial boundary components of A.

3.2. **Relation to Fukaya categories.** The definition of the combinatorial boundary component for a gentle algebra is motivated by the following proposition:

Proposition 3.2.1. *Suppose Σ is a surface with a collection of marked points $\Lambda \subset \partial\Sigma$, and a line field η. Let $\{L_i\}$ be a collection of Lagrangians such that the complement of $\bigsqcup_i L_i$ is a union of disks each of which has exactly one stop on its boundary. Then the combinatorial boundary components of the homologically smooth gentle algebra $A = \left(\bigoplus_{i,j} \hom(L_i, L_j)\right)^{op}$ are in natural bijection with the boundary components of $\partial\Sigma$. Furthermore, if a combinatorial boundary component b corresponds to a boundary component $B \subset \partial\Sigma$ then the number of forbidden threads in b is equal to the number of stops on B and the winding numbers match:*

$$w_\eta(B) = w(b).$$

Proof. Figure 2 shows an example of the way the surface Σ looks around a boundary component B. Assume first that there is at least one stop on B. Let

$$q_1(1), \dots, q_1(k_1), q_2(1), \dots, q_2(k_2), \dots, q_n(1), \dots, q_n(k_n)$$

be the endpoints of the Lagrangians ending on B, ordered compatibly with the orientation of B. Here we assume that there are no stops between $q_i(j)$ and $q_i(j+1)$ and there is exactly one stop s_i between $q_i(k_i)$ and $q_{i+1}(1)$, for $i \in \mathbb{Z}/n$. Then for every $i \in \mathbb{Z}/n$ we have a permitted thread $p_i = \beta_i(k_i - 1) \dots \beta_i(1)$, where $\beta_i(j)$ is the generator of A corresponding to the flow on B from $q_i(j)$ to $q_i(j+1)$. On the other hand, each stop s_i lies on a unique disk D, and by looking at the pieces of ∂D formed by other boundary components of Σ, we obtain a forbidden thread $f_i = \alpha_{m_i} \dots \alpha_1$ starting at the Lagrangian corresponding to $q_i(1)$ and ending at the one corresponding to $q_{i-1}(k_{i-1})$. Thus, we get a combinatorial boundary component of type I, $b = p_n f_n \dots p_1 f_1$.

The winding number of η along the arc passing through the stop, oriented in the opposite direction to the boundary direction, is determined using the constraint (2.1) to be

$$|f| = \sum_{i=1}^{n-1} |\alpha_i| - (n-2)$$

On the other hand, the winding number of η along the arc corresponding to the permitted thread p is simply $|p|$. Thus, we get the equality $w_\eta(B) = w(b)$.

In the case of a boundary component $B \subset \partial\Sigma$ with no stops, the sequence of flows between the corresponding ends of Lagrangians on B gives a permitted cycle, i.e., a combinatorial boundary component of type II. Again, the winding numbers match.

It is easy to see that in this way we get a bijection between the boundary components B and the combinatorial boundary components of A.

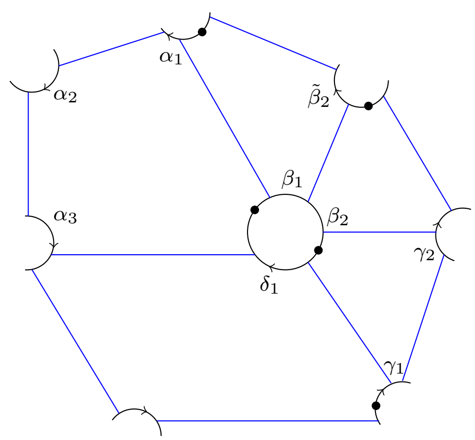


FIGURE 2. The boundary component is given by the cyclic sequence $p_2 f_2 p_1 f_1$ where $f_1 = \alpha_3\alpha_2\alpha_1$, $p_1 = \beta_2\beta_1$, $f_2 = \gamma_2\gamma_1$ and $p_2 = \delta_1$. Note that if instead of f_1, we considered the forbidden thread $\tilde{f}_1 = \tilde{\beta}_2\beta_1$, the condition $(\star)$ is violated.

□

Theorem 3.2.2. *(i) Given a homologically smooth graded gentle algebra A over a field $\mathbb{K}$ (with $|Q_1| > 0$), there exists a graded (connected) surface with stops $(\Sigma_A, \Lambda_A, \eta_A)$, with non-empty boundary and a derived equivalence*

$$D(A) \cong \mathcal{W}(\Sigma_A, \Lambda_A; \eta_A)$$

Furthermore, the AAG-invariants of A are given by the collection of pairs

$$(n_i, n_i - w_{\eta_A}(\partial_i\Sigma_A)),$$

where $(\partial_i\Sigma_A)_{i=1,\dots,N})$ are all boundary components of Σ_A and $n_i \in \mathbb{Z}_{\geq 0}$ is the number of marked points on $\partial_i\Sigma_A$.
(ii) One has

$$\chi(\Sigma_A) = \chi(Q) = |Q_0| - |Q_1|.$$

Proof. (i) We define a ribbon graph $\mathcal{R}_A$ whose vertices are in bijection with the collection of forbidden threads in Q, and whose edges are in bijection with vertices of Q.

Recall that there are precisely two forbidden threads that pass through a vertex of Q. The corresponding edge on $\mathcal{R}_A$ is defined to connect the two forbidden threads. Furthermore, we can equip the set of edges in $\mathcal{R}_A$ incident to a vertex f with a total ordering. Namely, the set of edges incident to a vertex f of $\mathcal{R}$ is in bijection with the set of vertices of Q which appear in the forbidden thread f. Hence, we can use the order in which these vertices appear in the forbidden thread. This linear order of edges incident to vertices of $\mathcal{R}_A$ induces a ribbon structure on $\mathcal{R}_A$, i.e., a cyclic order of edges incident to each vertex. Therefore, we can consider the associated thickened surface Σ_A such that $\mathcal{R}_A$ is embedded as a deformation retract of Σ_A. (A graph with the additional data of a linear ordering on the edges incident to a vertex is called a *ciliated fat graph* [9].)

Thus, to construct Σ_A we replace each vertex of $\mathcal{R}_A$ with a 2-disk $\mathbb{D}^2$ and each edge with a strip, a thin oriented rectangle $[-\epsilon, \epsilon] \times [0, 1]$, where the rectangles are attached to the boundary of the disks according to the given cyclic orders at the vertices. On the boundary of each disk associated to the vertex of $\mathcal{R}_A$ we also mark a point, called a *stop* as follows. If the linear order on edges incident to this vertex is given by $e_1 < e_2 < \ldots < e_k$, the stop e_0 appears in the circular order such that $e_k < e_0 < e_1$. We define Λ_A by taking the union of all such points. In particular, the cardinality of Λ_A, is equal to the number of forbidden threads in A.

We claim that the ribbon graph $\mathcal{R}_A$ and hence the associated surface Σ_A is connected. Indeed, for every vertex v of Q let $e(v)$ be the corresponding edge in $\mathcal{R}_A$, viewed as a subgraph in $\mathcal{R}_A$. Since Q is connected, it is enough to check that if v and v' are connected by an edge α in Q then $e(v)$ and $e(v')$ intersect in $\mathcal{R}_A$. Indeed, let f be a forbidden thread containing α (it always exists). Then f is a vertex of both $e(v)$ and $e(v')$. This proves our claim that $\mathcal{R}_A$ is connected.

Dual to the edges of $\mathcal{R}_A$ we obtain a disjoint collection of non-compact arcs L_v indexed by vertices of Q. Thus, Σ_A is a surface with non-empty oriented boundary, Λ_A is a set of marked points in its boundary, and $\{L_v : v \in Q_0\}$ is a collection pair-wise disjoint and non-isotopic Lagrangian arcs in $\Sigma_A \setminus \Lambda_A$. Furthermore, the complement

$$\Sigma_A \setminus \{\bigsqcup_v L_v\} = \bigcup_f D_f$$

is a union of disks D_f indexed by forbidden threads f in Q, with exactly one marked point on its boundary (see Examples 3.2.7, 3.2.8 below). In particular, the collection $\{L_v\}$ gives a generating set.

By construction, there is a bijection between arrows in the quiver Q and the generators of the endomorphism algebra $A_L := \bigoplus_{v,w} \hom(L_v, L_w)$ since each edge α in Q is in exactly one forbidden thread f, and the corresponding D_f has a flow associated to α. Furthermore, two flows $\alpha_1 : L_{v_2} \to L_{v_1}$ and $\alpha_2 : L_{v_3} \to L_{v_2}$ can be composed in A_L if and only if α_i is in a forbidden thread f_i, for $i = 1, 2$, such that the disks D_{f_1} and D_{f_2} are glued along the edge corresponding to v_2. But this means that the corresponding elements of A satisfy

$\alpha_2\alpha_1 \notin I$, as otherwise condition (3) of Definition 3.1.1 would be violated. This imples that A is naturally identified with A_L^{op} as an ungraded algebra.

We define the line field η_A on Σ_A as follows. We require that the line field is tangent to each L_v. Then it suffices to describe its restrictions to the disks D_f. Each D_f is a $2m$-gon as in Figure 1. The homotopy class of a line field on D_f is determined by the winding numbers θ_i along the boundary arcs of D_f, α_i, for $i = 1, \dots, (m-1)$, avoiding the unique stop (black in Figure 1) between the Lagrangians (blue in Figure 1). Indeed, the remaining winding number θ_m along the boundary arc that passes through the stop is determined by the condition $\sum_{i=1}^{m} \theta_i = m - 2$, and we can define $\eta_A|_{D_f}$ as the unique line field with these winding numbers. Now we set θ_i, for $i = 1, \dots, m-1$, to be the degree of the generator of A corresponding to α_i.

With this definition A and A_L^{op} are identified as graded algebras. Since we also know that the collection $\{L_v\}$ generates $\mathcal{W}(\Sigma_A, \Lambda_A; \eta_A)$, we conclude that

$$D(A) \cong \mathcal{W}(\Sigma_A, \Lambda_A; \eta_A).$$

Finally, the last statement follows from Proposition 3.2.1.

(ii) We have $\chi(\Sigma_A) = \chi(\mathcal{R}_A)$. Let us denote by $v(\mathcal{R}_A)$ and $e(\mathcal{R}_A)$ the numbers of vertices and edges in $\mathcal{R}_A$. We have $e(\mathcal{R}_1) = |Q_0|$, while $v(\mathcal{R}_A)$ is the number of forbidden threads. Let $f_1, \dots, f_m$ be all forbidden threads. Since every edge of Q belongs to the unique forbidden thread, we have

$$\sum \ell(f_i) = |Q_1|$$

(where $\ell(\cdot)$ is the length). On the other hand, since every vertex is contained in exactly two forbidden threads, we have

$$\sum (\ell(f_i) + 1) = 2|Q_0|.$$

Combining this with the previous formula we get

$$v(\mathcal{R}_A) = 2|Q_0| - |Q_1|,$$

so we deduce that $\chi(\mathcal{R}_A) = \chi(Q)$. □

Using formula (1.5) we derive the following property of the AAG-invariants.

Corollary 3.2.3. *Let $\{(n_i, m_i)\}_{i=1,\dots,d}$ be the AAG-invariants of a homologically smooth graded gentle algebra A. Then*

$$\sum_{i=1}^{d} (n_i - m_i + 2) = 4 - 4g,$$

where $g \geq 0$ is the genus of the corresponding surface Σ_A.

Corollary 3.2.4. *Given two homologically smooth graded gentle algebras A and B, assume there exists a homeomorphism of $\phi : \Sigma_A \to \Sigma_B$ inducing a bijection $\Lambda_A \to \Lambda_B$ such that $\phi_*(\eta_A)$ is homotopic to η_B, i.e., the invariants of η_A and η_B from Theorem 1.2.5 coincide. Then $D(A) \simeq D(B)$.*

As a particular case of the last Corollary, we can describe some cases when already looking at the AAG-invariants gives the derived equivalence.

Corollary 3.2.5. *Assume that A and B are homologically smooth graded gentle algebras, such that the AAG-invariants of A and B coincide (up to permutation) and are given by a collection $\{(n_i, m_i)\}_{i=1,\dots,d}$. Assume in addition that one of the following conditions holds:*
(a) $\sum_i (n_i - m_i + 2) = 4$*;*
(b) $\sum_i (n_i - m_i + 2) = 0$ *and* $\gcd(n_1 - m_1 + 2, \dots, n_d - m_d + 2) = 1$*;*
(c) $\sum_i (n_i - m_i + 2) < 0$ *and at least one of the numbers* $n_i - m_i$ *is odd.*
Then $D(A) \simeq D(B)$*.*

Proof. By Corollary 3.2.3, the three cases are distinguished by the genus $g(\Sigma_A)$: in case (a) it is 0, in case (b) it is 1, and in case (c) it is > 1. Now the assertion follows from the corresponding cases in Theorem 1.2.5. □

We can use Koszul duality to convert our results about homologically smooth graded gentle algebras into those about finite-dimensional gentle algebras. Namely let A be a finite-dimensional gentle algebra with grading in degree 0. Let $A^!$ be the Koszul dual gentle algebra (with respect to the generators given by the edges). We equip $A^!$ with the grading for which all edges have degree 1 (i.e., path-length grading). Then the result of Keller in [14, Sec. 10.5] ("exterior" case) gives an equivalence

$$D_f(A) \simeq D(A^!),$$

where $D_f(A)$ is the bounded derived category of finite-dimensional A-modules (and $D(A^!)$ is the perfect derived category of $A^!$ viewed as a dg-algebra, as before).

Furthermore, it is easy to check that the AAG-invariants of A and $A^!$ are the same. Thus, Corollary 3.2.5 leads to the following result.

Corollary 3.2.6. *Let A and B be finite-dimensional gentle algebras with grading in degree 0, such that the AAG-invariants of A and B coincide (up to permutation) and satisfy one of the conditions (a)–(c) of Corollary 3.2.5. Then*

$$D_f(A) \simeq D_f(B).$$

Example 3.2.7. Here is an example illustrating the construction of associating a surface to a gentle algebra. Consider the gentle algebra given in Figure 3.

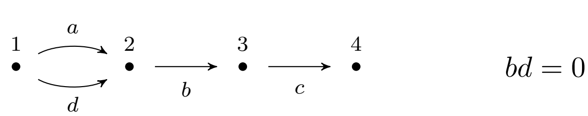


FIGURE 3. An example of gentle algebra

The forbidden threads are given by $\{a, bd, c, e_4\}$. The permitted threads are given by $\{cba, d, e_3, e_4\}$. The combinatorial boundary components are given by $\{p_3 f_3 p_2 f_2 p_1 f_1, p_4 f_4\}$ where, $f_1 = e_4$, $p_1 = e_4$, $f_2 = c$, $p_2 = e_3$, $f_3 = bd$, $p_3 = cba$, and $f_4 = a$, $p_4 = d$.

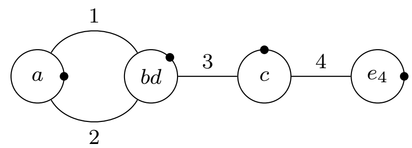


FIGURE 4. Ribbon graph associated to a gentle algebra

The associated ribbon graph is given in Figure 4, where the cyclic order at vertices are given by counter-clockwise rotation.

Figure 5 depicts the corresponding surface, together with the dual arcs L_1, L_2, L_3, L_4.

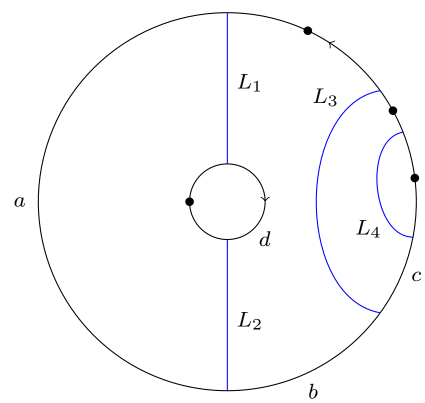
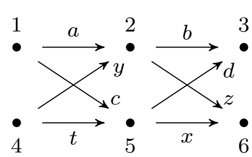


FIGURE 5. Surface associated to a gentle algebra

Example 3.2.8. Here is another example that produces a genus 1 surface with 2 boundary components. Consider the gentle algebra given by Figure 6.



$$za = by = xc = dt = 0$$

FIGURE 6. Another example of a gentle algebra

The forbidden threads are given by $\{za, by, xc, dt\}$, and the permitted threads are given by $\{ba, dc, xt, zy\}$. The combinatorial boundary components are given by $\{p_2 f_2 p_1 f_1, p_4 f_4 p_3 f_3\}$ where $f_1 = dt, p_1 = zy, f_2 = xc, p_2 = ba$, and $f_3 = za, p_3 = dc, f_4 = by, p_4 = xt$.

The corresponding surface is given in Figure 7.

Remark 3.2.9. An optimist's conjecture would be that conversely if A and B are homologically smooth graded gentle algebras which are derived equivalent, then there exists a homeomorphism $\phi : \Sigma_A \to \Sigma_B$ inducing a bijection $\Lambda_A \to \Lambda_B$ and such that $\phi_*(\eta_A)$ is homotopic to η_B. Note that to prove this, one needs to show that the topological type of $(\Sigma_A, \Lambda_A; \eta_A)$ is a derived invariant of A. This is encoded by the numerical invariants of η_A introduced in Theorem 1.2.5 (from which one can recover the topological type of the surface), together with the numbers of marked points on each boundary component.

Remark 3.2.10. In Theorem 3.2.2, it is possible to drop the assumption that A is smooth. Assume for simplicity that A is proper. In this case, the surface Σ would be glued together from the disks D_f associated to forbidden threads as before, and also disks D_c with an interior hole, associated with forbidden cycles. In other words, D_c is an annulus whose inner boundary component has no marked points and is not glued to anything, while its outer boundary component is connected by strips, corresponding to the vertices in c, to other disks (this boundary component of D_c still has no stops). In the presence of unmarked boundary components, there is a dual construction to the construction of partially wrapped Fukaya categories, $\mathcal{W}(\Sigma, \Lambda; \eta)$, namely, the infinitesimal wrapped Fukaya categories $\mathcal{F}(\Sigma, \Lambda; \eta)$, studied in [16]. Its objects are graded Lagrangians which do not end on the unmarked components of the boundary. Thus, for non-smooth proper gentle algebras, a version of Theorem 3.2.2 should state the equivalence

$$D(A) \simeq \mathcal{F}(\Sigma_A, \Lambda_A; \eta_A)$$

However, we have not checked that the collection of Lagrangians $\{L_v\}$ given by the construction in Theorem 3.2.2 (and modified as above) generates $\mathcal{F}(\Sigma_A, \Lambda_A; \eta_A)$.

4. Derived equivalences between stacky curves

4.1. **Chains.** Recall that in [16] we considered stacky curves $C(r_0, \dots, r_n; k_1, \dots, k_{n-1})$ obtained by gluing weighted projective lines

$$B(r_0, r_1), B(r_1, r_2), \dots, B(r_{n-1}, r_n)$$

into a chain, where $k_i \in (\mathbb{Z}/r_i)^*$ are used to determine the stacky structure of the nodes in this chain. We showed in [16, Thm. B] that the bounded derived category of coherent

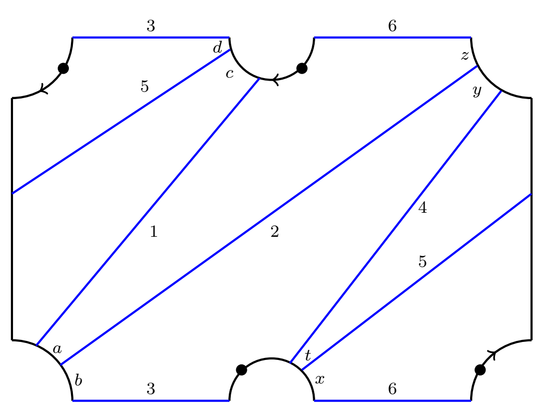


Figure 7. Genus 1 surface with 2 boundary components. Left-right and top-bottom are identified.

sheaves on such a stacky curve is equivalent to the partially wrapped Fukaya category of a surface obtained by a certain linear gluing of the annuli.

Namely, let $A(r,r')$ denote the annulus with ordered boundary components that has r marked points $p_1^-,\ldots,p_r^-$ on the first component and r' marked points $p_1^+,\ldots,p_{r'}^+$ on the second boundary component (the points are ordered cyclically compatibly with the orientation of the boundary). Given a collection of permutations $\sigma_i\in\mathfrak{S}_{r_i}$, $i=1,\ldots,n-1$, we consider the surface $\Sigma^{lin}(r_0,\ldots,r_n;\sigma_1,\ldots,\sigma_{n-1})$ obtained by gluing the annuli

$$A(r_0,r_1),A(r_1,r_2),\ldots,A(r_{n-1},r_n)$$

in the following way. For each $i=1,\ldots,n-1$, $j=1,\ldots,r_i$, we glue a small segment of the boundary around the marked point p_j^+ in $A(r_{i-1},r_i)$ with a small segment of the boundary around the point $p_{\sigma_i(j)}^-$ in $A(r_i,r_{i+1})$ by attaching a strip, as in Figure 8.

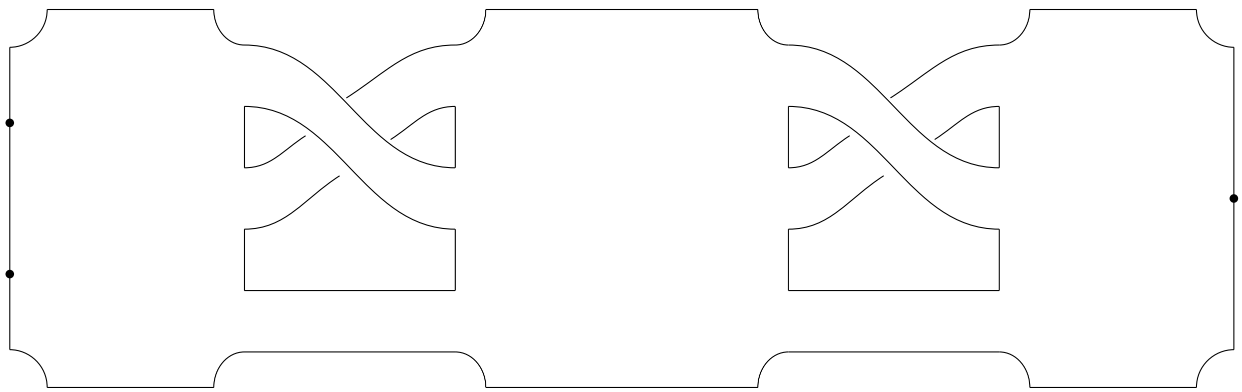


FIGURE 8. Surface glued from annuli (top and bottom are identified). $(r_0,r_1,r_2,r_3)=(2,3,3,1)$, $\sigma_1=\sigma_2:(1,2,3)\to(2,1,3)$

Note the resulting surface has two special boundary components equipped with r_0 and r_n marked points, respectively (there are no other marked points on the other boundary components). The boundary components that arise because of the gluing are from $n-1$ groups, so that components in the ith group are in bijection with cycles in the cycle decomposition of the commutator $[\sigma_i,\tau]\in\mathfrak{S}_{r_i}$, where τ is the cyclic permutation $j\mapsto j-1$.

We equip each annulus with the standard line field that has zero winding numbers on both boundary components. These line fields glue into a line field η on the surface $\Sigma^{lin}(r_0,\ldots,r_n;\sigma_1,\ldots,\sigma_{n-1})$. More precisely, we take η that corresponds to the horizontal direction in Figure 8.

It is easy to see that the boundary invariants of η are given as follows. For the two special boundary components the winding numbers are equal to zero, so the corresponding invariant is 2. For a boundary component corresponding to a k-cycle in the cycle decomposition of $[\sigma_i,\tau]$ the winding number is $-2k$, so the invariant is $2-2k$.

To get the surface related to the stacky curve $C(r_0,\ldots,r_n;k_1,\ldots,k_{n-1})$, we take in [16] the permutation σ_i to be the permutation $x\mapsto -k_ix$ of $\mathbb{Z}/r_i\mathbb{Z}$. Let us denote the resulting surface by $\Sigma^{lin}(r_0,\ldots,r_n;k_1,\ldots,k_{n-1})$. We equip it with r_0 and r_n stops on two special boundary components, and denote this set of stops as Λ_{r_0,r_n}. Now [16, Thm. B] states that

$$D^b(\mathrm{Coh}\, C(r_0,\dots,r_n;k_1,\dots,k_{n-1}))\cong \mathcal{W}(\Sigma^{lin}(r_0,\dots,r_n;k_1,\dots,k_{n-1}),\Lambda_{r_0,r_n};\eta).$$

Note that for each i the commutator $[\sigma_i,\tau]$ is given by $x\mapsto x+k_i+1 \,\mathrm{mod}(r_i)$, so its cycle decomposition has $p_i=\gcd(k_i+1,r_i)$ cycles of length r_i/p_i. Thus, the corresponding boundary invariants are $2,2$ (for the special boundary components) and for each $i=1,\dots,n-1$, the number $2-2r_i/p_i$ repeated p_i times.

The genus of the surface $\Sigma^{lin}(r_0,\dots,r_n;k_1,\dots,k_{n-1})$ is given by

$$g=\frac{1}{2}\sum_{i=1}^{n-1}(r_i-p_i).$$

4.1.1. *Genus* 0. Due to the above formula the genus of the surface is 0 precisely when $k_i=-1$ for all i. In this case all the other boundary components except for the two special ones have the zero invariant (since $p_i=r_i$). Thus, Theorem 1.2.5 implies that there exists a homeomorphism

$$\Sigma^{lin}(r_0,\dots,r_n;-1,\dots,-1)\simeq\Sigma^{lin}(r_0,r_1+\ldots+r_{n-1},r_n;-1)$$

preserving the line fields. In fact, this homeomorphism is easy to construct directly. This leads to a derived equivalence between the categories of coherent sheaves over $C(r_0,\dots,r_n;-1,\dots,-1)$ and $C(r_0,r_1+\ldots+r_{n-1},r_n;-1)$ (see [21]).

4.1.2. *Trade-off for balanced nodes.* More generally, let $I\subset[1,n-1]$ be the subset of indices i such that $k_i=-1$, and let $r_I=\sum_{i\in I}r_i$. Then, we have a homeomorphism

$$\Sigma^{lin}(r_0,\dots,r_n;k_1,\dots,k_{n-1})\simeq\Sigma^{lin}(r_0,r_I,(r_i)_{i\notin I},r_n;-1,(k_i)_{i\notin I})$$

preserving the line fields. This can be either derived from Theorem 1.2.5 as above or constructed directly. As before, this leads to a derived equivalence of the corresponding stacky curves.

4.1.3. *Genus* 1. The surfaces $\Sigma^{lin}(r_\bullet;k_\bullet)$ can have genus 1 only when $r_{i_0}-p_{i_0}=2$ for some $i_0\in[1,n-1]$ and $r_i=p_i$ for $i\neq i_0$. This can happen only when either $r_{i_0}=3$ or $r_{i_0}=4$ and $k_{i_0}=1$. These cases are distinguished by the presence of the boundary components with the invariant either -4 or -2. So in this case we do not any other equivalences except those due to the trade-offs for balanced nodes.

4.1.4. *Genus* ≥ 2. Because of the two special components with the boundary invariant 2, the Arf-invariant never appears. Thus, two surfaces $\Sigma^{lin}(r_\bullet;k_\bullet)$ and $\Sigma^{lin}(r'_\bullet;k'_\bullet)$ of genus $g\geq 2$ are homeomorphic as surfaces with a line field, whenever we have $r_0=r'_0$, $r_n=r'_n$ and the sequence $((r_1/p_1)^{p_1},\dots,(r_{n-1}/p_{n-1})^{p_{n-1}})$ differs from the corresponding sequence for $(r'_\bullet,k'_\bullet)$ by a permutation (here $(r_i/p_i)^{p_i}$ means the number r_i/p_i repeated p_i times).

For example, we can specialize to the case $n=2$, $r_0=r_2=0$, $r_1=r$. Note that the corresponding stacky curve $C(0,r,0;k)$ is the global quotient of the affine coordinate cross

$xy = 0$ by the μ_r-action $\zeta \cdot (x, y) = (\zeta^k x, \zeta y)$. We obtain that for $k, k' \in (\mathbb{Z}/r)^*$, such that $\gcd(k+1, r) = \gcd(k'+1, r)$, there exists an equivalence

$$D^b \operatorname{Coh}(C(0, r, 0; k)) \simeq D^b \operatorname{Coh}(C(0, r, 0; k')).$$

4.2. **Rings.** Now let us consider another class of stacky curves considered in [16], denoted by $R(r_1, \ldots, r_n; k_1, \ldots, k_n)$. They are defined by gluing the weighted projective lines $B(r_1, r_2), B(r_2, r_3), \ldots, B(r_n, r_1)$ into a ring, where as before $k_i \in (\mathbb{Z}/r_i)^*$ are used to determine the stacky structure of the nodes.

On the symplectic side we can modify our definition of the surfaces $\Sigma^{lin}(r_0, \ldots, r_n; \sigma_1, \ldots, \sigma_{n-1})$ as follows. Starting with the annuli $A(r_1, r_2), A(r_2, r_3), \ldots, A(r_n, r_1)$ we can glue them circularly using permutations $\sigma_1, \ldots, \sigma_n$. Thus, the corresponding surface could be represented similarly to Figure 8 but with the right and left ends identified (so that the corresponding boundary components disappear). We denote the resulting surface by $\Sigma^{cir}(r_1, \ldots, r_n; \sigma_1, \ldots, \sigma_n)$. Similarly to the case of a linear gluing it is equipped with a natural line field η that corresponds to the horizontal direction when the surface is depicted as on Figure 8.

By [16, Thm. B], we have an equivalence

$$D^b(\operatorname{Coh} R(r_1, \ldots, r_n; k_1, \ldots, k_n)) \cong \mathcal{W}(\Sigma^{cir}(r_1, \ldots, r_n; k_1, \ldots, k_n); \eta)$$

with the (fully) wrapped Fukaya category of the surface

$$\Sigma^{cir}(r_1, \ldots, r_n; k_1, \ldots, k_n) := \Sigma^{cir}(r_1, \ldots, r_n; \sigma(k_1), \ldots, \sigma(k_n)),$$

where σ_i is the permutation $x \mapsto -k_i x$ of $\mathbb{Z}/r_i\mathbb{Z}$. The genus of this surface is given by

$$g = 1 + \frac{1}{2}\sum_{i=1}^{n}(r_i - p_i).$$

4.3. **Case of irreducible stacky curve.** This is the case $n = 1$. Let $r = r_1$. Let us consider the case of $k \in \mathbb{Z}_r$ such that $\gcd(k+1, r) = 1$ (note that this is possible only when r is odd). Then the surface $\Sigma^{cir}(r; k)$ has genus $g = (r+1)/2$ and one boundary component with the winding number $-2r$, i.e., the invariant $2 - 2r$. Note that $2 - 2r$ is divisible by 4, so to determine the orbit of the line field under the mapping class group we have to calculate the corresponding Arf-invariant. This invariant will depend on k.

First, let us consider the case $k = 1$. Let us look at the simple curves α_i, $i = 1, \ldots, r-1$, depicted on Figure 9. In addition, we have two simple curves α and β corresponding to a vertical and horizontal line on Figure 9.

Then the classes $[\alpha]$, $[\beta]$ and $([\alpha_i])_{i=1,\ldots,r-1}$ give a basis of $H_1(\Sigma, \mathbb{Z}_2)$ with the intersection numbers

$$\alpha_i \cdot \alpha_j = 1 \bmod 2,\ i < j;\ \alpha \cdot \beta = 1;\quad \alpha_i \cdot \alpha = \alpha_i \cdot \beta = 0.$$

Furthermore, the winding number along each α_i is -2 so $q(\alpha_i) = 0$. On the other hand, the winding numbers along either α and β is zero. Thus, the corresponding space with a quadratic form over $\mathbb{Z}_2$ is a direct sum of V_{r-1} from Example 4.5.1 below and a 2-dimensonal space with the Arf-invariant 1.

Hence, the Arf-invariant is given in this case by $1 + \binom{(r-1)/2}{2} \bmod 2$.

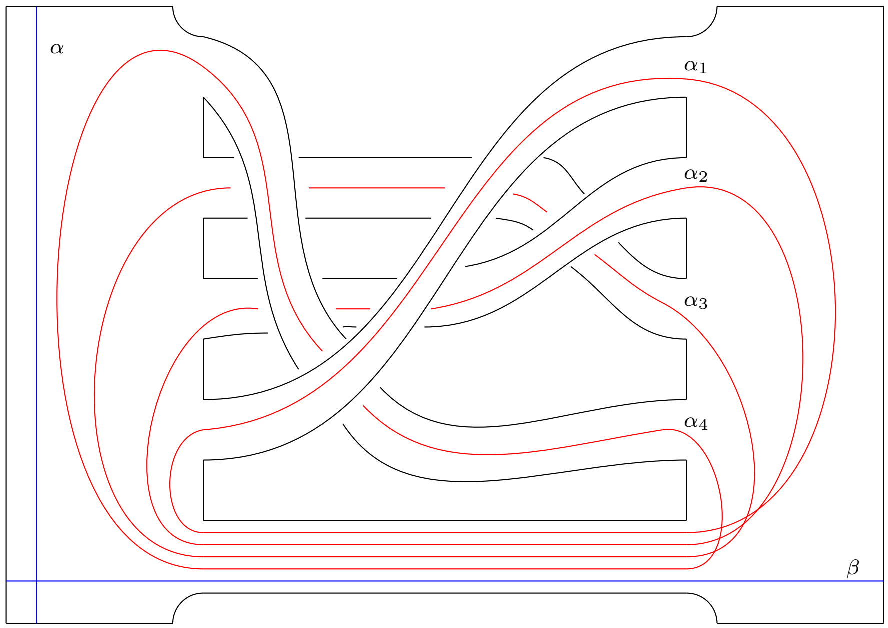


FIGURE 9. Circular gluing with $r = 5$, $k = 1$ (left-right, top-bottom are identified).

Next, assume in addition that r is not divisible by 3 and consider the case $k = 2$ (then $\gcd(k+1, r) = 1$). Then we claim that the classes $[\alpha_i]$, together with (α, β) still project to a basis of $H_1(\Sigma, \mathbb{Z}_2)$, however, their intersection numbers are now given by

$$\alpha_i \cdot \alpha_j = \begin{cases} 0, & i < j < i + r/2, \\ 1, & \text{otherwise}, \end{cases}$$

where $i < j$ (we still have $\alpha_i \cdot \alpha = \alpha_i \cdot \beta = 0$ and $\alpha \cdot \beta = 1$. It is easy to see that by renumbering the classes (α_i) as follows:

$$\alpha'_1 = \alpha_{(r-1)/2}, \ldots, \alpha'_{(r-1)/2} = \alpha_1, \alpha'_{(r-1)/2+1} = \alpha_{r-1}, \ldots, \alpha'_{r-1} = \alpha'_{(r-1)/2+1}$$

we get the quadratic form of Example 4.5.2. Hence, the Arf invariant is given by $1 + (r - 1)/2 \bmod 2$. Thus, we deduce the following derived equivalence.

Proposition 4.3.1. *Assume that $r \geq 7$ is not divisible by 3 and $r \equiv \pm 1 \bmod (8)$. Then the stacky curves $C^{ring}(r; 1)$ and $C^{ring}(r; 2)$ are derived equivalent.*

4.4. **Merging two nodes into one.** Let us fix an odd r. Then the surfaces $\Sigma^{cir}(r, r; 1, 1)$ and $\Sigma^{cir}(2r; 1)$ are homeomorphic: they both have genus r and 2 boundary components. One can ask whether they are homeomorphic as surfaces with line fields. The boundary invariant on each component is equal to $2 - 2r$, so we need to look at the Arf-invariant.

Proposition 4.4.1. *The Arf-invariant of the form associated to the line field on $\Sigma^{cir}(r, r; 1, 1)$ is equal to 1. The Arf-invariant of the form associated to the line field on $\Sigma^{cir}(2r; 1)$ is* *equal to* $(r+1)/2 \bmod 2$. *Hence, if* $r \equiv 1 \bmod(4)$ *then the stacky curves* $C^{ring}(r,r;1,1)$ *and* $C^{ring}(2r;1)$ *are derived equivalent.*

Proof. In the case of the surface $\Sigma^{cir}(r,r;1,1)$ we have two collections of simple curves $(\alpha_1,\ldots,\alpha_{r-1})$, $(\alpha'_1,\ldots,\alpha'_{r-1})$ associated with each of the two segments where the gluing happens. In addition we have two standard curves α and β as before. So the corresponding quadratic space will be a direct sum of two copies of V_{r-1} together with the 2-dimensional space spanned by (α,β). Thus, the Arf-invariant is equal to 1.

For the surface $\Sigma^{cir}(2r;1)$ we have in addition to α and β the curves $\alpha_1,\ldots,\alpha_{2r-1}$ defined as before. However, the class $[\alpha_1]+\ldots+[\alpha_{2r-1}]$ comes from the homology of the boundary, so passing to the quotient by this class we get the direct sum of the quadratic space $\overline{V}_{2r-2}$ from Example 4.5.1 with the 2-dimensional space spanned by (α,β). Hence, the Arf-invariant is equal to $1+(r-1)/2 \bmod(2)$.

Thus, the Arf-invariants match exactly when $(r-1)/2 \equiv 0 \bmod(2)$. □

4.5. **Computation of Arf-invariants.**

Example 4.5.1. Let V_n be a $\mathbb{Z}_2$-vector space with the basis $\alpha_1,\ldots,\alpha_n$, and the even pairing given by $\alpha_i\cdot\alpha_j = 1$ for $i\neq j$. Let q be the unique quadratic form in $\mathrm{Quad}(V_n)$ such that $q(\alpha_i)=0$ for all i.

First, assume that n is even. Then we claim that this pairing is nondegenerate and the Arf-invariant of q is given by
$$A(q) = \binom{n/2}{2} \bmod 2.$$
Indeed, it is enough to prove that the Gauss sum
$$G(q) := \sum_{x\in V_n} (-1)^{q(x)}$$
is equal to $\pm 2^{n/2}$. Then the sign will determine the Arf-invariant. It is easy to see that $q(x) = (-1)^{\binom{k}{2}}$, where k is the number of nonzero coordinates of x. Thus, we have
$$G(q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{\binom{k}{2}}.$$
Now we observe that
$$(-1)^{\binom{k}{2}} = \frac{1-i}{2}\cdot i^k + \frac{1+i}{2}\cdot(-i)^k,$$
where $i=\sqrt{-1}$. Thus, we have
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{\binom{k}{2}} = \frac{1-i}{2}\cdot(1+i)^n + \frac{1+i}{2}\cdot(1-i)^n =$$
$$2^{n/2}\cdot[\frac{1-i}{2}\cdot i^{n/2} + \frac{1+i}{2}\cdot(-i)^{n/2}] = 2^{n/2}\cdot(-1)^{\binom{n/2}{2}}.$$

Now, let us assume that n is odd. Then the vector $v_0 = \sum_{k=1}^{n}\alpha_k$ lies in the kernel of the pairing and $q(v_0) = \binom{n}{2} \bmod 2$. Thus, if we assume in addition that $n \equiv 1 \bmod 4$

then we have $q(v_0) = 0$ and so the form q descends to a well-defined quadratic form $\overline{q}$ on $\overline{V}_{n-1} = V_n/\langle v_0\rangle$. We claim that its Arf-invariant is

$$A(\overline{q}) = \frac{n-1}{4} \bmod 2.$$

Indeed, again we will consider the Gauss sum

$$G(\overline{q}) := \sum_{x\in\overline{V}} (-1)^{\overline{q}(x)}.$$

We have

$$G(\overline{q}) = \frac{1}{2}\cdot G(q) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}(-1)^{\binom{k}{2}} =$$
$$\frac{1-i}{4}\cdot(1+i)^n + \frac{1+i}{4}(1-i)^n = (-4)^{(n-1)/4}.$$

Example 4.5.2. Now let V be a $\mathbb{Z}_2$-vector space with the basis $\alpha_1,\dots,\alpha_n$, where $n\ge 4$ is even, the even pairing given by the rule

$$\alpha_i\cdot\alpha_j = \begin{cases} 1, & i<j<i+n/2,\\ & 0, j\ge i+n/2,\end{cases}$$

and the quadratic form q in $\mathrm{Quad}(V)$ such that $q(\alpha_i)=0$ for all i. Assume also that $n\not\equiv 2 \bmod (3)$. Then we claim that the pairing is nondegenerate and

$$A(q) = n/2 \bmod 2.$$

We will prove this by relating (V,q) with another quadratic form. For every $k\ge 0$, such that $k\not\equiv 2\bmod(3)$, let us consider a $\mathbb{Z}_2$-vector space W_k with the basis $\beta_1,\gamma_1,\dots,\beta_k,\gamma_k$, the even pairing given by the rule

$$\beta_i\cdot\beta_j = 1 \text{ for } i\ne j;\quad \gamma_i\cdot\gamma_j = 1 \text{ for } i\ne j;$$
$$\beta_i\cdot\gamma_j = 1 \text{ for } i\le j;\quad \beta_i\cdot\gamma_j = 0 \text{ for } i>j,$$

and the quadratic form q_k in $\mathrm{Quad}(W_k)$ such that $q_k(\beta_i) = q(\gamma_i) = 1$ for all i.

First, we will prove that $A(q) = A(q_{n/2-2})$ and then we will prove that

$$A(q_k) = k \bmod 2 \qquad (4.1)$$

To relate (V,q) with $(W_{n/2-2}, q_{n/2-2})$ let us consider the 2-dimensional isotropic subspace $I\subset V$ spanned by α_1 and α_n. Then we have $q|_I\equiv 0$, so the Arf-invariant of q is equal to that of the induced quadratic form on $I^\perp/I$. Now setting

$$\gamma_i = \alpha_2 + \alpha_{2+i},\quad \beta_i = \alpha_{n/2+1} + \alpha_{n/2+1+i},$$

for $i = 1,\dots,n/2-2$, we get an identification of $I^\perp/I$ with $W_{n/2-2}$, compatible with the quadratic forms.

To prove (4.1) we use induction on k. It is easy to check that $A(q_1) = 1$ (and $A(q_0) = 0$ for trivial reasons), so it is enough to establish the formula

$$A(q_k) = A(q_{k-3}) + 1.$$

To this end we consider the 2-dimensional isotropic subspace $J \subset W_k$ spanned by $\beta_k + \gamma_1$ and $\beta_1 + \beta_k + \gamma_k$. We have $q_k|_J = 0$, and our formula follows from the identification

$$J^\perp / J \simeq W_{k-3} \oplus W_1,$$

where the standard basis of W_{k-3} corresponds to the elements

$$(\beta_2 + \beta_{2+i} \bmod J, \gamma_2 + \gamma_{2+i} \bmod J)_{1 \le i \le k-3}$$

while a copy of W_1 spanned by $\beta_k \bmod J$ and $\gamma_k \bmod J$.

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