

Corrigendum to: “Geometric Composition in Quilted Floer Theory”

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As we explain below, an error in Lemma 11 of [5] affects the main results of that paper (Theorems 1, 2 and 3 of [5]). In this note we show that, under an additional assumption of monotonicity for cylinders (assumptions (A1), (A2), and (A3) below for Theorems 1, 2 and 3 respectively), the proofs can be fixed easily and the results remain valid. The additional monotonicity assumption on cylinders holds for the applications of our results that have appeared in [4] and [6], hence those are not affected.

In Lemma 11 of [5], we identified the topological type of a bubble resulting from breaking at the Y-end of our geometric composition map, Y-map, with a disk mapping to $M_0^- \times M_1 \times M_1 \times M_2^-$ with boundary on $L_{01} \times L_{12}^-$, where $L_{01} \subset M_0^- \times M_1$ and $L_{12}^- \subset M_1 \times M_2^-$ are Lagrangian boundary conditions. As a consequence of monotonicity assumptions on disks, it was argued that this bubble cannot exist. This identification of the topological type of the bubble at the Y-end is wrong. Indeed, the construction given in Lemma 11 of [5] shows that the topological type of this bubble at the Y-end can be identified with an annulus with one boundary component mapping to $L_{01} \times L_{12}^-$ and the other mapping to $\iota(L_{01} \times L_{12}^-)$ where $\iota : M_0^- \times M_1 \times M_1 \times M_2^- \rightarrow M_0^- \times M_1 \times M_1 \times M_2^-$ is the involution given by switching the two factors of M_1 . Here, we add the assumption of monotonicity for these annuli that is needed to run the proofs of the main theorems given in [5] in full generality.

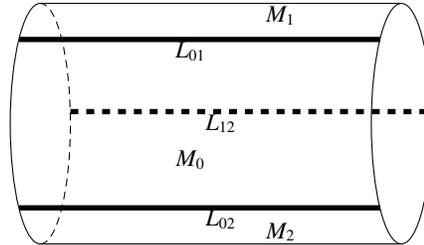


Figure 1: Cylinder bubble at the Y-end.

The bubble at the Y-end is a quilted cylinder, depicted in Figure (1), given by a quilted map:

$$\delta_i(s, t) : \mathbb{R} \times [0, 1] \rightarrow M_i, \quad i = 0, 1, 2$$

with cyclic Lagrangian boundary conditions $(L_{01}, L_{12}, L_{02}^-)$. Thus, we have

$$(\delta_0(s, 1), \delta_1(s, 0)) \in L_{01} \quad , \quad (\delta_1(s, 1), \delta_2(s, 0)) \in L_{12} \quad , \quad (\delta_2(s, 1), \delta_0(s, 0)) \in L_{02}^-.$$

As in [5], at the incoming and outgoing ends this converges exponentially to the Morse-Bott intersection $(L_{01} \times L_{12}) \cap (L_{02} \times \Delta_{M_1}^-) \subset M_0^- \times M_1 \times M_1^- \times M_2$, where $\Delta_{M_1}^- \subset M_1 \times M_1^-$ is the diagonal correspondence. Note that this intersection is diffeomorphic to L_{02} by the assumption that $L_{02} = L_{01} \circ L_{12}$ is given by an embedded geometric composition. By folding, we can in turn view this quilted cylinder as a strip:

$$(1) \quad \delta : \mathbb{R} \times [0, 1] \rightarrow M_0^- \times M_1 \times M_1^- \times M_2$$

with Lagrangian boundary conditions $(L_{01} \times L_{12}, L_{02} \times \Delta_{M_1}^-)$.

Our main purpose in this note is to rule out bubbles corresponding to δ maps as above that may appear in the boundary of the 0- and 1-dimensional components of the moduli space $\mathcal{M}_J(\underline{x}, \underline{y})$ of quilts used in the definition of the Y-map (see Definition 5 of [5]) so as to ensure that the Y-map is a chain map. To this end, the following additional assumptions (A1), (A2), and (A3) must be made in Theorems 1, 2 and 3 of [5] respectively:

In the notation of Theorems 1, 2 and 3 of [5], let $Z = M_{r-1}^- \times M_r \times M_r^- \times M_{r+1}$ with the symplectic form $\omega = (-\omega_{r-1}) \oplus \omega_r \oplus (-\omega_r) \oplus \omega_{r+1}$.

(A1) (aspherical) For any topological cylinder $\nu : S^1 \times [0, 1] \rightarrow Z$ with $\nu_{S^1 \times \{0\}} \subset L_{r-1} \times L_r$ and $\nu_{S^1 \times \{1\}} \subset (L_{r-1} \circ L_r) \times \Delta_{M_r}$, we have $\int \nu^* \omega = 0$.

(A2) ($\tau > 0$ positive monotone) For any topological cylinder $\nu : S^1 \times [0, 1] \rightarrow Z$ with $\nu_{S^1 \times \{0\}} \subset L_{r-1} \times L_r$ and $\nu_{S^1 \times \{1\}} \subset (L_{r-1} \circ L_r) \times \Delta_{M_r}$, we have

$$\int \nu^* \omega = \tau I_{Maslov}(\nu^*(L_{r-1} \times L_r), \nu^*((L_{r-1} \circ L_r) \times \Delta_{M_r})).$$

(A3) ($\tau < 0$ strongly negative monotone) For any topological cylinder $\nu : S^1 \times [0, 1] \rightarrow Z$ with $\nu_{S^1 \times \{0\}} \subset L_{r-1} \times L_r$ and $\nu_{S^1 \times \{1\}} \subset (L_{r-1} \circ L_r) \times \Delta_{M_r}$ we have

$$\int \nu^* \omega = \tau I_{Maslov}(\nu^*(L_{r-1} \times L_r), \nu^*((L_{r-1} \circ L_r) \times \Delta_{M_r})),$$

and if $\int \nu^* \omega > 0$, then $I_{Maslov}(\nu^*(L_{r-1} \times L_r), \nu^*((L_{r-1} \circ L_r) \times \Delta_{M_r})) < -\dim(L_{r-1} \circ L_r) + 1$.

In all cases, the relevant cylinders for Lagrangian Floer theory will have the property that the arc $\nu|_{\{s\} \times [0, 1]}$ is isotopic relative to its boundary to a point in $(L_{r-1} \times L_r) \cap (L_{r-1} \circ L_r) \times \Delta_M$ for any, hence all, $s \in S^1$. Henceforth, we will tacitly assume this.

We continue by letting $M_{r-1} = M_0$, $M_r = M_1$ and $M_{r+1} = M_2$ and $L_{01} = L_{r-1}$, $L_{12} = L_r$ and $L_{02} = L_{r-1} \circ L_r$.

Note that the limits of δ at the incoming and outgoing ends has to be in the same path-component of the intersection $(L_{01} \times L_{12}) \cap (L_{02} \times \Delta_{M_1}^-)$, since the possible limits of δ at the ends are $\{(m_0, m_1, m_1, m_2) | (m_0, m_1) \in L_{01}, (m_1, m_2) \in L_{12}\} \cong L_{02}$ and there is a seam in L_{02} originally connecting the incoming and outgoing ends of the quilted cylinder. Thus, by concatenating an arc that lies in the Morse-Bott intersection to any strip δ as in (1), we may assume that δ converges exponentially to the same point in the Morse-Bott intersection at each of its ends, which in view of removal of singularities ([11, Lemma 2.5] or [8, Sec 2.3]), we may consider as a cylinder $\nu : S^1 \times [0, 1] \rightarrow M_0^- \times M_1 \times M_1^- \times M_2$ with $\nu_{S^1 \times \{0\}} \subset L_{01} \times L_{12}$ and $\nu_{S^1 \times \{1\}} \subset L_{02} \times \Delta_{M_1}^-$. The following relations between δ and ν are immediate:

$$\int \delta^* \omega = \int \nu^* \omega \quad , \quad I_{Maslov}(\delta) = I_{Maslov}(\nu)$$

Note that even though the homotopy class of the map ν depends on the choice of the arc in $(L_{01} \times L_{12}) \cap (L_{02} \times \Delta_{M_1}^-)$, the symplectic area and the Maslov index of the resulting cylinder is independent of that choice (see [7] for the definition of Maslov index in the clean intersection case).

Recall that Fredholm index is given by the sum of Maslov index and the dimension of the Morse-Bott boundary conditions (cf. Section 2.4 in [5]). Thus, we have:

$$index(\delta) = I_{Maslov}(\delta) + \dim(L_{02}) = I_{Maslov}(\nu) + \dim(L_{02})$$

With these assumptions in place, the bubbling at the Y-end can be avoided in the situations of Theorems 1, 2 and 3 of [5] :

Correction to proof of Theorem 1: Assumption (A1) implies that any bubble δ at the Y-end must have zero area. In view of holomorphicity of δ , this implies that δ is constant.

Correction to proof of Theorem 2: Assumption (A2) implies that any bubble δ at the Y-end must have positive $I_{Maslov}(\delta)$ in view of holomorphicity of δ , hence $I_{Maslov}(\delta) \geq 2$ by orientability of our Lagrangians. Therefore, the Fredholm index $index(\delta) \geq 2 + \dim(L_{02})$.

Now if we have a bubbled configuration (u, δ) in the index 1 moduli space $\mathcal{M}_J^1(\underline{x}, \underline{y})$ of Y-maps, with a bubble at the Y-end, using the gluing theorem for Fredholm indices for Morse-Bott ends on the original Y-map one sees that

$$1 = index((u, \delta)) = index(u) + index(\delta) - \dim(L_{02})$$

which implies

$$index(u) \leq -1.$$

Hence, this configuration cannot exist in view of transversality.

Correction to proof of Theorem 3: Assumption (A3) implies that any bubble δ at the Y-end must have Fredholm index $index(\delta) = I_{Maslov}(\delta) + \dim(L_{02}) < 1$. We can always ensure transversality for the moduli space that δ belongs to by choosing our J to be t -dependent near the Y-end [3]. Furthermore, $index(\delta)$ cannot be zero, since translations in the s -direction contribute one dimension to the moduli space. Therefore, we conclude that $index(\delta) < 0$, which violates transversality.

Remark 1 *In the following two cases of interest, the assumptions (A1), (A2) and (A3) introduced above follow from the original assumptions made in Theorems 1, 2 and 3 of [5].*

- (1) *Suppose that at least one of the inclusion induced maps $\pi_1(L_{r-1} \times L_r) \rightarrow \pi_1(Z)$ or $\pi_1((L_{r-1} \circ L_r) \times \Delta_{M_r}) \rightarrow \pi_1(Z)$ has torsion image. In this case, after possibly taking a cover, an annulus ν as in (A1) or (A2) can be capped off in Z by a disk along one of the Lagrangians to obtain a disk with boundary on the other Lagrangian. The assumptions (A1) and (A2) can then be deduced from the corresponding assumptions made for disks in Theorems 1 and 2 of [5]. The geometric composition isomorphism theorem of [5] in this case was used in [6].*
- (2) *Suppose that the $\delta_1 : \mathbb{R} \times [0, 1] \rightarrow M_1$ part of the quilted cylinder is homotopic (respecting the boundary conditions) to a strip which is constant along the t -coordinate. In this case, after homotopy, we can assume that the quilted cylinder $(\delta_0, \delta_1, \delta_2)$ is such that δ_1 is independent of t . The maps (δ_0, δ_2) then give a new quilted cylinder with two parallel seams labelled by L_{02} . This can in turn be folded to obtain a disk*

$$\tilde{\delta} : D \rightarrow M_0^- \times M_2$$

with boundary mapping to L_{02} such that

$$Area(\delta) = Area(\tilde{\delta}) \quad , \quad I_{Maslov}(\delta) = I_{Maslov}(\tilde{\delta})$$

The area equality is clear. For the equality of Maslov indices, see the proof of Lemma 2.1.2 of [9]. Now, the assumptions (A1), (A2) and (A3) can be deduced from the corresponding assumptions for disks in $M_0^- \times M_2$ with boundary on L_{02} . These follow from assumptions made in Theorem 1 and 2 in the aspherical and positively monotone cases. In the strongly negative case one has to check the strong negativity condition on disks in $M_0^- \times M_2$ with boundary on L_{02} to deduce (A3). Note that this condition appears already in checking that the Floer complex $CF(L_0, L_{02}, L_2)$ is well-defined and is a special case of (A3) in general.

This situation arises, for example, when L_{12} is a fibred coisotropic: it includes into M_2 as a coisotropic submanifold and fibres over M_1 . The geometric composition isomorphism of [5] in this case was used in [4] and more recently also in [2].

We end with the following remark:

Remark 2 *The quilted cylinder bubbles at the Y-end play a similar role to so-called ‘‘figure-eight’’ bubbles whose role have been emphasised in [1] in obstructing the geometric composition*

isomorphism in quilted Floer theory. On the other hand, analytically, the quilted cylinder bubbles are better behaved (as they have the standard boundary conditions that approach to each other transversely at the ends). In particular, removal of singularities in this case is established in the standard way in Floer theory (see for ex. [11, Lemma 2.5]).

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Added in proof: After this corrigendum was written, a paper by Wehrheim [10] appeared which falsely claims that the application in [4] of Theorem 3 is not valid, contrary to Remark 1 above. We do not endorse this view.

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