ERRATUM TO “THE RELATIVE BREUIL-KISIN CLASSIFICATION OF p-DIVISIBLE GROUPS AND FINITE FLAT GROUP SCHEMES” \[Kim14\]

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It was pointed out by E. Lau that Definition 3.1 and Proposition 3.8 in \[?\] need to be corrected as follows:

1. In the definition of Dieudonné crystal \[?\], Definition 3.1(3)], the condition on the filtration \[?\], Definition 3.1(3)] should be replaced by requiring \(\text{Fil}^1 E_X\) to be an admissible filtration, as introduced by Grothendieck \[\text{Gro74}, \text{Ch. V, } \S 3\]. We recall the definition below (cf. Definition 1), and verify that the Hodge filtration for any \(p\)-divisible group \(G\) over \(X\) is admissible (cf. Lemma 3), which makes \(D^+(G)\) a Dieudonné crystal according to the corrected definition.

2. The statement of \[?\], Proposition 3.8] holds for the corrected definition of Dieudonné crystal. In \(\S 2\) we explain how to correct the proof of \[?\], Proposition 3.8].

In other words, Dieudonné crystals with the corrected definition have the claimed semilinear algebraic interpretation in terms of filtered Frobenius modules (cf. \[?\], Proposition 3.8]), therefore the rest of the results of \[?\] hold true for the corrected definition of Dieudonné crystals.

The author is deeply grateful to E. Lau for informing the author of this error and directing him to \[\text{Gro74}\].

1. Admissible Filtration: Correction of \[?\], Definition 3.1(3)]

Let \(S \to R\) be a divided power thickening in characteristic \(p\). Then the \(p\)th power map \(\varphi : S \to S\) factors through \(R\), since \(\varphi\) kills any divided power ideal in characteristic \(p\). We let \(\phi : R \to S\) denote the map factoring \(\varphi\).

Let \(\overline{X}\) be a scheme of characteristic \(p\). For any quasi-coherent sheaf \(\mathcal{F}\) on \(\overline{X}\) we define a crystal of quasi-coherent \(O_{\overline{X}/\mathbb{Z}_p}\)-modules \(\Phi^* \mathcal{F}\) killed by \(p\), as follows: for any open affine subscheme \(\text{Spec } R \subset \overline{X}\) and a compatible divided power thickening \(S \to R\) as before, we have

\[
\phi^* (\mathcal{F} / p \mathcal{F}) = \mathcal{S} / p \mathcal{S} \otimes_{\mathcal{S}, \phi, R} \mathcal{F} (R) = \mathcal{S} / p \mathcal{S} \otimes_{\mathcal{S}, \phi, R} \mathcal{F} (R) \]

where \(\mathcal{S}\) is the isomorphism \(\varphi^* (\mathcal{F} / p \mathcal{F}) \cong \Phi^* (\mathcal{E})\); indeed, for any compatible divided power thickening \(S \to R\) as before, we have

\[
\varphi^* (\mathcal{E} / p \mathcal{E}) (S) = S / p S \otimes_{\varphi, S} \mathcal{E} (S) \cong S / p S \otimes_{\phi, R} \mathcal{E} (R) = \Phi^* (\mathcal{E}) (S);
\]

cf. \[\text{Gro74}, \text{Ch. IV, } \S 3.4\].

We may clearly extend the definition of \(\Phi^* \mathcal{F}\) and the isomorphism \(\varphi^* (\mathcal{E} / p \mathcal{E}) \cong \Phi^* (\mathcal{E})\) when \(\overline{X}\) is a formal scheme of characteristic \(p\).

From now on, we let \(\mathcal{X}\) be a formal scheme over \(\text{Spf } \mathbb{Z}_p\), and set \(\mathcal{X} := \mathcal{X} \times_{\text{Spf } \mathbb{Z}_p} \text{Spec } \mathbb{F}_p\). Let \(\mathcal{E}\) be a crystal of locally free \(O_{\mathcal{X}/\mathbb{Z}_p}\)-modules equipped with \(F\) and \(V\) as in \[?\], Definition 3.1(2)]; in particular, if \(\mathcal{X}\) is of characteristic \(p\), then \(F : \varphi^* \mathcal{E} \to \mathcal{E}\)
and $V : \mathcal{E} \to \varphi^*\mathcal{E}$ are such that $FV = p$ and $VF = p$. Let $\mathcal{E}_X$ denote the restriction of $\mathcal{E}_\mathcal{X}$ to $\mathcal{X}$.

**Definition 1.** (Cf. [Gro74], Ch. V, §3.) A subbundle $\text{Fil}^1 \mathcal{E}_X \subset \mathcal{E}_X$ over $\mathcal{X}$ is said to be an admissible filtration if $\Phi^*(\text{Fil}^1 \mathcal{E}_\mathcal{X})$ is the kernel of $F : \varphi^*(\mathcal{E}/p\mathcal{E}) \to \mathcal{E}/p\mathcal{E}$ under the identification $\varphi^*(\mathcal{E}/p\mathcal{E}) \cong \Phi^*(\mathcal{E}_\mathcal{X})$, where $\text{Fil}^1 \mathcal{E}_\mathcal{X}$ is the restriction of $\text{Fil}^1 \mathcal{E}_\mathcal{X}$ to $\mathcal{X}$.

Admissible filtrations have the following concrete description. For simplicity, let us assume that $\mathcal{X} = \mathcal{X} = \text{Spec} R$. Then a filtration $\text{Fil}^1 \mathcal{E}_X \subset \mathcal{E}_X$ is admissible if and only if for any divided power thickening $S \to R$ with $pS = 0$, the following $S$-submodule

$$S \otimes_{\varphi,R} \text{Fil}^1 \mathcal{E}_X(R) \subset S \otimes_{\varphi,R} \mathcal{E}(R) \cong \varphi^*\mathcal{E}(S)$$

is the kernel of $F : \varphi^*\mathcal{E}(S) \to \mathcal{E}(S)$, where $\varphi : R \to S$ is the map factoring the $p$th power map on $S$.

**Remark 2.** Let $\mathcal{X}$ be any formal scheme over $\text{Spf} \mathbb{Z}_p$. It is clear that if $\text{Fil}^1 \mathcal{E}_X \subset \mathcal{E}_X$ is an admissible filtration then $\varphi^*\text{Fil}^1 \mathcal{E}_\mathcal{X}$ is the kernel of $F : \varphi^*\mathcal{E}_\mathcal{X} \to \mathcal{E}_\mathcal{X}$; in other words, any admissible filtration satisfies [?, Definition 3.1(3)]. When $\mathcal{X}$ is such that $\mathcal{X}$ is a scheme locally admitting a $p$-basis, then [?, Definition 3.1(3)] is equivalent to admissibility by [BM90, Proposition 1.3.3]. On the other hand, admissibility is in general stronger than [?, Definition 3.1(3)], especially when $\mathcal{X}$ is a non-reduced scheme (for example, if $\mathcal{X} = \text{Spf} \mathcal{E}_K$ where $\mathcal{E}_K$ is a ramified extension of $\mathbb{Z}_p$).

Indeed, if $\mathcal{X} = \text{Spec} k[G]/(\epsilon^2)$ for a perfect field $k$ of characteristic $p$, then any lift of an admissible filtration over $\text{Spec} k$ generates the kernel of $F : \varphi^*\mathcal{E}_\mathcal{X} \to \mathcal{E}_\mathcal{X}$. On the other hand, there is at most one admissible filtration for the following reason. Consider $S := k[\epsilon]/(\epsilon^{2p})$ with the usual divided power structure on $(\epsilon^2)$. Then since $\varphi : R \to S$ is injective, $\text{Fil}^1 \mathcal{E}_\mathcal{X}$ is uniquely determined by $\Phi^*(\text{Fil}^1 \mathcal{E}_\mathcal{X})(S)$, which is uniquely determined by $F$ by admissibility.

**Lemma 3.** Let $G$ be a $p$-divisible group over a formal scheme $\mathcal{X}$ over $\text{Spf} \mathbb{Z}_p$. Then the Hodge filtration $\text{Fil}^1 \mathcal{E}(S) \subset \mathcal{E}(S)$ is admissible.

**Proof.** We may assume that $\mathcal{X} = \text{Spec} R$ where $R$ is a ring of characteristic $p$. For any divided power thickening $S \to R$ of characteristic $p$, we want to show that

$$S \otimes_{\varphi,R} \text{Fil}^1 \mathcal{E}(S) \subset \varphi^*\mathcal{E}(S)$$

is the kernel of $F$. When $S = R$, this was proved in [BM82, Proposition 4.3.10]. The case with general $S$ can be reduced to the case with $S = R$ by choosing a $S$-lift $G_S$ of $G$; here, we use the fact that the natural isomorphism $D^*(G_S)(S) \cong D^*(G)(S)$ ([BM90, Théorème 3.1.7]) and the Hodge filtration are functorial and commute with base change.

### 2. Correction of [?, Proposition 3.8]

The statement of [?, Proposition 3.8] holds for the corrected definition of Dieudonné crystal, which can be proved as follows. By the last paragraph of the proof of [?, Proposition 3.8], it suffices to handle the case when $p$ is nilpotent in $R$. Note that admissibility could be checked after replacing $R$ with $R/pR$. By the second paragraph of the proof of [?, Proposition 3.8], it remains to show the following lemma:

**Lemma 4.** Assume that $p$ is nilpotent in $R$. Let $(\mathcal{E}, F, V, \text{Fil}^1 \mathcal{E}(R))$ be a tuple satisfying (1) and (2) of [?, Definition 3.1], and consider $(\mathcal{M}, \varphi_M, \text{Fil}^1 \mathcal{M})$ associated to it by the recipe of [?, Remark 3.7]. Then $\varphi_M(\text{Fil}^1 \mathcal{M})$ generates $p\mathcal{M}$. If the Hodge filtration $\text{Fil}^1 \mathcal{E}(R)$ is admissible. The converse holds when $\mathcal{D}$ is constructed as in [?, Remark 3.5].
Proof. Let $\text{Fil}^1 \varphi_* M \subset \varphi_* M$ and $\text{Fil}^1 \varphi^*(M/pM) \subset \varphi^*(M/pM)$ respectively denote the submodules generated by the image of $\text{Fil}^1 M$. Then we claim that $\varphi_* (\text{Fil}^1 M)$ generates $pM$ if and only if $\text{Fil}^1 \varphi^*(M/pM)$ is the kernel of $1 \otimes \varphi_* M : \varphi^*(M/pM) \to M/pM$; indeed, the first condition is satisfied if and only if $\text{Fil}^1 \varphi_* M$ is the image of the map induced by $V$, which can be checked modulo $p$. We now conclude using $\ker F = \text{im} V$ in $\varphi^*(\mathcal E/p\mathcal E)$.

The natural isomorphism $\varphi^*(M/pM) \cong \hat{D}/\hat{pD} \otimes_{\phi,R/pR} \mathcal E(R/pR)$ restricts to $\text{Fil}^1 \varphi^*(M/pM) \cong \hat{D}/\hat{pD} \otimes_{\phi,R/pR} \text{Fil}^1 \mathcal E(R/pR)$. If $\mathcal E(R)$ is admissible, then $\text{Fil}^1 \varphi^*(M/pM)$ is the kernel of $1 \otimes \varphi_* M : \varphi^*(M/pM) \to M/pM$, which shows the first claim.

Now, assume that $\hat{D}$ is constructed as in [?, Remark 3.5] and $\text{Fil}^1 \varphi^*(M/pM)$ is the kernel of $1 \otimes \varphi_* M : \varphi^*(M/pM) \to M/pM$. Let $S \to R/pR$ be a divided power thickening with $pS = 0$. By construction of $\hat{D}$, there exists a divided power morphism $\hat{D}/\hat{pD} \to S$ lifting the natural projection onto $R$ (cf. [?, Remark 3.5, Lemma 2.1]), so we have a natural isomorphism $\mathcal E(S) \cong S \otimes_{\hat{D}} M$. Since $\phi : R/pR \to \hat{D}/\hat{pD}$ factors $\phi : R/pR \to S$, we have

$$S \otimes_{\hat{D}/\hat{pD}} \text{Fil}^1 \varphi^*(M/pM) \cong S \otimes_{\phi,R/pR} \text{Fil}^1 \mathcal E(R/pR) \subset \varphi^* \mathcal E(S),$$

which is the kernel of $F : \varphi^* \mathcal E(S) \to \mathcal E(S)$ since $\text{Fil}^1 \varphi^*(M/pM)$ is the kernel of $F : \varphi^* \mathcal E(\hat{D}/\hat{pD}) \to \mathcal E(\hat{D}/\hat{pD})$. In other words, $\text{Fil}^1 \mathcal E(R)$ is admissible. \hfill $\Box$

\section*{References}


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