# ERRATUM TO "THE RELATIVE BREUIL-KISIN CLASSIFICATION OF $p$-DIVISIBLE GROUPS AND FINITE FLAT GROUP SCHEMES" [Kim14]

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It was pointed out by E. Lau that Definition 3.1 and Proposition 3.8 in [**?**] need to be corrected as follows:

(1) In the definition of Dieudonné crystal [**?**, Definition 3.1], the condition on the filtration [**?**, Definition 3.1(3)] should be replaced by requiring $\mathrm{Fil}^1 \mathscr{E}_{\mathfrak{X}}$ to be an *admissible* filtration, as introduced by Grothendieck [Gro74, Ch. V, §3]. We recall the definition below (*cf.* Definition 1), and verify that the Hodge filtration for any $p$-divisible group $G$ over $\mathfrak{X}$ is admissible (*cf.* Lemma 3), which makes $\mathbb{D}^*(G)$ a Dieudonné crystal according to the *corrected* definition.

(2) The statement of [**?**, Proposition 3.8] holds for the *corrected* definition of Dieudonné crystal. In §2, we explain how to correct the proof of [**?**, Proposition 3.8].

In other words, Dieudonné crystals with the corrected definition have the claimed semilinear algebraic interpretation in terms of filtered Frobenius modules (*cf.* [**?**, Proposition 3.8]), therefore the rest of the results of [**?**] hold true for the corrected definition of Dieudonné crystals.

The author is deeply grateful to E. Lau for informing the author of this error and directing him to [Gro74].

## 1. Admissible filtration: correction of [?, Definition 3.1(3)]

Let $S \twoheadrightarrow R$ be a divided power thickening in characteristic $p$. Then the $p$th power map $\varphi : S \to S$ factors through $R$, since $\varphi$ kills any divided power ideal in characteristic $p$. We let $\phi : R \to S$ denote the map factoring $\varphi$.

Let $\overline{\mathfrak{X}}$ be a scheme of characteristic $p$. For any quasi-coherent sheaf $\mathscr{F}$ on $\overline{\mathfrak{X}}$ we define a crystal of quasi-coherent $\mathcal{O}_{\overline{\mathfrak{X}}/\mathbb{Z}_p}$-modules $\Phi^*\mathscr{F}$ killed by $p$, as follows: for any open affine subscheme $\operatorname{Spec} R \subset \overline{\mathfrak{X}}$ and a compatible divided power thickening $S \twoheadrightarrow R$ with $p$ nilpotent, we set $(\Phi^*\mathscr{F})(S) := S/pS \otimes_{\phi,R} \mathscr{F}(R)$, where $\phi : R \to S/pS$ is as defined above. (The crystal $\Phi^*\mathscr{F}$ was denoted by $\varphi^*\mathscr{F}$ in [Gro74, Ch. IV, §3.3].)

We continue to assume that $\overline{\mathfrak{X}}$ is a scheme of characteristic $p$, and let $\mathscr{E}_{\overline{\mathfrak{X}}}$ denote the locally free $\mathcal{O}_{\overline{\mathfrak{X}}}$-module obtained from the pull back of $\mathscr{E}$ to the Zariski topos. Then we have a natural isomorphism $\varphi^*(\mathscr{E}/p\mathscr{E}) \cong \Phi^*(\mathscr{E}_{\overline{\mathfrak{X}}})$; indeed, for any compatible divided power thickening $S \twoheadrightarrow R$ as before, we have

$$\varphi^*(\mathscr{E}/p\mathscr{E})(S) = S/pS \otimes_{\varphi,S} \mathscr{E}(S) \cong S/pS \otimes_{\phi,R} \mathscr{E}(R) = \Phi^*(\mathscr{E}_{\overline{\mathfrak{X}}})(S);$$

*cf.* [Gro74, Ch. IV, §3.4].

We may clearly extend the definition of $\Phi^*\mathscr{F}$ and the isomorphism $\varphi^*(\mathscr{E}/p\mathscr{E}) \cong \Phi^*(\mathscr{E}_{\overline{\mathfrak{X}}})$ when $\overline{\mathfrak{X}}$ is a formal scheme of characteristic $p$.

From now on, we let $\mathfrak{X}$ be a formal scheme over $\operatorname{Spf} \mathbb{Z}_p$, and set $\overline{\mathfrak{X}} := \mathfrak{X} \times_{\operatorname{Spf} \mathbb{Z}_p} \operatorname{Spec} \mathbb{F}_p$. Let $\mathscr{E}$ be a crystal of locally free $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$-modules equipped with $F$ and $V$ as in [**?**, Definition 3.1(2)]; in particular, if $\mathfrak{X}$ is of characteristic $p$, then $F : \varphi^*\mathscr{E} \to \mathscr{E}$

and $V : \mathscr{E} \to \varphi^*\mathscr{E}$ are such that $FV = p$ and $VF = p$. Let $\mathscr{E}_{\overline{\mathfrak{X}}}$ denote the restriction of $\mathscr{E}_{\mathfrak{X}}$ to $\overline{\mathfrak{X}}$.

**Definition 1.** (*Cf.* [Gro74, Ch. V, §3].) A subbundle $\mathrm{Fil}^1 \mathscr{E}_{\mathfrak{X}} \subset \mathscr{E}_{\mathfrak{X}}$ over $\mathfrak{X}$ is said to be an *admissible filtration* if $\Phi^*(\mathrm{Fil}^1 \mathscr{E}_{\overline{\mathfrak{X}}})$ is the kernel of $F : \varphi^*(\mathscr{E}/p\mathscr{E}) \to \mathscr{E}/p\mathscr{E}$ under the identification $\varphi^*(\mathscr{E}/p\mathscr{E}) \cong \Phi^*(\mathscr{E}_{\overline{\mathfrak{X}}})$, where $\mathrm{Fil}^1 \mathscr{E}_{\overline{\mathfrak{X}}}$ is the restriction of $\mathrm{Fil}^1 \mathscr{E}_{\mathfrak{X}}$ to $\overline{\mathfrak{X}}$.

Admissible filtrations have the following concrete description. For simplicity, let us assume that $\mathfrak{X} = \overline{\mathfrak{X}} = \mathrm{Spec}\, R$. Then a filtration $\mathrm{Fil}^1 \mathscr{E}_{\mathfrak{X}} \subset \mathscr{E}_{\mathfrak{X}}$ is admissible if and only if for any divided power thickening $S \twoheadrightarrow R$ with $pS = 0$, the following $S$-submodule

$$S \otimes_{\phi, R} \mathrm{Fil}^1 \mathscr{E}_{\mathfrak{X}}(R) \subset S \otimes_{\phi, R} \mathscr{E}(R) \cong \varphi^*\mathscr{E}(S)$$

is the kernel of $F : \varphi^*\mathscr{E}(S) \to \mathscr{E}(S)$, where $\phi : R \to S$ is the map factoring the $p$th power map on $S$.

*Remark* 2. Let $\mathfrak{X}$ be any formal scheme over $\mathrm{Spf}\, \mathbb{Z}_p$. It is clear that if $\mathrm{Fil}^1 \mathscr{E}_{\mathfrak{X}} \subset \mathscr{E}_{\mathfrak{X}}$ is an admissible filtration then $\varphi^* \mathrm{Fil}^1 \mathscr{E}_{\overline{\mathfrak{X}}}$ is the kernel of $F : \varphi^*\mathscr{E}_{\overline{\mathfrak{X}}} \to \mathscr{E}_{\overline{\mathfrak{X}}}$; in other words, any admissible filtration satisfies [**?**, Definition 3.1(3)]. When $\mathfrak{X}$ is such that $\overline{\mathfrak{X}}$ is a scheme locally admitting a $p$-basis, then [**?**, Definition 3.1(3)] is equivalent to admissibility by [BM90, Proposition 1.3.3]. On the other hand, admissibility is in general stronger than [**?**, Definition 3.1(3)], especially when $\overline{\mathfrak{X}}$ is a non-reduced scheme (for example, if $\mathfrak{X} = \mathrm{Spf}\, \mathscr{O}_K$ where $\mathscr{O}_K$ is a ramified extension of $\mathbb{Z}_p$). Indeed, if $\overline{\mathfrak{X}} = \mathrm{Spec}\, k[\epsilon]/(\epsilon^2)$ for a perfect field $k$ of characteristic $p$, then any lift of admissible filtration over $\mathrm{Spec}\, k$ generates the kernel of $F : \varphi^*\mathscr{E}_{\overline{\mathfrak{X}}} \to \mathscr{E}_{\overline{\mathfrak{X}}}$. On the other hand, there is at most one admissible filtration for the following reason. Consider $S := k[\ epsilon]/(\epsilon^{2p})$ with the usual divided power structure on $(\epsilon^2)$. Then since $\phi : R \to S$ is injective, $\mathrm{Fil}^1 \mathscr{E}_{\overline{\mathfrak{X}}}$ is uniquely determined by $\Phi^*(\mathrm{Fil}^1 \mathscr{E}_{\overline{\mathfrak{X}}})(S)$, which is uniquely determined by $F$ by admissibility.

**Lemma 3.** *Let $G$ be a $p$-divisible group over a formal scheme $\mathfrak{X}$ over* $\mathrm{Spf}\, \mathbb{Z}_p$*. Then the Hodge filtration* $\mathrm{Fil}^1 \mathbb{D}^*(G)_{\mathfrak{X}} \subset \mathbb{D}^*(G)_{\mathfrak{X}}$ *is admissible.*

*Proof.* We may assume that $\mathfrak{X} = \mathrm{Spec}\, R$ where $R$ is a ring of characteristic $p$. For any divided power thickening $S \twoheadrightarrow R$ of characteristic $p$, we want to show that

$$S \otimes_{\phi, R} \mathrm{Fil}^1 \mathbb{D}^*(G)(R) \subset \varphi^*\mathbb{D}^*(G)(S)$$

is the kernel of $F$. When $S = R$, this was proved in [BBM82, Proposition 4.3.10]. The case with general $S$ can be reduced to the case with $S = R$ by choosing a $S$-lift $G_S$ of $G$; here, we use the fact that the natural isomorphism $\mathbb{D}^*(G_S)(S) \cong \mathbb{D}^*(G)(S)$ ([BM90, Théorème 3.1.7]) and the Hodge filtration are functorial and commute with base change. □

## 2. Correction of [?, Proposition 3.8]

The statement of [**?**, Proposition 3.8] holds for the *corrected* definition of Dieudonné crystal, which can be proved as follows. By the last paragraph of the proof of [**?**, Proposition 3.8], it suffices to handle the case when $p$ is nilpotent in $R$. (Note that admissibility could be checked after replacing $R$ with $R/pR$.) By the second paragraph of the proof of [**?**, Proposition 3.8], it remains to show the following lemma:

**Lemma 4.** *Assume that $p$ is nilpotent in $R$. Let $(\mathscr{E}, F, V, \mathrm{Fil}^1 \mathscr{E}(R))$ be a tuple satisfying (1) and (2) of* [**?**, Definition 3.1]*, and consider $(\mathcal{M}, \varphi_{\mathcal{M}}, \mathrm{Fil}^1 \mathcal{M})$ associated to it by the recipe of* [**?**, Remark 3.7]. *Then $\varphi_{\mathcal{M}}(\mathrm{Fil}^1 \mathcal{M})$ generates $p\mathcal{M}$. if the Hodge filtration* $\mathrm{Fil}^1 \mathscr{E}(R)$ *is admissible. The converse holds when $\widehat{D}$ is constructed as in* [**?**, Remark 3.5].

*Proof.* Let $\mathrm{Fil}^1 \varphi^*\mathcal{M} \subset \varphi^*\mathcal{M}$ and $\mathrm{Fil}^1 \varphi^*(\mathcal{M}/p\mathcal{M}) \subset \varphi^*(\mathcal{M}/p\mathcal{M})$ respectively denote the submodules generated by the image of $\mathrm{Fil}^1 \mathcal{M}$. Then we claim that $\varphi_{\mathcal{M}}(\mathrm{Fil}^1 \mathcal{M})$ generates $p\mathcal{M}$ if and only if $\mathrm{Fil}^1 \varphi^*(\mathcal{M}/p\mathcal{M})$ is the kernel of $1 \otimes \varphi_{\mathcal{M}} : \varphi^*(\mathcal{M}/p\mathcal{M}) \to \mathcal{M}/p\mathcal{M}$; indeed, the first condition is satisfied if and only if $\mathrm{Fil}^1 \varphi^*\mathcal{M}$ is the image of the map induced by $V$, which can be checked modulo $p$. We now conclude using $\ker F = \mathrm{im}\, V$ in $\varphi^*(\mathscr{E}/p\mathscr{E})$.

The natural isomorphism $\varphi^*(\mathcal{M}/p\mathcal{M}) \cong \widehat{D}/p\widehat{D} \otimes_{\phi,R/pR} \mathscr{E}(R/pR)$ restricts to $\mathrm{Fil}^1 \varphi^*(\mathcal{M}/p\mathcal{M}) \cong \widehat{D}/p\widehat{D} \otimes_{\phi,R/pR} \mathrm{Fil}^1 \mathscr{E}(R/pR)$. If $\mathrm{Fil}^1 \mathscr{E}(R)$ is admissible, then $\mathrm{Fil}^1 \varphi^*(\mathcal{M}/p\mathcal{M})$ is the kernel of $1 \otimes \varphi_{\mathcal{M}} : \varphi^*(\mathcal{M}/p\mathcal{M}) \to \mathcal{M}/p\mathcal{M}$, which shows the first claim.

Now, assume that $\widehat{D}$ is constructed as in [**?**, Remark 3.5] and $\mathrm{Fil}^1 \varphi^*(\mathcal{M}/p\mathcal{M})$ is the kernel of $1 \otimes \varphi_{\mathcal{M}} : \varphi^*(\mathcal{M}/p\mathcal{M}) \to \mathcal{M}/p\mathcal{M}$. Let $S \twoheadrightarrow R/pR$ be a divided power thickening with $pS = 0$. By construction of $\widehat{D}$, there exists a divided power morphism $\widehat{D}/p\widehat{D} \to S$ lifting the natural projection onto $R$ (*cf.* [**?**, Remark 3.5, Lemma 2.1]), so we have a natural isomorphism $\mathscr{E}(S) \cong S \otimes_{\widehat{D}} \mathcal{M}$. Since $\phi : R/pR \to \widehat{D}/p\widehat{D}$ factors $\phi : R/pR \to S$, we have

$$S \otimes_{\widehat{D}/p\widehat{D}} \mathrm{Fil}^1 \varphi^*(\mathcal{M}/p\mathcal{M}) \cong S \otimes_{\phi,R/pR} \mathrm{Fil}^1 \mathscr{E}(R/pR) \subset \varphi^*\mathscr{E}(S),$$

which is the kernel of $F : \varphi^*\mathscr{E}(S) \to \mathscr{E}(S)$ since $\mathrm{Fil}^1 \varphi^*(\mathcal{M}/p\mathcal{M})$ is the kernel of $F : \varphi^*\mathscr{E}(\widehat{D}/p\widehat{D}) \to \mathscr{E}(\widehat{D}/p\widehat{D})$. In other words, $\mathrm{Fil}^1 \mathscr{E}(R)$ is admissible. □

## References


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