Optimal stopping without Snell envelopes

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Abstract

This paper proves the existence of optimal stopping times via elementary functional analytic arguments. The problem is first relaxed into a convex optimization problem over a closed convex subset of the unit ball of the dual of a Banach space. The existence of optimal solutions then follows from the Banach–Alaoglu compactness theorem and the Krein–Milman theorem on extreme points of convex sets. This approach seems to give the most general existence results known to date. Applying convex duality to the relaxed problem gives a new dual problem and optimality conditions in terms of martingales that dominate the reward process.

Keywords. optimal stopping, Banach spaces, duality

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1 Introduction

Given a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual hypotheses, let R be an optional process of class (D), and consider the optimal stopping problem

$$\text{maximize} \quad ER_\tau \quad \text{over} \quad \tau \in \mathcal{T}, \tag{OS}$$

where $\mathcal{T}$ is the set of stopping times with values in $[0,T]\cup\{T+\}$. When $T=\infty$, the interval $[0,T]$ is interpreted as the one-point compactification of the positive reals. Unless specified otherwise, all processes in this paper are assigned the value zero at $T+$, so the role of $T+$ is to allow not to stop at all. This essentially means that R is assumed nonnegative. This does not restrict generality since, when R is of class (D), (OS) can be written in terms of a positive reward process. Indeed, by [15], there exists a closed martingale M that minorizes R, so $ER_\tau = E(R-M)_\tau + EM_0$.

Optimal stopping times are rarely unique and, without further conditions, they need not exist (take any deterministic process R whose supremum is not attained). Theorem II.2 of Bismut and Skalli [8] establishes the existence for bounded reward processes R such that $R \geq \overleftarrow{R}$ and $\vec{R} \leq {}^pR$, where for $t \in [0,T]$, $\overleftarrow{R}_T := 0$ and

$$\vec{R}_t := \lim_{s \nearrow t} \sup_{r \in (s,t)} R_r \quad \text{and} \quad \overleftarrow{R}_t := \lim_{s \searrow t} \sup_{r \in (t,s)} R_s,$$

the *left-* and *right-upper limits* of R, respectively, and ${}^pR$ is the predictable projection of R.

In order to extend the above, we follow Bismut [7] and first consider the "optimal quasi-stopping problem"

$$\text{maximize} \quad E[R_\tau + \vec{R}_{\tilde{\tau}}] \quad \text{over} \quad (\tau, \tilde{\tau}) \in \hat{\mathcal{T}}, \tag{OQS}$$

where $\hat{\mathcal{T}}$ is the set of *quasi-stopping times* ("split stopping time" in Dellacherie and Meyer [12, page 409]) defined by

$$\hat{\mathcal{T}} := \{(\tau, \tilde{\tau}) \in \mathcal{T} \times \mathcal{T}_p \mid \tilde{\tau} > 0,\ \tau \vee \tilde{\tau} = T+\},$$

where $\mathcal{T}_p$ is the set of predictable times. When R is cádlág , $\vec{R}$ coincides with the left limit R_- of R, and our formulation of the quasi-optimal stopping coincides with that of Bismut [7]. An advantage of (OQS) is that it may admit solutions even when (OS) does not. It also provides an indirect way of proving existence of solutions for (OS); see Theorem 13 below. Existence results for (OQS) has applications also in the theory of Markov processes; see [7, Section 3]. Assuming that R is a bounded cádlág process, Bismut [7] shows that the optimum values of (OS) and (OQS) coincide and that (OQS) admits solutions.

This paper generalizes the existence results of [8] and [7] by employing simple functional analytic arguments building on Banach spaces of stochastic processes from [18]. For (OQS), we relax the assumption of [7] that the paths of the reward process R are cádlág . We prove existence under the much weaker assumption that the paths of R are right-upper semicontinuous in the sense that $R \geq \overleftarrow{R}$. As a corollary, we obtain the existence for (OS) when, in addition, $\vec{R} \leq {}^pR$. This generalizes the existence result [8, Theorem II.2] by relaxing the boundedness assumption on R as suggested already on page 301 of [8].

Our existence proofs are based on functional analytical arguments that avoid the use of Snell envelopes which are used in most analyses of optimal stopping. Our strategy is to first look at a convex relaxation of the problem. This turns out be a linear optimization problem over a compact convex set of random measures whose extremal points can be identified with (quasi-)stopping times. As soon as the objective function is upper semicontinuous on this set, Krein-Milman theorem gives the existence of (quasi-)stopping times. Sufficient conditions for upper semicontinuity are obtained as a simple application of the main result of Perkkiö and Trevino [19]. The overall approach was suggested already on page 287 of Bismut [6] in the case of optimal stopping. We extended the strategy (and provide explicit derivations) to quasi-optimal stopping for a merely right-upper semicontinuous reward process.

The last section of the paper develops a dual problem and optimality conditions for (OS) and (OQS). We find that the optimum values of (OS) and (OQS) coincide without any path properties, thus extending [7, Proposition 1.2]. The dual variables turn out to be martingales that dominate R. As a simple consequence, we generalize the duality result of Davis and Karatzas [10] to reward processes that are merely of class (D) without any path properties.

2 Regular processes

In this section, the reward process R is assumed to be *regular*, i.e. an optional cádlág process of class (D) such that the left limit R_- and the predictable projection ${}^pR$ of R are indistinguishable; see e.g. [5] or [12, Remark 50.d]. Recall that a measurable process y is of class (D) if the set $\{y_\tau \,|\, \tau \in \mathcal{T}\}$ is uniformly integrable. The *predictable projection* of such a y is the unique (up to indistinguishability) predictable process ${}^py$ such that

$$E[y_\tau \mid \mathcal{F}_{\tau-}] = {}^py_\tau \quad P\text{-a.s.}$$

for all predictable times τ. Here both y and ${}^py$ are assigned the value zero at $T+$ and $\mathcal{F}_{\tau-} := \mathcal{F}_0 \vee \sigma(\{A \cap \{t < \tau\} \mid A \in \mathcal{F}_{t-},\ t \in [0,T]\})$; see [12, Section VI.2] for further details. Our analysis will be based on the fact that the space of regular processes is a Banach space whose dual can be identified with optional measures of essentially bounded variation; see Theorem 1 below. The class of regular processes is quite general as it includes e.g. all continuous, Levy and Feller processes as soon as they are of class (D); see [17, Remark 1].

The space M of Radon measures on $[0,T]$ may be identified with the space X_0 of left-continuous functions of bounded variation on $[0,T] \cup \{T+\}$ such that $x_0 = 0$. Indeed, for every $x \in X_0$, there exists a unique $Dx \in M$ such that $x_t = Dx([0,t))$ for all $t \in [0,T] \cup \{T+\}$. Here $[0,T+) := [0,T]$. Thus $x \mapsto Dx$ defines a linear isomorphism between X_0 and M. Similarly, the space $\mathcal{M}^\infty$ of optional random measures with essentially bounded total variation may be identified with the space $\mathcal{N}_0^\infty$ of adapted processes x with $x \in X_0$ almost surely and $Dx \in \mathcal{M}^\infty$.

The space C of continuous functions on $[0,T]$ is a Banach space under the supremum norm $\|\cdot\|$. Its dual can be identified with the space M and the dual norm is the total variation norm $\|\cdot\|_{TV}$. Let $L^1(C)$ be the space of (not necessarily adapted) continuous processes y with $E\|y\| < \infty$. The norm $E\|y\|$ makes $L^1(C)$ into a Banach space whose dual can be identified with the space $L^\infty(M)$ of random measures whose pathwise total variation is essentially bounded. The following result is essentially from [5]; see [17, Theorem 8] or [18, Corollary 16]. It provides the functional analytic setting for analyzing optimal stopping with regular processes. Recall that the *optional projection* ${}^oy$ of a measurable process y of class (D) is the unique (up to indistinguishability) optional process such that

$$E[y_\tau \mid \mathcal{F}_\tau] = {}^oy_\tau \quad P\text{-a.s.}$$

for all stopping times τ. Here both y and ${}^o y$ are assigned the value zero at $T+$ and $\mathcal{F}_\tau := \{A \in \mathcal{F} \mid A \cap \{\tau \le t\} \in \mathcal{F}_t \ \forall t \in [0,T]\}$; see [12, Section VI.2] for further details.

Theorem 1. *The space $\mathcal{R}^1$ of regular processes equipped with the norm*

$$\|y\|_{\mathcal{R}^1} := \sup_{\tau \in \mathcal{T}} E|y_\tau|$$

is Banach and its dual can be identified with $\mathcal{M}^\infty$ through the bilinear form

$$\langle y, u\rangle = E\int_{[0,T]} y du.$$

The optional projection is a continuous surjection of $L^1(C)$ to $\mathcal{R}^1$ and its adjoint is the embedding of $\mathcal{M}^\infty$ to $L^\infty(M)$. The norm of $\mathcal{R}^1$ is equivalent to

$$p(y) := \inf_{z \in L^1(C)} \{E\|z\| \mid {}^o z = y\}$$

which has the dual representation

$$p(y) = \sup\{\langle y, u\rangle \mid \text{ess sup}(\|u\|_{TV}) \le 1\}.$$

We first write the optimal stopping problem (OS) as

$$\text{maximize} \quad \langle R, Dx\rangle \quad \text{over} \quad x \in \mathcal{C}_e,$$

where

$$\mathcal{C}_e := \{x \in \mathcal{N}^\infty_{0+} \,|\, x_t \in \{0,1\}\},$$

where $\mathcal{N}^\infty_{0+}$ denotes the nondecreasing processes of $\mathcal{N}^\infty_0$. The equation $\tau(\omega) = \inf\{t \in [0,T] \mid x_{t+}(\omega) \ge 1\}$, where the infimum over the empty set is defined as $T+$, gives a one-to-one correspondence between the elements of $\mathcal{T}$ and $\mathcal{C}_e$. Consider also the convex relaxation of minimizing $\langle R, Dx\rangle$ over the set

$$\mathcal{C} := \{x \in \mathcal{N}^\infty_{0+} \,|\, x_{T+} \le 1\}.$$

The relaxation can be written as

$$\text{maximize} \quad E[\int_{[0,T]} R dx] \quad \text{over} \quad x \in \mathcal{C}, \tag{ROS}$$

which makes sense for any reward process R regular or not.

Clearly, $\mathcal{C}_e \subset \mathcal{C}$ so the optimum value of optimal stopping is dominated by the optimum value of the relaxation. Given $x \in \mathcal{C}$ with $x_{T+} = 1$, the function $S : \Omega \times [0,1] \to [0,T]$ given by $S(\omega,\alpha) := \inf\{t \in [0,T] \mid x_t(\omega) \ge \alpha\}$ is adapted, nondecreasing and left-continuous in α, so it is a *randomized stopping time* in the sense of Baxter and Chacon [2, Section 2]. Edgar, Millet and Sucheston [13] employed the approach of [2] in stopping of Banach space-valued processes. Recent applications of randomized stopping times can be found in Belomestny and Krätschmer [4] who studied optimal stopping under model uncertainty. Our formulation in terms of increasing processes on $[0,T]\cup\{T+\}$ is closer to those of Bismut [7] and Touzi and Vieille [24] who identified randomized stopping times with right-continuous increasing processes but the corresponding optional measures are the same as ours. As we will see, the analysis of optimal stopping via Theorem 1 is very simple.

Recall that $x\in\mathcal{C}$ is an *extreme point* of $\mathcal{C}$ if it cannot be expressed as a convex combination of two points of $\mathcal{C}$ different from x.

Lemma 2. *The set $\mathcal{C}$ is convex, $\sigma(\mathcal{N}_0^\infty,\mathcal{R}^1)$-compact and $\mathcal{C}_e$ is the set of its extreme points.*

Proof. The set $\mathcal{C}$ is a closed convex subset of the unit ball that $\mathcal{N}_0^\infty$ has as the dual of the Banach space $\mathcal{R}^1$. The compactness thus follows from Banach-Alaoglu. It is easily shown that the elements of $\mathcal{C}_e$ are extreme points of $\mathcal{C}$. On the other hand, if $x\in\mathcal{C}\setminus\mathcal{C}_e$ there exists an $\bar{s}\in(0,1)$ such that the processes

$$x^1_t:=\frac{1}{\bar{s}}[x_t\wedge\bar{s}]\quad\text{and}\quad x^2_t:=\frac{1}{1-\bar{s}}[(x_t-\bar{s})\vee 0]$$

are different elements of $\mathcal{C}$. Since $x=\bar{s}x^1+(1-\bar{s})x^2$, it is not an extreme point of $\mathcal{C}$. □

Since the function $x\mapsto\langle R,Dx\rangle$ is continuous, the compactness of $\mathcal{C}$ in Lemma 2 implies that the maximum in (ROS) is attained. The fact that the maximum is attained at a genuine stopping time follows from the characterization of the extreme points in Lemma 2 and the classical Krein-Milman theorem or the following variant of it; see e.g. [9, Theorem 25.9].

Theorem 3 (Bauer's maximum principle)**.** *In a locally convex Hausdorff topological vector space, an upper semicontinuous (usc) convex function on a compact convex set K attains its maximum at an extremal point of K.*

Combining Lemma 2 and Theorem 3 gives the following.

Theorem 4. *An optimal stopping time in* (OS) *exists for every $R\in\mathcal{R}^1$.*

The above seems to have been first proved in Bismut and Skalli [8, Theorem I.3], which says that a stopping time defined in terms of the Snell envelope of the regular process R is optimal. Their proof assumes bounded reward R but they note on page 301 that it actually suffices that R be of class (D). The proof of Bismut and Skalli builds on the (nontrivial) existence of a Snell envelope and further limiting arguments involving sequences of stopping times. In contrast, our proof is based on elementary functional analytic arguments in the Banach space setting of Theorem 1, which is of independent interest.

Note that x solves the relaxed optimal stopping problem if and only if R is *normal* to $\mathcal{C}$ at x, i.e. if $R\in\partial\delta_{\mathcal{C}}(x)$ or equivalently $x\in\partial\sigma_{\mathcal{C}}(R)$, where

$$\sigma_{\mathcal{C}}(R)=\sup_{x\in\mathcal{C}}\langle R,Dx\rangle.$$

Here, ∂ denotes the *subdifferential* of a function; see e.g. [21]. If R is nonnegative, we have $\sigma_{\mathcal{C}}(R) = \|R\|_{\mathcal{R}^1}$ (by Krein–Milman) and the optimal solutions of the relaxed stopping problem are simply the subgradients of the $\mathcal{R}^1$-norm at R.

3 Càdlàg processes

While the class of regular processes is quite large it excludes e.g. semimartingales whose BV-part is discontinuous; see [17, Remark 1]. This section extends the previous section to optimal quasi-stopping problems when the reward process R is merely *cádlág and of class* (D). In this case, optimal stopping times need not exist (see the discussion on page 1) but we will prove the existence of a quasi-stopping time by functional analytic arguments analogous to those in Section 2.

Even without any path properties, we have the following.

Lemma 5. *If R is of class (D), then the optimum values of* (OS) *and* (OQS) *coincide.*

Proof. Let $(\tau, \tilde{\tau}) \in \hat{\mathcal{T}}$ and $\epsilon > 0$. Let (τ_k) be as in Lemma 17 and $M := {}^p(\mathbb{1}\vec{R}_{\tilde{\tau}})$. By [11, Theorem IV.90], $\vec{R}$ is predictable, so

$$M_{\tilde{\tau}} = E[\vec{R}_{\tilde{\tau}} \mid \mathcal{F}_{\tilde{\tau}-}] = \vec{R}_{\tilde{\tau}}.$$

Moreover, M is left-continuous, so, by the definition of $\vec{R}$, the projection on Ω of the optional set

$$\{(\omega, t) \in \Omega \times [0, T] \mid t \in (\tau_k(\omega), \tilde{\tau}(\omega)), R_t(\omega) - M_t(\omega) + \epsilon > 0\}$$

has measure $P(\{\tilde{\tau} < T+\})$. By Lemma 16, there exists $\sigma_k \in \mathcal{T}$ with $\sigma_k \in (\tau_k, \tilde{\tau})$ and $R_{\sigma_k} > M_{\sigma_k} - \epsilon$ on $\{\sigma_k < T+\}$ and $P(\sigma_k < T+) \geq P(\tilde{\tau} < T+) - 1/k$. Since M is of class (D) and $1_{\{\sigma_k < T+\}} M_{\sigma_k} \to 1_{\{\tilde{\tau} < T+\}} M_{\tilde{\tau}}$ almost surely, we get

$$\begin{aligned}
\limsup E[R_{\sigma_k}] &= \limsup E[1_{\{\sigma_k < T+\}} R_{\sigma_k}] \\
&\geq \limsup E[1_{\{\sigma_k < T+\}} (M_{\sigma_k} - \epsilon)] \\
&\geq E[1_{\{\tilde{\tau} < T+\}} M_{\tilde{\tau}}] - \epsilon \\
&= E[\vec{R}_{\tilde{\tau}}] - \epsilon,
\end{aligned}$$

where the second inequality follows from Fatou's lemma. Defining $\bar{\sigma}_k = \sigma_k \wedge \tau$, we get

$$\limsup E[R_{\bar{\sigma}_k}] \geq E[R_\tau + \vec{R}_{\tilde{\tau}}] - \epsilon.$$

Since $(\tau, \tilde{\tau}) \in \hat{\mathcal{T}}$ and $\epsilon > 0$, this completes the proof. □

The space D of cádlág functions on $[0, T]$ is a Banach space under the supremum norm $\| \cdot \|$. The space of purely discontinuous Borel measures will be denoted by $\tilde{M}$. The dual of D can be identified with $M \times \tilde{M}$ through the bilinear form

$$\langle y, (u, \tilde{u}) \rangle := \int_{[0,T]} y du + \int_{[0,T]} y_- d\tilde{u}$$

and the dual norm is given by

$$\sup_{y\in D}\left\{\int_{[0,T]} ydu+\int_{[0,T]} y_-d\tilde u \,\middle|\, \|y\|\le 1\right\}=\|u\|_{TV}+\|\tilde u\|_{TV}.$$

This can be deduced from [20, Theorem 1] or seen as the deterministic special case of [12, Theorem VII.65] combined with [12, Remark VII.4(a)].

The following result from [18] provides the functional analytic setting for analyzing quasi-stopping problems with cádlág processes of class (D).

Theorem 6. *The space $\mathcal{D}^1$ of optional cádlág processes of class (D) equipped with the norm*

$$\|y\|_{\mathcal{D}^1}:=\sup_{\tau\in\mathcal{T}} E|y_\tau|$$

is Banach and its dual can be identified with

$$\hat{\mathcal{M}}^\infty:=\{(u,\tilde u)\in L^\infty(M\times\tilde M)\mid u \text{ is optional, } \tilde u \text{ is predictable}\}$$

through the bilinear form

$$\langle y,(u,\tilde u)\rangle=E\left[\int_{[0,T]} ydu+\int_{[0,T]} y_-d\tilde u\right].$$

The optional projection is a continuous surjection of $L^1(D)$ to $\mathcal{D}^1$ and its adjoint is the embedding of $\hat{\mathcal{M}}^\infty$ to $L^\infty(M\times\tilde M)$. The norm of $\mathcal{D}^1$ is equivalent to

$$p(y):=\inf_{z\in L^1(D)}\{E\|z\|\mid {}^o z=y\},$$

which has the dual representation

$$p(y)=\sup\{\langle y,(u,\tilde u)\rangle\mid \operatorname{ess\,sup}(\|u\|_{TV}+\|\tilde u\|_{TV})\le 1\}.$$

The space $M\times\tilde M$ may be identified with the space $\hat X_0$ of (not necessarily left-continuous) functions $x:\mathbb{R}_+\to\mathbb{R}$ of bounded variation which are constant on $(T,\infty]$ and have $x_0=0$. Indeed, every $x\in\hat X_0$ can be written uniquely as

$$x_t=Dx([0,t))+\tilde Dx([0,t]),$$

where $\tilde Dx\in\tilde M$ and $Dx\in M$ are the measures associated with the functions $\tilde x_t:=\sum_{s\le t}(x_s-x_{s-})$ and $x-\tilde x$, respectively. The linear mapping $x\mapsto(Dx,\tilde Dx)$ defines an isomorphism between $\hat X_0$ and $M\times\tilde M$. The value of x for $t>T$ will be denoted by x_{T+}. Similarly, the space $\hat{\mathcal{M}}^\infty$ may be identified with the space $\hat{\mathcal{N}}_0^\infty$ of predictable processes x with $x\in\hat X_0$ almost surely and $(Dx,\tilde Dx)\in\hat{\mathcal{M}}^\infty$.

Problem (OQS) can be written as

$$\text{maximize}\quad\langle R,(Dx,\tilde Dx)\rangle\quad\text{over}\quad x\in\hat{\mathcal{C}}_e,$$

where

$$\hat{\mathcal{C}}_e := \{x \in \hat{\mathcal{N}}^\infty_{0+} \,|\, x_t \in \{0,1\}\},$$

and $\hat{\mathcal{N}}^\infty_{0+}$ denotes the nondecreasing processes of $\hat{\mathcal{N}}^\infty_0$. Indeed, the equations $\tau(\omega) = \inf\{t \in [0,T] \mid x_{t+}(\omega) \geq 1\}$ and $\tilde{\tau}(\omega) = \inf\{t \in [0,T] \mid x_t - x_{t-}(\omega) \geq 1\}$ give a one-to-one correspondence between the elements of $\hat{\mathcal{T}}$ and $\hat{\mathcal{C}}_e$.

Consider also the convex relaxation of minimizing $\langle R, (Dx, \tilde{D}x)\rangle$ over the set

$$\hat{\mathcal{C}} := \{x \in \hat{\mathcal{N}}^\infty_{0+} \,|\, x_{T+} \leq 1\}.$$

The relaxation can be written as

$$\text{maximize} \quad E[\int_{[0,T]} R dx + \int_{[0,T]} \vec{R} \tilde{d}x] \quad \text{over} \quad x \in \hat{\mathcal{C}}, \tag{ROQS}$$

where the second integral is that of $\vec{R}$ with respect to the measure $\tilde{D}x$. Note that problem (ROQS) makes sense for any reward process R cádlág or not.

Lemma 7. *The set $\hat{\mathcal{C}}$ is convex, $\sigma(\hat{\mathcal{M}}^\infty, \mathcal{D}^1)$-compact and the set of quasi-stopping times $\hat{\mathcal{C}}_e$ is its extreme points. Moreover, the set of stopping times is $\sigma(\hat{\mathcal{M}}^\infty, \mathcal{D}^1)$-dense in $\hat{\mathcal{C}}_e$ and, thus, $\mathcal{C}$ is $\sigma(\hat{\mathcal{M}}^\infty, \mathcal{D}^1)$-dense in $\hat{\mathcal{C}}$.*

Proof. The set $\hat{\mathcal{C}}$ is a closed convex set of the unit ball that $\hat{\mathcal{N}}^\infty_0$ has as the dual of the Banach space $\mathcal{D}^1$. The compactness thus follows from Banach-Alaoglu. It is easily shown that the elements of $\hat{\mathcal{C}}_e$ are extreme points of $\hat{\mathcal{C}}$.

If $x \notin \hat{\mathcal{C}}_e$, there exist $\bar{s} \in (0,1)$ such that

$$x^1_t := \frac{1}{\bar{s}}[x_t \wedge \bar{s}], \qquad x^2_t := \frac{1}{1-\bar{s}}[(x_t - \bar{s}) \vee 0]$$

are distinguishable processes that belong to $\hat{\mathcal{C}}$. Since $x = \bar{s}x^1 + (1-\bar{s})x^2$, x is not an extremal in $\hat{C}$.

To prove the last claim, let $(\tau, \tilde{\tau})$ be a quasi-stopping time and let (τ^ν) be as in Lemma 17. We then have

$$\langle (\delta_{\tau\wedge\tau^\nu}, 0), y\rangle \to \langle (\delta_\tau, \delta_{\tilde{\tau}}), y\rangle$$

for every $y \in \mathcal{D}^1$. □

Just like in Section 2, a combination of Lemma 7 and Theorem 3 gives the following existence result which was established in Bismut [7] using more elaborate techniques based on the existence of Snell envelopes.

Theorem 8. *If $R \in \mathcal{D}^1$, then an optimal quasi-stopping time in* (OQS) *exists and the optimal values of* (OS), (OQS), (ROS) *and* (ROQS) *are all equal.*

As another implication of Lemma 7 and Theorem 6, we recover the following result of Bismut which says that the seminorms in Theorem 6 are not just equivalent but equal.

Theorem 9 ([5, Theorem 4])**.** *For every $y \in \mathcal{D}^1$,*

$$\|y\|_{\mathcal{D}^1} = \inf_{z \in L^1(D)} \{E\|z\|_D \mid {}^o z = y\}.$$

Proof. The expression on the right is the seminorm p in Theorem 6 with the dual representation

$$p(y) = p(|y|) = \sup_{x \in \hat{\mathcal{C}}} \langle |y|, (Dx, \tilde{D}x) \rangle$$

which, by Theorem 8, equals the left side. □

Combining Theorem 9 with Theorem 1 gives a simple proof of the following.

Theorem 10 ([5, Theorem 3])**.** *For every $y \in \mathcal{R}^1$,*

$$\|y\|_{\mathcal{R}^1} = \inf_{z \in L^1(C)} \{E\|z\|_D \mid {}^o z = y\}.$$

Proof. By Jensen's inequality, the left side is less than the right which is the seminorm p in Theorem 1 with the dual representation

$$\begin{aligned} p(y) &= \sup\{\langle y, u \rangle \mid \text{ess sup}(\|u\|_{TV}) \le 1\} \\ &\le \sup\{\langle y, (u, \tilde{u}) \rangle \mid \text{ess sup}(\|u\|_{TV} + \|\tilde{u}\|_{TV}) \le 1\} \\ &= \sup_{x \in \hat{\mathcal{C}}} \langle |y|, (Dx, \tilde{D}x) \rangle, \end{aligned}$$

which, again by Theorem 8, equals the left side. □

4 Non-cádlág processes

This section gives a further extension to cases where the reward process is not necessarily cádlág but merely of class (D) and *right-upper semicontinuous* (right-usc) in the sense that $R \ge \overleftarrow{R}$. In this case, the objective function of the relaxed quasi-optimal stopping problem (ROQS) may be discontinuous. Lemma 11 below says that it is, nevertheless, upper semicontinuous, so Bauer's maximum principle, Theorem 3, still applies. Beyond cádlág processes, right-usc processes include, e.g., pointwise infima of cádlág processes. Moreover, we get existence of solutions of (OS) for processes of the form $R = g(Z)$, where Z is regular and g is a finite convex function; see Remark 1 below. Such processes are not necessarily regular so the results of Section 2 do not apply.

Given any measurable process R,

$$\hat{\mathcal{J}}(u, \tilde{u}) := \begin{cases} E\left[\int_{[0,T]} R du + \int_{[0,T]} \vec{R} d\tilde{u}\right] & \text{if } (u, \tilde{u}) \in \hat{\mathcal{M}}^\infty_+ \\ -\infty & \text{otherwise} \end{cases}$$

defines an extended real-valued function on $\hat{\mathcal{M}}^\infty$. The relaxed optimal quasi stopping problem (ROQS) can be written as

$$\text{minimize} \quad \hat{\mathcal{J}}(Dx, \tilde{D}x) \quad \text{over} \quad x \in \hat{\mathcal{C}}.$$

Lemma 11. *If R is right-usc and of class (D), then $\hat{\mathcal{J}}$ is $\sigma(\hat{\mathcal{M}}^\infty, \mathcal{D}^1)$-usc.*

Proof. By [15, Theorem 2], there exists a measurable process z such that $R = {}^o z$ and $\sup_t z_t \in L^1$. It follows that $|R| \leq M$, where $M \in \mathcal{D}^1$ is the optional projection of the pathwise constant process $r = \sup_t z_t$. Thus, the first example in [19, Section 8] implies, with obvious changes of signs, that $\hat{\mathcal{J}}$ is usc. □

Combining Lemma 11 with Theorem 3 gives the existence of a relaxed quasi-stopping time at an extreme point of $\mathcal{C}$ which, by Lemma 7, is a quasi-stopping time. We thus obtain the following.

Theorem 12. *If R is right-usc and of class (D), then* (OQS) *has a solution and the optimum values of* (OQS) *and* (ROQS) *are equal.*

We have not been able find the above result in the literature but it can be derived from Theorem 2.39 of El Karoui [14] on "divided stopping times" (temps d'arret divisés). A recent analysis of divided stopping times can be found in Bank and Besslich [1]. These works extend Bismut's approach on optimal quasi-stopping by dropping the assumption of right-continuity and augmenting quasi-stopping times with a third component that acts on the right limit of the reward process. Much like Bismut's approach, [14, 1] build on the existence of a Snell envelope.

Theorem 12 yields the existence of an optimal stopping time when the reward process R is *subregular* in the sense that it is right-usc, of class (D) and $\vec{R} \leq {}^p R$.

Theorem 13. *If R is subregular, then* (OS) *has a solution and its optimum value equals that of* (OQS).

Proof. Clearly, the optimum value of (OQS) is at least that of (OS) while for subregular R,

$$E[R_\tau + \vec{R}_{\tilde{\tau}}] \leq E[R_\tau + {}^p R_{\tilde{\tau}}] = E[R_\tau + R_{\tilde{\tau}}] = ER_{\tau \wedge \tilde{\tau}},$$

where the first equality holds by the definition of predictable projection. The claim now follows from Theorem 12. □

The above seems to have been first established in Bismut and Skalli [8, Section II] for bounded R (again, they mention on page 301 that, instead of boundedness, it would suffice to assume that R is of class (D)).

Remark 1. *Regularity properties are preserved under compositions with convex functions much like martingale properties. Specifically, if R is regular and g is a real-valued convex function on $\mathbb{R}$ then $g(R)$ is subregular as soon as it is of class (D). Indeed, for any $\tau \in \mathcal{T}_p$, conditional Jensen's inequality gives*

$$E[g(\vec{R}_\tau)\mathbb{1}_{\tau<+\infty}] = E[g({}^p R_\tau)\mathbb{1}_{\tau<+\infty}] \leq E[g(R_\tau)\mathbb{1}_{\tau<+\infty}].$$

Similarly, if R is subregular and g is a real-valued increasing convex function, then $g(R)$ is subregular as soon as the composition is of class (D).

5 Duality

We end this paper by giving optimality conditions and a dual problem for the optimal stopping problems. The derivations are based on the duality framework of [21] for convex optimization problems in general locally convex vector spaces. The results below establish the existence of dual solutions without assuming the existence of optimal (quasi-)stopping times. They hold without any path properties as long as the reward process R is of class (D).

We denote the space of martingales of class (D) by $\mathcal{R}^1_m$.

Theorem 14. *Let R be of class (D). Then the optimum values of* (ROQS) *and* (ROS) *both equal that of*

$$\inf\{EM_0 \mid M \in \mathcal{R}^1_m,\ R \leq M\}, \tag{DOS}$$

where the infimum is finite and attained. An $x \in \hat{\mathcal{C}}$ is optimal in (ROQS) *if and only if there exists $M \in \mathcal{R}^1_m$ with $R \leq M$ and*

$$\int_{[0,T]} (M - R)dx + \int_{[0,T]} (M_- - \vec{R})d\tilde{x} = 0, \tag{1}$$

$$x_{T+} = 1 \quad or \quad M_T = 0 \tag{2}$$

almost surely. In particular, $x \in \mathcal{C}$ is optimal in (ROS) *if and only if there exists $M \in \mathcal{R}^1_m$ with $R \leq M$ and*

$$\int_{[0,T]} (M - R)dx = 0,$$

$$x_{T+} = 1 \quad or \quad M_T = 0$$

almost surely.

Proof. By [15, Theorem 2], there exists a measurable process z such that $R = {}^o z$ and $E[\sup_t z_t] < \infty$. Clearly, z is dominated by the pathwise constant process $r := \sup_t z_t$, so $R \leq {}^o r$ and $\vec{R} \leq ({}^o r)_-$. By [18, Lemma 6], $({}^o r)_- = {}^p(r_-) = {}^p r$. It follows that

$$\hat{\mathcal{J}}(Du, D\tilde{u}) \leq E\left[\int_{[0,T]} r du + \int_{[0,T]} r d\tilde{u}\right] = E[\sup_t z_t(\|u\|_{TV} + \|\tilde{u}\|_{TV})]$$

for any $(u, \tilde{u}) \in \hat{\mathcal{M}}^\infty_+$ so

$$\hat{\mathcal{J}}(Dx, \tilde{D}x) \leq E[\sup_t z_t x_{T+}] \tag{3}$$

for any $x \in \hat{\mathcal{N}}^\infty_{0+}$.

The optimum value and optimal solutions of (ROQS) coincide with those of

$$\underset{x \in \hat{\mathcal{N}}^\infty_0}{\text{maximize}} \quad E\left[\hat{\mathcal{J}}(Dx, \tilde{D}x) - \rho(x_{T+} - 1)^+\right], \tag{4}$$

where $\hat{\mathcal{J}}$ is defined in Lemma 11 and $\rho := \sup_t z_t + 1$. Indeed, if x is feasible in (ROQS), then the second term in (4) disappears and we get the objective of (ROQS). On the other hand, if x is feasible in (4) then $\bar{x} := x \wedge 1$ is feasible in (ROQS) and since $x - \bar{x}$ is an increasing process with $(x-\bar{x})_{T+} = (x_{T+}-1)^+$, we get

$$\begin{aligned}\hat{\mathcal{J}}(D\bar{x}, \tilde{D}\bar{x}) &= \hat{\mathcal{J}}(Dx, \tilde{D}x) - \hat{\mathcal{J}}(D(x-\bar{x}), \tilde{D}(x-\bar{x}))\\ &\geq \hat{\mathcal{J}}(Dx, \tilde{D}x) - E\rho(x_{T+}-1)^+,\end{aligned}$$

so infeasible solutions of (ROQS) are never optimal in (4). The upper bound (3) implies that the optimum value in (4) is finite.

Problem (4) fits the general conjugate duality framework of [21] with $U = L^\infty$, $Y = L^1$ and

$$F(x, w) = -\hat{\mathcal{J}}(Dx, \tilde{D}x) + E\rho(x_{T+} + w - 1)^+.$$

By [21, Theorem 22], $w \to F(0, w)$ is continuous on L^∞ in the Mackey topology that it has as the dual of L^1. Thus, by [21, Theorem 17], the optimum value of (4) coincides with the infimum of the dual objective function

$$g(y) := -\inf_{x \in \hat{\mathcal{N}}_0^\infty} L(x, y),$$

where $L(x, y) := \inf_{w \in L^\infty}\{F(x, w) - Ewy\}$, and moreover, the infimum of g is attained. By the interchange rule [22, Theorem 14.60],

$$\begin{aligned}L(x,y) &= \begin{cases} +\infty & \text{if } x \notin \hat{\mathcal{N}}_{0+}^\infty,\\ -\hat{\mathcal{J}}(Dx, \tilde{D}x) + E\left[\inf_{w\in\mathbb{R}}\{\rho(x_{T+}+w-1)^+ - wy\}\right] & \text{otherwise}\end{cases}\\ &= \begin{cases} +\infty & \text{if } x \notin \hat{\mathcal{N}}_{0+}^\infty,\\ -\hat{\mathcal{J}}(Dx, \tilde{D}x) + E\left[x_{T+}y - y - \delta_{[0,\rho]}(y)\right] & \text{otherwise.}\end{cases}\end{aligned}$$

By [18, Lemma 6],

$$\begin{aligned}E[x_{T+}y] &= E[\int_{[0,T]}(y\mathbb{1})dx + \int_{[0,T]}(y\mathbb{1})d\tilde{x}]\\ &= E[\int_{[0,T]}{}^o(y\mathbb{1})dx + \int_{[0,T]}{}^p(y\mathbb{1})d\tilde{x}]\\ &= E[\int_{[0,T]} M dx + \int_{[0,T]} M_- d\tilde{x}]\end{aligned}$$

where $M := {}^o(y\mathbb{1}) \in \mathcal{R}^1_m$. Thus, since $y = M_T$,

$$L(x,y) = \begin{cases} +\infty & \text{if } x \notin \hat{\mathcal{N}}_{0+}^\infty,\\ E[\int_{[0,T]}(M-R)dx + \int_{[0,T]}(M_- - \vec{R})d\tilde{x}] - EM_T & \text{if } x \in \hat{\mathcal{N}}_{0+}^\infty \text{ and } 0 \leq M_T \leq \rho,\\ -\infty & \text{otherwise.}\end{cases}$$

If $M \geq R$ and $0 \leq M_T \leq \rho$, then $M_- \geq \vec{R}$ and $\inf_x L(x,y) = -EM_T$. If $M \not\geq R$, Lemma 16 gives a $\tau \in \mathcal{T}$ with $E[M_\tau - R_\tau] < 0$. Taking $x = \lambda(\delta_\tau, 0)$ and letting $\lambda \nearrow \infty$ then gives $\inf_x L(x,y) = -\infty$, so the dual objective becomes

$$g(y) = \begin{cases} EM_0 & \text{if } 0 \leq M_T \leq \rho,\ M \geq R, \\ +\infty & \text{otherwise.} \end{cases}$$

In summary, the optimum value of (OQS) equals that of (DOS).

The dual problem of (OS) is obtained similarly by defining

$$F(x,w) = -\mathcal{J}(Dx) + E\rho(x_{T+} + w - 1)^+.$$

The function $w \to F(0,w)$ is again Mackey-continuous on L^∞ and one finds that the dual is again (DOS). Thus, the optimum value of (OS) equals that of (DOS).

As to the optimality conditions, the equivalence of (e) and (f) in Theorem 15 of [21] says that x is optimal in (4) and y is optimal in the dual if and only if

$$0 \in \partial_x L(x,y) \quad \text{and} \quad 0 \in \partial_y[-L](x,y).$$

The first condition means that x minimizes $L(\cdot, y)$, or equivalently, $x \in \hat{\mathcal{N}}^\infty_{0+}$, $M \geq R$ and

$$\int_{[0,T]} (M - R)dx = 0, \quad \int_{[0,T]} (M_- - \vec{R})d\tilde{x} = 0 \quad P\text{-a.s.}$$

By the interchange rule for subdifferentials ([21, Theorem 21c]), the latter is equivalent to (2). $\square$

Corollary 15. *If R is of class (D), then the optimum values of* (OS), (OQS), (ROS) *and* (ROQS) *are all equal. A quasi-stopping time $(\tau, \tilde{\tau}) \in \hat{\mathcal{T}}$ is optimal in* (OQS) *if and only if there exists $M \in \mathcal{R}^1_m$ with*

$$R \leq M,\ M_\tau = R_\tau,\ M_{\tilde{\tau}_-} = \vec{R}_{\tilde{\tau}}$$

and almost surely either $\tau \wedge \tilde{\tau} < T+$ or $M_T = 0$. In particular, a stopping time $\tau \in \mathcal{T}$ is optimal in (OS) *if and only if there exists $M \in \mathcal{R}^1_m$ with*

$$R \leq M,\ M_\tau = R_\tau$$

and almost surely either $\tau < T+$ or $M_T = 0$.

Proof. It suffices to prove the first claim since then, the optimality conditions for (OQS) and (OS) follow from Theorem 14. Note first that, since each $M \in \mathcal{R}^1_m$ is right-continuous, the optimum value of (DOS) is unaffected if we replace R by its right-upper semicontinuous hull defined by $\bar{R}_{T+} := 0$, $\bar{R}_T := R_T$ and

$$\bar{R}_t := \lim_{s \searrow t} \sup_{r \in [t,s)} R_s \quad \forall t < T.$$

It is easily seen that $\bar{R}$ is right-usc. By [11, Theorem IV.90], it is optional. Since (DOS) is feasible, there exists an $M \in \mathcal{R}^1_m$ with $R \leq M$ so $\bar{R}$ is of class (D). Thus, by Theorem 12, the optimum value of (DOS) equals the optimum value of (OQS) for the reward process $\bar{R}$. By Lemma 5, this equals the optimum value of (OS) for $\bar{R}$. It remains to show that the optimum value of (OS) not affected when if we replace R by its right-upper semicontinuous hull.

Let $\tau \in \mathcal{T}$ and $\epsilon > 0$. Let $\epsilon' > 0$ be such that $E[1_A \bar{R}_\tau] < \epsilon$ whenever $P(A) \leq \epsilon'$. The random variable $\bar{R}_\tau$ is $\mathcal{F}_\tau$-measurable so the set

$$S = \{(\omega, t) \in \Omega \times [0, T] \mid t \geq \tau(\omega),\ R_t(\omega) \geq \bar{R}_{\tau(\omega)}(\omega) - \epsilon\}$$

is optional. Indeed, it is the upper level set of the optional process $Q_t := 1_{\{t \geq \tau\}}(R_t - \bar{R}_\tau + \epsilon) - 1_{\{t<\tau\}}$. By the optional section theorem (see e.g. [11, Theorem IV.84]), there exists a $\tau' \in \mathcal{T}$ such that $(\omega, \tau'(\omega)) \in S$ when $\tau'(\omega) < T+$ and $P(\tau' < T+) \geq P(\tau < T+) - \epsilon'$. Since $\{\tau' < T+\} \subset \{\tau < T+\}$, we get $P(\{\tau < T+\} \setminus \{\tau < T+\}) \leq \epsilon'$ and

$$\begin{aligned} ER_{\tau'} &= E[1_{\{\tau' < T+\}} R_{\tau'}] \\ &\geq E[1_{\{\tau' < T+\}} (\bar{R}_\tau - \epsilon)] \\ &= E[\bar{R}_\tau - 1_{\{\tau < T+\} \setminus \{\tau' < T+\}} \bar{R}_\tau - 1_{\{\tau' < T+\}} \epsilon] \\ &\geq E\bar{R}_\tau - 2\epsilon. \end{aligned}$$

Since $\tau \in \mathcal{T}$ and $\epsilon > 0$ arbitrary, this completes the proof. □

Note that if Y is the Snell envelope of R (the smallest supermartingale that dominates R), then the martingale part M in the Doob–Meyer decomposition $Y = M - A$ is dual optimal. Indeed, if M was not dual optimal, there would exist a martingale $\bar{M} \geq R$ with $E\bar{M}_0 < EM_0 = EY_0$, so Y would not be the smallest supermartingale dominating R.

Note also that for any martingale $M \in \mathcal{R}^1_m$,

$$\sup_{\tau \in \mathcal{T}} ER_\tau = \sup_{\tau \in \mathcal{T}} E(R_\tau + M_T - M_\tau) \leq E \sup_{t \in [0,T]} (R_t + M_T - M_t),$$

where the last expression is dominated by EM_0 if $R \leq M$. Thus,

$$\begin{aligned} \sup_{\tau \in \mathcal{T}} ER_\tau &\leq \inf_{M \in \mathcal{R}^1_m} E \sup_{t \in [0,T]} (R_t + M_T - M_t) \\ &\leq \inf_{M \in \mathcal{R}^1_m} \{E \sup_{t \in [0,T]} (R_t + M_T - M_t) \,|\, R \leq M\} \\ &\leq \inf_{M \in \mathcal{R}^1_m} \{EM_0 \,|\, R \leq M\}, \end{aligned}$$

where, by Theorem 8, the last expression equals the first one as soon as R is of class (D). The optimum value of the stopping problem then equals

$$\inf_{M \in \mathcal{R}^1_m} E \sup_{t \in [0,T]} (R_t + M_T - M_t). \tag{5}$$

This was obtained in [10] under the assumptions that the reward process is right-continuous and that its pathwise supremum is integrable. We have relaxed both conditions by dropping all path properties and relaxing the integrability to class (D). Rogers [23], Haugh and Kogan [16] and Becker, Cheridito and Jentzen [3] used (5) to numerically compute upper bounds for optimal stopping problems.

Appendix

Lemmas 16 and 17 below state the optional section theorem and the existence of announcing sequences, respectively, for the compactified time interval $[0, T]$. Let $a : [0, 1] \to [0, T]$ be an increasing homeomorphism. Given $y : [0, T] \to \mathbb{R}$, let $\Phi(y) : [0, \infty) \to \mathbb{R}$ be defined by

$$\Phi(y)_t = \begin{cases} y_{a^{-1}(t)} & \text{if } t \leq 1 \\ y_T & \text{if } t > 1. \end{cases}$$

We define a filtration $(\mathcal{G}_t)$ on $[0, \infty)$ by $\mathcal{G}_t := \mathcal{F}_{a(t)}$ for $t \in [0, 1]$ and $\mathcal{G}_t := \mathcal{F}_T$ for $t > 1$. If y is a process on $\Omega \times [0, T]$, $\Phi(y)$ is a process on $\Omega \times [0, \infty)$. If y is right-continuous (left-continuous), $\Phi(y)$ is right-continuous (left-continuous). By the monotone class theorem, $\Phi(y)$ is $(\mathcal{G}_t)$-optional (predictable) if y is $(\mathcal{F}_t)$-optional (predictable). In particular, Φ maps the optional (predictable) σ-algebra of $\Omega \times [0, T]$ to the optional (predictable) σ-algebra of $\Omega \times [0, \infty)$.

On $\Omega \times [0, T]$, the optional section theorem takes the following form.

Lemma 16. *Given optional $A \subseteq \Omega \times [0, T]$ and $\epsilon > 0$, there exists $\tau \in \mathcal{T}$ such that $(\omega, \tau(\omega)) \in A$ on $\{\tau < T+\}$ and $P(\tau < T+) \geq P(\{\omega \mid \exists t : (\omega, t) \in A\}) - \epsilon$.*

Proof. The set $B := \{(\omega, t) \in \Omega \times [0, 1] \mid (\omega, a(t)) \in A\}$ is optional, since $\mathbb{1}_B = \Phi(\mathbb{1}_A)$ on $\Omega \times [0, 1]$. By the optional section theorem ([11, Theorem IV.84]), there exists a $(\mathcal{G}_t)$-stopping time σ such that $(\omega, \sigma(\omega)) \in B$ on $\{\sigma < \infty\}$ and $P(\sigma < \infty) \geq P(\{\omega \mid \exists t : (\omega, t) \in B\}) - \epsilon$, so we may take $\tau := 1_{\{\sigma \leq 1\}} a(\sigma) + 1_{\{\sigma > 1\}} T+$. □

Given a predictable $\tilde{\tau} \in \mathcal{T}$, there exists (see, e.g., [11, Theorem IV.77]) a sequence (τ^ν) of stopping times with values in $[0, T)$ such that $\tau^\nu < \tau$ and $\tau^\nu \nearrow \tau$ on $\{\tilde{\tau} < T+\}$. Such sequences cannot, however, distinguish between the sets $\{\tilde{\tau} = T\}$ and $\{\tilde{\tau} = T+\}$. Using the compactness of $[0, T]$, we get the following characterization of general predictable $\tilde{\tau} \in \mathcal{T}$.

Lemma 17. *Given a predictable $\tilde{\tau} \in \mathcal{T}$, there exists a nondecreasing $(\tau_k)_{\nu=1}^\infty \subset \mathcal{T}$ such that $\tau_k < \tilde{\tau}$ and $\tau_k \nearrow \tilde{\tau}$ on $\{\tilde{\tau} < T+\}$ and $P(\{\tau_k < T+\}) \searrow P(\{\tilde{\tau} < T+\})$.*

Proof. Defining

$$\sigma := \begin{cases} a^{-1}(\tilde{\tau}) & \text{on } \{\tilde{\tau} < T+\} \\ +\infty & \text{otherwise} \end{cases}$$

we have $\mathbb{1}_{[\sigma, \infty)} = \Phi(\mathbb{1}_{[\tilde{\tau}, T]})$, so σ is $(\mathcal{G}_t)$-predictable time. By [11, Theorem IV.77], there exists a $(\mathcal{G}_t)$-announcing sequence σ_k of σ, so we may take $\tau_k := 1_{\{\sigma_k \leq 1\}} a(\sigma_k) + 1_{\{\sigma_k > 1\}} T+$. □

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