Optimal investment and contingent claim valuation in illiquid markets

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Abstract

This paper extends basic results on arbitrage bounds and attainable claims to illiquid markets and general swap contracts where both claims and premiums may have multiple payout dates. Explicit consideration of swap contracts is essential in illiquid markets where the valuation of swaps cannot be reduced to the valuation of cumulative claims at maturity. We establish the existence of optimal trading strategies and the lower semicontinuity of the optimal value of optimal investment under conditions that extend the no-arbitrage condition in the classical linear market model. All results are derived with the "direct method" without resorting to duality arguments.

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1 Introduction

In complete markets, prices of contingent claims are uniquely determined by the replication argument but in incomplete markets, subjective factors such as market expectations and risk preferences come in. Moreover, in incomplete markets, the classical replication argument has two distinct generalizations leading to two distinct notions of a "price". The first one is relevant in financial supervision and accounting, where one is often interested in the least amount of capital that would allow for covering a claim at an acceptable level of risk. In banking, such a value is sometimes called the "economic capital" while in insurance, the terms "technical provisions" and "reserving" are often used; see e.g. Article 76 of the Solvency II Directive 2009/138/EC of the European Parliament. Such an accounting value is not necessarily a price at which an agent would be willing to sell or buy a claim. Offered prices are better described by the *indifference principle* which says that the seller of a claim charges (at least) a price that allows

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him to sell the claim without increasing the risk of his existing financial position. While accounting values are meant to describe objective hedging costs, indifference prices depend on an agent's financial position before the trade. Indeed, it is intuitively clear that the price an agent would be willing to pay e.g. for an insurance contract depends on his exposure to the insured risk. Indifference pricing has been widely applied in various areas of finance and insurance; see e.g. Bühlmann [7], Hodges and Neuberger [28] or Bielecki et al. [5] for a small sample. We refer the reader to Carmona [8] for further references on the topic.

This article studies reserving and indifference pricing in illiquid markets where wealth cannot be transferred quite freely, neither between assets nor through time, due to nonlinear trading costs and portfolio constraints. Such features are often encountered in practice but they cause complications in the classical theory of optimal investment and contingent claim valuation. In particular, in markets without a perfectly liquid "numeraire" asset, the timing of claim payments is an important issue. Indeed, financial liabilities in banking and insurance often require payments at several points in time. Similarly, much of trading in practice consists of exchanging sequences of cash flows (swaps, coupon paying bonds, insurance contracts, ...). This is in sharp contrast with traditional models of financial mathematics which assume the existence of a "cash account" that can be used to finance trading costs and claim payments all of which are then settled at a single maturity date. Admitting constraints and/or frictions on all assets (including cash) corresponds better to reality but requires new techniques of analysis. Mathematically, without a perfectly liquid numeraire, the wealth process cannot be expressed as a stochastic integral of the portfolio process with respect to the price process.

This paper gives arbitrage bounds for reservation values and general swap contracts where both claims and premiums may have multiple payout dates. Our bounds account for nonlinearities but reduce to the usual arbitrage bounds in the classical perfectly liquid market model. For replicable claims, the arbitrage bounds collapse to a single point and both reservation values and indifference prices coincide with traditional replication based prices when the premium consists of a lump sum payment at start date.

In addition to the usual measurability properties, we only assume that the trading costs and portfolio constraints are convex. This is quite natural e.g. in limit order markets where marginal prices of market orders are increasing functions of the traded amount. Rather than aiming at analytical pricing formulas we study general relations between optimal investment and valuation of contingent claims in general convex market models in finite discrete time. Many of the properties of reservation values and indifference swap rates are derived by purely algebraic arguments using elementary convex analysis. This is done by analyzing an optimal investment problem parameterized by liabilities with multiple payment dates. The problem extends many earlier formulations on liquid market models and models with proportional transaction costs. Existence of solutions and the lower semicontinuity of the optimal value function are established under conditions that extend the no-arbitrage condition beyond the classical linear market model.

2 Optimal investment

A fundamental problem in financial economics is that of managing investments so that their proceeds cover given liabilities as well as possible. This is important not only in wealth management but also in pricing and hedging of financial products and in reserving for financial liabilities; see Dybvig and Marshall [22] or Davis [15] for general discussion. Mathematical theory of optimal investment with liabilities has a long history; see e.g. [14, 17, 25, 15, 29, 42, 4] and the references there. The purpose of this section is to study basic properties of such problems in the presence of illiquidity effects. Specifically, we will formulate optimal investment problems using the market model from [43, 45] where trading costs may depend nonlinearly on traded amounts and portfolios may be subject to constraints. The results of this section form the basis for reserving and indifference swap rates that will be studied in Sections 3 and 4 below.

Consider a financial market where a finite set J of assets can be traded over finite discrete time $t = 0, \dots, T$. Trading costs will be described by an adapted sequence $S = (S_t)_{t=0}^T$ of convex functions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$. More precisely, we assume that each S_t is a $\mathcal{B}(\mathbb{R}^J) \otimes \mathcal{F}_t$-measurable real-valued function on $\mathbb{R}^J \times \Omega$ such that $S_t(\cdot, \omega)$ is convex, lower semicontinuous and $S_t(0, \omega) = 0$ for every ω. The value $S_t(x, \omega)$ represents the cost of buying a portfolio $x \in \mathbb{R}^J$ at time t in state ω. As usual, we interpret negative costs as revenue. We have assumed that the cost functions S_t are real-valued only for notational simplicity. Our main results extend immediately to extended real-valued functions. This is relevant e.g. when S_t models the cost of a market order in a limit order market with finite depth on the ask side.

The classical model of perfectly liquid markets corresponds to $S_t(x, \omega) = s_t(\omega) \cdot x$ where $s = (s_t)_{t=0}$ is an $(\mathcal{F}_t)_{t=0}^T$-adapted sequence of price vectors independent of traded amounts. Markets with proportional transaction costs and/or bid-ask-spreads can be modeled with

$$S_t(x, \omega) = \sup\{s \cdot x \mid s \in [\underline{s}_t(\omega), \bar{s}_t(\omega)]\},$$

where the $(\mathcal{F}_t)_{t=0}^T$-adapted $\mathbb{R}^J$-valued processes $\underline{s}$ and $\bar{s}$ give the bid- and ask-prices, respectively; see Jouini and Kallal [33]. Convex trading costs arise naturally also in modern limit order markets, where the cost of a "market order" is nonlinear and convex in the traded amount; see [40]. Çetin and Rogers [9] studied a two-asset model with a cash-account and an illiquid asset subject to convex trading costs. Writing $x = (x^0, x^1)$ their model corresponds to

$$S_t(x, \omega) = x^0 + s_t(\omega)\psi(x^1),$$

where s is a strictly positive adapted process and ψ is an extended real-valued convex function with $\psi(0) = 0$. It follows from Proposition 14.44(d) and Corollary 14.46 of [48] that such functions are $\mathcal{B}(\mathbb{R}^J) \otimes \mathcal{F}_t$-measurable. We have assumed for simplicity that the cost functions S_t are real-valued but extended real-valued functions would pose no real difficulties in our approach. Allowing S_t to take on the value $+\infty$ could be used to model e.g. situations where the limit order book has finite depth on the sell side.

We will describe trading strategies by $(\mathcal{F}_t)_{t=0}^T$-adapted $\mathbb{R}^J$-valued processes $x = (x_t)_{t=0}^T$, where x_t is the portfolio held over period $(t, t+1]$. We model portfolio constraints by an adapted sequence $D = (D_t)_{t=0}^T$ of sets in $\mathbb{R}^J$. That D is adapted, means that each D_t is $\mathcal{F}_t$-measurable in the sense that

$$\{\omega \in \Omega \,|\, D_t(\omega) \cap U \neq \emptyset\} \in \mathcal{F}_t$$

for every open $U \subset \mathbb{R}^J$. We assume that $D_t(\omega)$ are closed convex with $0 \in D_t(\omega)$ so that the zero portfolio is always feasible. The classical unconstrained model corresponds to $D \equiv \mathbb{R}^J$ while short sale constraints, studied e.g. in Cvitanic and Karatzas [12] and Jouini and Kallal [32] are given by $D \equiv \mathbb{R}^J_+$. More general portfolio constraints have been studied e.g. in [41, 49, 19, 13].

Since we do not assume the existence of perfectly liquid numeraire asset, it is important to distinguish between payments at different dates. We will denote the linear space of $(\mathcal{F}_t)_{t=0}^T$-adapted sequences of cash-flows by

$$\mathcal{M} = \{(c_t)_{t=0}^T \,|\, c_t \in L^0(\Omega, \mathcal{F}_t, P)\},$$

where $L^0(\Omega, \mathcal{F}_t, P)$ is the space of (equivalence classes of) $\mathcal{F}_t$-measurable real-valued random variables. Given $c \in \mathcal{M}$ and proper[1], nondecreasing, convex functions $\mathcal{V}_t$ on $L^0(\Omega, \mathcal{F}_t, P)$, we will study the problem

$$\text{minimize} \quad \sum_{t=0}^T \mathcal{V}_t(S_t(\Delta x_t) + c_t) \quad \text{over} \quad x \in \mathcal{N}_D, \tag{ALM}$$

where

$$\mathcal{N}_D := \{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^J),\ x_t \in D_t,\ t = 0, \dots, T-1,\ x_T = 0\}$$

and $\Delta x_t := x_t - x_{t-1}$. We always define $x_{-1} = 0$ so that the elements of $\mathcal{N}_D$ describe trading strategies that start and end at liquidated positions. We allow $\mathcal{V}_t$ to be extended real-valued but assume that $\mathcal{V}_t(0) = 0$.

Problem (ALM) can be interpreted as an *asset-liability management* problem where one looks for a trading strategy $x \in \mathcal{N}_D$ whose proceeds fit the liabilities $c \in \mathcal{M}$ as well as possible. The functions $\mathcal{V}_t$ measure the *disutility* (regret, loss, ...) caused by the net expenditure $S_t(\Delta x_t) + c_t$ of updating the portfolio and paying out the claim c_t at time t. We allow c to take both positive as well as negative values. In particular, $-c_0$ may be interpreted as the *initial wealth* while the subsequent components c_t may describe financial liabilities or random endowments depending on their sign. Problem (ALM) can also be interpreted as an optimal consumption-investment problem with random endowment $-c_t$ and consumption $-S_t(\Delta x_t) - c_t$ at time t.

Despite its simple appearance, problem (ALM) covers many familiar instances of optimal investment problems. In particular, when[2] $\mathcal{V}_t = \delta_{L^0_-}$ for

[1] An extended real-valued function is *proper* if it is not identically $+\infty$ and if it never takes on the value $-\infty$.

[2] Here and in what follows, δ_C denotes the *indicator function* of a set C: $\delta_C(x)$ equals 0 or $+\infty$ depending on whether $x \in C$ or not.

$t < T$, we can write it as

$$\begin{array}{lll} \text{minimize} & \mathcal{V}_T(S_T(\Delta x_T) + c_T) \quad \text{over} \quad x \in \mathcal{N}_D \\ \text{subject to} & S_t(\Delta x_t) + c_t \le 0, \quad t = 0, \dots, T-1. \end{array} \tag{1}$$

This is an illiquid version of the classical utility maximization problem. In the classical linear market model, this can be written in a more familiar form.

Example 1 (Numeraire and stochastic integration) *Consider problem* (1) *and assume, as e.g. in the model of Çetin and Rogers [9], that there is a perfectly liquid asset (numeraire), say* $0 \in J$*, such that*

$$S_t(x, \omega) = s^0_t(\omega)x^0 + \tilde{S}_t(\tilde{x}, \omega) \quad \textit{and} \quad D_t(\omega) = \mathbb{R} \times \tilde{D}_t(\omega),$$

where $x = (x^0, \tilde{x})$*,* s^0 *is a strictly positive adapted process and* $\tilde{S}$ *and* $\tilde{D}$ *are the cost process and the constraints for the remaining risky assets* $\tilde{J} = J \setminus \{0\}$*. Expressing all costs in terms of asset* 0*, we may assume* $s^0 \equiv 1$*. We can then use the budget constraint in* (1) *to substitute out* x^0 *from the problem formulation. Indeed, defining*

$$x^0_t = x^0_{t-1} - \tilde{S}_t(\Delta\tilde{x}_t) - c_t \quad t = 0, \dots, T-1,$$

the budget constraint holds as an equality for $t = 1, \dots, T-1$*,*

$$x^0_{T-1} = -\sum_{t=0}^{T-1} \tilde{S}_t(\Delta\tilde{x}_t) - \sum_{t=0}^{T-1} c_t$$

and

$$S_T(\Delta x_T) + c_T = \Delta x^0_T + \tilde{S}_T(\Delta\tilde{x}_T) + c_T = x^0_T + \sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{x}_t) + \sum_{t=0}^{T} c_t.$$

Recalling that $x_T = 0$ *for* $x \in \mathcal{N}_D$*, problem* (1) *can thus be written as*

$$\text{minimize} \qquad \mathcal{V}_T\left(\sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{x}_t) + \sum_{t=0}^{T} c_t\right) \quad \text{over} \quad x \in \mathcal{N}_D$$

much like in Çetin and Rogers [9, Equation (2.9)]. In the presence of a perfectly liquid numeraire asset, the timing of payments is thus irrelevant. Furthermore, in the linear case $\tilde{S}_t(\tilde{x}) = \tilde{s}_t \cdot \tilde{x}_t$*, we can express the accumulated trading costs as a stochastic integral,*

$$\sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{x}_t) = \sum_{t=0}^{T} \tilde{s}_t \cdot \Delta\tilde{x}_t = -\sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta\tilde{s}_{t+1}.$$

We then recover constrained discrete-time versions of the utility maximization problems studied e.g. in [36, 17, 37, 3, 4]. In [36, 37, 3], the financial position of the agent was described solely in terms of an initial endowment $w \in \mathbb{R}$ *without future liabilities. This corresponds to* $c_0 = -w$ *and* $c_t = 0$ *for* $t > 0$*.*

We will denote the optimum value of problem (ALM) by $\varphi(c)$. That is,

$$\varphi(c) := \inf_{x \in \mathcal{N}_D} \sum_{t=0}^{T} \mathcal{V}_t(S_t(\Delta x_t) + c_t).$$

This is an extended real-valued function on the space $\mathcal{M}$ of adapted sequences of claims. In the totally risk averse case where $\mathcal{V}_t = \delta_{L^0_-}$ for all $t = 0, \ldots, T$, we get $\varphi = \delta_{\mathcal{C}}$, where

$$\mathcal{C} = \{c \in \mathcal{M} \mid \exists x \in \mathcal{N}_D : \; S_t(\Delta x_t) + c_t \leq 0 \; P\text{-a.s.} \quad t = 0, \ldots, T\}, \tag{2}$$

the set of claim processes that can be *superhedged* without cost. On the other hand, since the functions $\mathcal{V}_t$ are nondecreasing, we can write the value function φ in terms of $\mathcal{C}$ as

$$\begin{aligned}
\varphi(c) &= \inf_{x \in \mathcal{N}_D} \sum_{t=0}^{T} \mathcal{V}_t(S_t(\Delta x_t) + c_t) \\
&= \inf_{x \in \mathcal{N}_D, d \in \mathcal{M}} \left\{ \sum_{t=0}^{T} \mathcal{V}_t(d_t) \,\middle|\, d_t \geq S_t(\Delta x_t) + c_t \right\} \\
&= \inf_{d \in \mathcal{M}} \{ \mathcal{V}(d) \mid c - d \in \mathcal{C} \} \\
&= \inf_{d \in \mathcal{C}} \mathcal{V}(c - d),
\end{aligned} \tag{3}$$

where

$$\mathcal{V}(d) := \sum_{t=0}^{T} \mathcal{V}_t(d_t). \tag{4}$$

If we ignore the financial market so that $\mathcal{C} = \mathcal{M}_- := \{c \in \mathcal{M} \mid c_t \geq 0 \; P\text{-a.s.} \; \forall t\}$, we simply have $\varphi = \mathcal{V}$.

The set $\mathcal{C}$ plays the same role as "the set of contingent claims super-replicable at price 0" defined in Delbaen and Schachermayer [18] for the classical perfectly liquid market model with a cash-account. Whereas [18] studied contingent claims with a single payout date, Rockafellar and Dermody [20, 21], Jouini and Napp [34], Jaschke and Küchler [31] and more recently Madan [39] study claims with multiple payout dates in conical market models without a cash-account. In conical market models, the elements of $\mathcal{C}$ remain superhedgeable without cost when multiplied by positive constants but in general, this scalability property is lost due to nonlinear trading costs and nonconical portfolio constraints.

While the set $\mathcal{C}$ consists of the claims that can be superhedged without cost, its *recession cone*

$$\mathcal{C}^\infty = \{c \in \mathcal{M} \mid \bar{c} + \alpha c \in \mathcal{C} \quad \forall \bar{c} \in \mathcal{C}, \; \forall \alpha > 0\}$$

consists of the claims that can be superhedged without cost in unlimited amounts. Clearly, $\mathcal{M}_- \subseteq \mathcal{C}^\infty$. The following lemma summarizes some key properties of the value function φ and the set $\mathcal{C}$.

Lemma 2 *The value function* φ *is convex,* $\varphi(0) \leq 0$ *and*

$$\varphi(\bar{c} + c) \leq \varphi(\bar{c}) \quad \forall \bar{c} \in \mathcal{M},\ c \in \mathcal{C}^\infty.$$

In particular, the set $\mathcal{C}$ *is convex and* $\mathcal{C}^\infty \subseteq \mathcal{C}$. *If the trading costs* S_t *are sublinear and portfolio constraints* D_t *are conical, then* $\mathcal{C}$ *is a cone and* $\mathcal{C}^\infty = \mathcal{C}$.

Proof. The convexity of φ follows from the representation (3). Indeed, the convexity of S_t and D_t implies that $\mathcal{C}$ is a convex set (see [43, Lemma 4.1]) while the convexity of the disutility functions $\mathcal{V}_t$ implies that the function $\mathcal{V}$ defined in (4) is convex on $\mathcal{M}$. If $c \in \mathcal{C}^\infty$, then $\bar{c} + c - d \in \mathcal{C}$ whenever $\bar{c} - d \in \mathcal{C}$ so that

$$\inf_{d \in \mathcal{M}} \{ \mathcal{V}(d) \mid \bar{c} + c - d \in \mathcal{C} \} \leq \inf_{d \in \mathcal{M}} \{ \mathcal{V}(d) \mid \bar{c} - d \in \mathcal{C} \},$$

which means that $\varphi(\bar{c} + c) \leq \varphi(\bar{c})$. The second claim follows by applying the first part of the lemma to the case where $\mathcal{V}_t = \delta_{L^0_-}$ for all t and $\bar{c} = 0$. When S_t are sublinear and D_t are conical, the set $\mathcal{C}$ is a cone, by [43, Lemma 4.1]. When $\mathcal{C}$ is a convex cone, we have $\bar{c} + \alpha c \in \mathcal{C}$ for every $\bar{c} \in \mathcal{C}$, $c \in \mathcal{C}$ and $\alpha > 0$ as is easily verified. This implies $\mathcal{C} \subseteq \mathcal{C}^\infty$ which completes the proof. □

We will say that a claim $c \in \mathcal{M}$ is *redundant* if

$$c \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty),$$

i.e. if $\bar{c} + \alpha c \in \mathcal{C}$ for every $\bar{c} \in \mathcal{C}$ and every $\alpha \in \mathbb{R}$. In particular, arbitrary positive as well as negative multiples of redundant claims can be superhedged without cost. By Lemma 2, the optimal value of (ALM) is invariant with respect to redundant claims. In the terminology of convex analysis, $\mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ is the "lineality space" of $\mathcal{C}$; see [47, page 65]. Redundancy generalizes the notion of "replicability/attainability" in linear market models; see Examples 3 and 4 below.

A market model (S, D) satisfies the *no-arbitrage* condition if

$$\mathcal{C} \cap \mathcal{M}_+ = \{0\}, \tag{NA}$$

where $\mathcal{M}_+ = \{c \in \mathcal{M} \mid c_t \geq 0\ P\text{-a.s.}\ \forall t\}$. Violation of this condition would mean that there exist nonzero nonnegative claims that could be costlessly superhedged by trading in financial markets.

Example 3 *When* $S_t(x) = s_t \cdot x$ *and* $D_t = \mathbb{R}^J$, *a claim* $c \in \mathcal{M}$ *is redundant if there is an* $x \in \mathcal{N}_D$ *such that* $s_t \cdot \Delta x_t + c_t = 0$. *The converse holds under the no-arbitrage condition.*

Proof. By Lemma 2, the set $\mathcal{C}$ is a cone and $\mathcal{C}^\infty = \mathcal{C}$. Thus, a claim $c \in \mathcal{M}$ is redundant iff $c \in \mathcal{C}$ and $-c \in \mathcal{C}$, which means that there exist $x^1, x^2 \in \mathcal{N}_D$ such that

$$s_t \cdot \Delta x^1_t + c_t \leq 0 \quad \text{and} \quad s_t \cdot \Delta x^2_t - c_t \leq 0 \quad \forall t. \tag{5}$$

If there is an $x \in \mathcal{N}_D$ such that $s_t \cdot \Delta x_t + c_t = 0$, then (5) holds with $x^1 = x$ and $x^2 = -x$. On the other hand, (5) implies $s_t \cdot \Delta(x_t^1 + x_t^2) \leq 0$, where equality must hold under (NA). Combining this equality with (5), we get

$$c_t \leq -s_t \cdot \Delta x_t^1 = s_t \cdot \Delta x_t^2 \leq c_t$$

so both x^1 and $-x^2$ satisfy the condition $s_t \cdot \Delta x_t + c_t = 0$. □

Redundancy generalizes the classical notion of "attainability" in the classical linear model with a cash account; see e.g. [23, Definition 5.25] or [18, Definition 2.2.1].

Example 4 (Numeraire and stochastic integration) *In the classical linear model with $S_t(x) = x_0 + \tilde{s}_t \cdot \tilde{x}$ and $D_t = \mathbb{R}^J$ (see Example 1), we have*

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}\}$$

and a claim $c \in \mathcal{C}$ is redundant if $\sum_{t=0}^{T} c_t$ is "attainable at price 0" (see e.g. [18, Definition 2.2.1]) in the sense that there exists an $x \in \mathcal{N}_D$ such that

$$\sum_{t=0}^{T} c_t = \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1} \tag{6}$$

The converse holds under the (NA) *condition which, in this linear model, can be stated as*

$$x \in \mathcal{N}_D : \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1} \geq 0 \implies \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1} = 0.$$

Proof. The expression for $\mathcal{C}$ follows from Example 1 by recalling that the value function φ becomes the indicator of $\mathcal{C}$ when $\mathcal{V}_t = \delta_{L^0_-}$ for every $t = 0, \ldots, T$. By Lemma 2, the set $\mathcal{C}$ is a cone and $\mathcal{C}^\infty = \mathcal{C}$. Thus, a claim $c \in \mathcal{M}$ is redundant iff $c \in \mathcal{C}$ and $-c \in \mathcal{C}$, which means that there exist $x^1, x^2 \in \mathcal{N}_D$ such that

$$\sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} \tilde{x}_t^1 \cdot \Delta \tilde{s}_{t+1} \quad \text{and} \quad -\sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} \tilde{x}_t^2 \cdot \Delta \tilde{s}_{t+1}. \tag{7}$$

If there is an $x \in \mathcal{N}_D$ such that (6) holds, then (7) holds with $x^1 = x$ and $x^2 = -x$. On the other hand, (7) implies

$$\sum_{t=0}^{T-1} (\tilde{x}_t^1 + \tilde{x}_t^2) \cdot \Delta \tilde{s}_{t+1} \geq 0.$$

where equality must hold under (NA). Combining this equality with (7), we see that both $\tilde{x}^1$ and $-\tilde{x}^2$ satisfy (6). □

3 Reserving

In financial reporting and supervision of financial institutions, one is often interested in determining the least amount of capital that would allow for covering a given financial liability at an acceptable level of risk. If a liability is described by a random sequence $c \in \mathcal{M}$ of future payments, the number

$$\pi^0(c) = \inf\{\alpha \in \mathbb{R} \,|\, \varphi(c - \alpha p^0) \leq 0\},$$

where $p^0 = (1, 0, \dots, 0)$ gives the least amount of initial capital one needs in order to construct a *hedging strategy* $x \in \mathcal{N}_D$ whose proceeds cover a sufficient part of c so that the risk associated with the residual is no higher than the risk from doing nothing at all (recall that $\mathcal{V}_t(0) = 0$, by assumption). It is natural to assume that $c_0 = 0$ since a nonzero value of c_0 would just add directly to the required initial capital $\pi^0(c)$. We will call $\pi^0(c)$ the *reservation value* for $c \in \mathcal{M}$. Our definition of $\pi^0(c)$ is analogous to definition (2.2) in Davis et al. [16] who analyzed the pricing of European options under transaction costs.

Assuming that the infimums in the definition of π^0 and in expression (3) for φ are attained for every $c \in \mathcal{M}$ (see Theorem 10 and Proposition 12 below), we have

$$\begin{aligned}\pi^0(c) &= \inf\{\alpha \,|\, \exists d \in \mathcal{C} : \ \mathcal{V}(c - \alpha p^0 - d) \leq 0\} \\ &= \inf_{d \in \mathcal{C}} \mathcal{W}(c - d), \end{aligned} \tag{8}$$

where $\mathcal{W}(c) = \inf\{\alpha \,|\, \mathcal{V}(c - \alpha p^0) \leq 0\}$. The general structure is similar to that studied in Barrieu and El Karoui [2], Föllmer and Schied [23, Section 8.2] and Cherny and Madan [10] where claims with a single payout date were studied. Jaschke and Küchler [31] allowed for general claims with multiple payout dates but still assumed that the set $\mathcal{C}$ is a cone.

If the infimum in the definition of φ is attained and if $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$ as in (1), the reservation value $\pi^0(c)$ is given by the optimum value of the convex optimization problem

$$\begin{aligned} &\text{minimize} && S_0(x_0) + c_0 \quad \text{over} \quad x \in \mathcal{N}_D, \\ &\text{subject to} && S_t(\Delta x_t) + c_t \leq 0, \quad t = 1, \dots, T-1, \\ &&& \mathcal{V}_T(S_T(\Delta x_T) + c_T) \leq 0. \end{aligned} \tag{9}$$

The same idea is behind the classical Black–Scholes–Merton option pricing model [6] where the price of an option is defined as the least amount of initial capital that allows for the construction of a trading strategy whose terminal value equals the payout of the option. The reservation value π^0 can be interpreted much like a *risk measure* as defined in Artzner, Delbaen, Eber and Heath [1]. Unlike in [1], however, we have not assumed the existence of a perfectly liquid numeraire asset so the natural domain of definition for π^0 is the space $\mathcal{M}$ of sequences of cash-flows. This is important in practice where financial liabilities often have several payout dates. The above approach has been applied in the valuation of pension insurance portfolios in [27].

In the deterministic case, we recover the classical discounting principle from actuarial mathematics.

Example 5 (Actuarial best estimate) *Assume that* c *and* S *are deterministic with* $S_t(x,\omega) = x/P_t$ *where* P_t *is the price at time* 0 *of a zero-coupon bond maturing at time* t. *If* $\mathcal{V}_T(\alpha) > 0$ *for positive constants* α, *problem* (9) *can be written as*

$$\begin{aligned} &\text{minimize} \quad && x_0 + c_0 \quad \text{over} \quad x \in \mathcal{N}_D,\\ &\text{subject to} \quad && \Delta x_t/P_t + c_t \le 0, \quad t = 1,\dots,T, \end{aligned}$$

where $\mathcal{N}_D$ *now consists of sequences* $(x_t)_{t=0}^T$ *of real numbers with* $x_T = 0$. *This can be solved explicitly for* $x_t = \sum_{s=t+1}^T P_s c_s$ *and the optimum value*

$$\pi^0(c) = \sum_{s=0}^T P_s c_s.$$

Such formulas are frequently encountered in actuarial mathematics where c_t *is usually defined as the expected claims to be paid at time* t. *In that context, the above expression is known as the "best estimate" of the claims.*

It should be noted, however, that the "best estimate" is appropriate only for the valuation of deterministic cash-flows. For uncertain ones, it can result in quite unreasonable values. For example, the "best estimate" of a European call option is usually much higher than its market or Black–Scholes value. Nevertheless, the "best estimate" is still widely used in the insurance industry and it forms the basis of the regulatory standard in the Solvency II Directive 2009/138/EC of the European Parliament.

When $\mathcal{V}_t = \delta_{L^0_-}$ for all t, so that $\varphi = \delta_{\mathcal{C}}$, the reservation value becomes the *superhedging cost*

$$\pi^0_{\sup}(c) = \inf\{\alpha \,|\, c - \alpha p^0 \in \mathcal{C}\}$$

which gives the least amount of initial capital required for delivering c without any risk of losing money. The superhedging cost has been studied extensively in liquid market models; see e.g. [23, 18]. Extensions to the illiquid case can be found in [52, 35, 45, 44]. Analogously, we define the *subhedging cost* by

$$\pi^0_{\inf}(c) = \sup\{\alpha \,|\, \alpha p^0 - c \in \mathcal{C}\}.$$

While the reservation value $\pi^0(c)$ depends on the disutility functions $\mathcal{V}_t$, the super- and subhedging costs are essentially independent of such subjective factors and they depend on the probability measure P only through its null sets. The following summarizes some basic properties of the reservation value.

Theorem 6 *The function* π^0 *is convex,* $\pi^0(0) \le 0$ *and*

$$\pi^0(c + c') \le \pi^0(c) \quad \forall c \in \mathcal{M},\ \forall c' \in \mathcal{C}^\infty.$$

We always have $\pi^0(c) \le \pi^0_{\sup}(c)$ *and if* $\pi^0(0) = 0$, *then*

$$\pi^0_{\inf}(c) \le \pi^0(c) \le \pi^0_{\sup}(c)$$

with equalities throughout if $c - \alpha p^0 \in \mathcal{C} \cap (-\mathcal{C})$ for some $\alpha \in \mathbb{R}$ in which case $\pi^0(c) = \alpha$.

Proof. Defining $\mathcal{A} = \{c \in \mathcal{M} \,|\, \varphi(c) \leq 0\}$, we have

$$\pi^0(c) = \inf\{\alpha \,|\, c - \alpha p \in \mathcal{A}\}.$$

By Lemma 2, $\mathcal{A}$ is a convex set with $0 \in \mathcal{A}$ and $c + c' \in \mathcal{A}$ for every $c \in \mathcal{A}$ and $c' \in \mathcal{C}^\infty$. This implies the first part of the statement.

Since $\delta_{L^0_-}$ is the greatest among all convex nondecreasing functions $\mathcal{V}_t$ with $\mathcal{V}_t(0) = 0$, we have $\varphi \leq \delta_{\mathcal{C}}$ and thus,

$$\begin{aligned}\pi^0(c) &= \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \leq 0\}\\ &\leq \inf\{\alpha \,|\, \delta_{\mathcal{C}}(c - \alpha p^0) \leq 0\}\\ &\leq \inf\{\alpha \,|\, c - \alpha p^0 \in \mathcal{C}\} = \pi^0_{\sup}(c).\end{aligned}$$

Since $\pi^0_{\inf}(c) = -\pi^0_{\sup}(-c)$, we also have $\pi^0_{\inf}(c) \leq -\pi^0(-c)$. By convexity,

$$\pi^0(0) = \pi^0(\frac{1}{2}c + \frac{1}{2}(-c)) \leq \frac{1}{2}\pi^0(c) + \frac{1}{2}\pi^0(-c)$$

so if $\pi^0(0) \geq 0$ we get $-\pi^0(-c) \leq \pi^0(c)$, which completes the proof of the inequalities. If $c - \alpha p^0 \in \mathcal{C} \cap (-\mathcal{C})$ for some $\alpha \in \mathbb{R}$, we get $\pi^0_{\sup}(c) \leq \alpha \leq \pi^0_{\inf}(c)$. □

The convexity and monotonicity properties of the reservation value π^0 make it reminiscent of convex risk measures. Moreover, if $c' \in \mathcal{M}$ is such that $c' - \alpha p^0$ is redundant for some $\alpha \in \mathbb{R}$, then

$$\pi^0(c + c') = \pi^0(c) + \alpha,$$

which extends the "translation invariance" property of risk measures to claims with multiple payout dates. Indeed, if $c' - \alpha p^0 \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$, Theorem 6 gives

$$\pi^0(c) = \pi^0(c + c' - \alpha' p^0) = \pi^0(c + c') - \alpha'.$$

The condition $\pi^0(0) = 0$ means that the reservation value for zero liabilities is zero. It holds, in particular, if $\mathcal{V}_0 = \delta_{L^0_-}$, the cost function S_0 is nondecreasing and if short selling is prohibited, i.e. if $D_0 \subseteq R^J_+$. In the totally risk averse case with $\mathcal{V}_t = \delta_{L^0_-}$ for all t, the condition $\pi^0(0) \geq 0$ means that it is not possible to superhedge the zero claim when starting from a strictly negative initial wealth. This is weaker than the no-arbitrage condition; see Section 6.

In general, the reservation value $\pi^0(c)$ depends on the probability measure P and the disutility functions $\mathcal{V}_t$ (views and preferences). By the last part of Theorem 6, however, the reservation value is independent of such subjective factors when $c - \alpha p^0 \in \mathcal{C} \cap (-\mathcal{C})$ for some $\alpha \in \mathbb{R}$.

Example 7 (Numeraire and stochastic integration) *Consider again the classical model of liquid markets in Example 4. We get*

$$\pi^0_{\sup}(c) = \inf\{\alpha \,|\, \exists x \in \mathcal{N}_D : \sum_{t=0}^{T} c_t \le \alpha + \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}\}.$$

Since now $\mathcal{C}^\infty = \mathcal{C}$, by Lemma 2, we have $c - \alpha p^0 \in \mathcal{C} \cap (-\mathcal{C})$ if and only if $c - \alpha p^0$ is redundant. By Example 4, this holds if there is an $x \in \mathcal{N}_D$ such that

$$\sum_{t=0}^{T} c_t = \alpha + \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}.$$

Under the no-arbitrage condition, the converse holds.

Proof. In the classical linear model, we have, by Example 4, that

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \sum_{t=0}^{T} c_t \le \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}\}$$

so

$$\begin{aligned}\pi^0_{\sup}(c) &= \inf\{\alpha \,|\, c - \alpha p^0 \in \mathcal{C}\} \\ &= \inf\{\alpha \,|\, \exists x \in \mathcal{N}_D : \sum_{t=0}^{T} (c_t - \alpha p^0_t) \le \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}\}.\end{aligned}$$

Since $p^0 = (1, 0, \dots, 0)$, this gives the expression for $\pi^0_{\sup}(c)$. The rest follow from the characterization of redundancy in Example 4. □

4 Indifference swap rates

Much of trading in practice consists of exchanging sequences of cash-flows. This is the case in various swap and insurance contracts where both claim and premium payments may be uncertain and they are paid over several points in time. The timing of payments matters since in practice one cannot postpone all payments by going short in the cash account. This section extends the indifference pricing principle to general swap contracts in illiquid markets. We show, in particular, that indifference swap rates lie between arbitrage bounds suitably generalized to account for nonlinear illiquidity effects and general premium processes.

Consider an agent with liabilities $\bar{c} \in \mathcal{M}$ (recalling that negative values of c are interpreted as income) and the possibility to enter a *swap contract* where the agent agrees to deliver an additional sequence $c \in \mathcal{M}$ of random payments in exchange for receiving another sequence proportional to a given *premium process* $p \in \mathcal{M}$. The lowest *swap rate* (premium rate) that would allow the agent to enter the contract without worsening his risk-return profile is given by

$$\pi_s(\bar{c}, p; c) := \inf\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} + c - \alpha p) \le \varphi(\bar{c})\}$$

where, as in the previous section, φ is the optimal value function of (ALM). Similarly,

$$\pi_b(\bar{c}, p; c) := \sup\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} + \alpha p - c) \leq \varphi(\bar{c})\}$$

gives the highest swap rate the agent would accept for taking the opposite side of the trade. Clearly, $\pi_b(\bar{c}, p; c) = -\pi_s(\bar{c}, p; -c)$.

When $p = (1, 0, \dots, 0)$ and $c = (0, \dots, 0, c_T)$, we recover the more traditional pricing problem where one looks for an initial payment that compensates for delivering a claim with a single payment date. The value of $\pi_s(\bar{c}, p; c)$ still depends on the agent's existing liabilities $\bar{c}$ and is, in general, different from the reservation value $\pi^0(c)$ studied in the previous section. In an *interest rate swap*, one has $p = (N, \dots, N)$ and $c = (r_0 N, \dots, r_T N)$, where N is the notional and r_t the floating rate. In a *credit default swap*, the premium process p is a constant sequence until the default of the reference entity after which it is zero while the claim c is zero except at the first t after default when it pays a fixed amount. *Collateralized debt obligations* also fit the above format. More examples can be found in the insurance industry. Traditional risk neutral pricing approaches do not apply to such contracts since the traded cash-flows cannot be replicated by trading; see e.g. Bielecki et al. [5] for further discussion. Indifference prices are related but different, in general, from the well-known "fair price" (marginal utility price) of Davis [14]; see [15, Section 4.3] for details and further references.

In order to derive arbitrage bounds for indifference swap rates, the traditional bounds have to be appropriately generalized to account for illiquidity and general premium processes $p \in \mathcal{M}$. Given $c \in \mathcal{M}$, we define *super-* and *subhedging swap rates*

$$\pi_{\sup}(p; c) = \inf\{\alpha \,|\, c - \alpha p \in \mathcal{C}^\infty\} \quad \text{and} \quad \pi_{\inf}(p; c) = \sup\{\alpha \,|\, \alpha p - c \in \mathcal{C}^\infty\}.$$

Clearly, $\pi_{\inf}(p; c) = -\pi_{\sup}(p; -c)$. When $\mathcal{C}$ is a cone, we have $\mathcal{C}^\infty = \mathcal{C}$, by Lemma 2. Thus, when $\mathcal{C}$ is a cone and $p = (1, 0, \dots, 0)$, the functions $\pi_{\sup}(p; \cdot)$ and $\pi_{\inf}(p; \cdot)$ coincide with the super- and subhedging costs $\pi^0_{\sup}$ and $\pi^0_{\inf}$ studied in the previous section. While the indifference swap rates depend on the disutility functions $\mathcal{V}_t$ and the initial liabilities $\bar{c}$, the super- and subhedging swap rates are independent of such subjective factors and they depend on the probability measure P only through its null sets.

Theorem 8 *Let $\bar{c}, p \in \mathcal{M}$. The function $\pi_s(\bar{c}, p; \cdot)$ is convex, $\pi_s(\bar{c}, p; 0) \leq 0$ and*

$$\pi_s(\bar{c}, p; c + c') \leq \pi_s(\bar{c}, p; c) \quad \forall c \in \mathcal{M},\ \forall c' \in \mathcal{C}^\infty.$$

We always have $\pi_s(\bar{c}, p; c) \leq \pi_{\sup}(p; c)$ and if $\pi_s(\bar{c}, p; 0) = 0$, then

$$\pi_{\inf}(p; c) \leq \pi_b(\bar{c}, p; c) \leq \pi_s(\bar{c}, p; c) \leq \pi_{\sup}(p; c)$$

with equalities throughout if $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some $\alpha \in \mathbb{R}$ in which case $\pi_s(\bar{c}, p; c) = \alpha$.

Proof. Defining $\mathcal{A}(\bar{c}) = \{c \in \mathcal{M} \,|\, \varphi(\bar{c} + c) \leq \varphi(\bar{c})\}$, we have

$$\pi_s(\bar{c}, p; c) = \inf\{\alpha \,|\, c - \alpha p \in \mathcal{A}(\bar{c})\}.$$

By Lemma 2, $\mathcal{A}(\bar{c})$ is a convex set with $0 \in \mathcal{A}(\bar{c})$ and $c + c' \in \mathcal{A}(\bar{c})$ for every $c \in \mathcal{A}(\bar{c})$ and $c' \in \mathcal{C}^\infty$. This implies the first part of the statement.

If $c - \alpha p \in \mathcal{C}^\infty$, Lemma 2 gives $\varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})$ and thus $\pi_s(\bar{c}, p; c) \leq \alpha$. Taking the infimum over such α, we get $\pi_s(\bar{c}, p; c) \leq \pi_{\sup}(p; c)$. By convexity,

$$\pi_s(\bar{c}, p; 0) \leq \frac{1}{2}\pi_s(\bar{c}, p; c) + \frac{1}{2}\pi_s(\bar{c}, p; -c),$$

so that $\pi_b(\bar{c}, p; c) \leq \pi_s(\bar{c}, p; c)$ when $\pi_s(\bar{c}, p; 0) \geq 0$. Since $\pi_{\inf}(p; c) = -\pi_{\sup}(p; -c)$ and $\pi_b(\bar{c}, p; c) = -\pi_s(\bar{c}, p; -c)$ we also get $\pi_{\inf}(p; c) \leq \pi_b(\bar{c}, p; c)$, which completes the proof of the chain of inequalities. If $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some α, we get $\pi_{\sup}(p; c) \leq \alpha$ and $\pi_{\inf}(p; c) \geq \alpha$ so the inequalities must hold as equalities. $\square$

The condition $\pi_s(\bar{c}, p; 0) = 0$ means that one cannot deliver strictly positive multiples of the premium p without worsening the optimal value of (ALM). In the totally risk averse case with $\mathcal{V}_t = \delta_{L^0_-}$ for all t and with $p = (1, 0, \ldots, 0)$, the condition $\pi_s(0, p; 0) = 0$ means that it is not possible to superhedge the zero claim when starting from a strictly negative initial wealth. Such a condition has been called "no-arbitrage of the second type" in the context of perfectly liquid markets [30] and "no strong arbitrage" in fixed-income markets with illiquidity effects [20, 21].

The inequality $\pi_b(\bar{c}, p; c) \leq \pi_s(\bar{c}, p; c)$ means that two agents with identical characteristics have no reason to trade with each other. Differences in the financial position, market expectations and/or risk preferences (as described by $\bar{c}$ and $\mathcal{V}_t$, respectively) may provide incentive for trading. If $c - \alpha p$ is redundant for some α, then $\pi_s(\bar{c}, p; c)$ equals $\pi_{\sup}(p; c)$ which is independent of such subjective factors. Moreover, the convexity of $\pi_{\sup}(p; \cdot)$ and the concavity of $\pi_{\inf}(p; \cdot)$ imply that swap rates are linear in c on the linear subspace of claims c such that $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$. These are well-known facts in liquid market models with a cash account and premiums of the form $p = (1, 0, \ldots, 0)$; see e.g. Biagini et al. [4, Proposition 4.2]. Rouge and El Karoui [50, Proposition 2.1] gives an extension to a more abstract market model.

While redundant claims relate indifference prices to classical "risk neutral prices", their practical significance should not be over-estimated.

Remark 9 *If there is an arbitrage-free price process s such that $s_t \cdot x < S_t(x)$ for $x \neq 0$, then $\mathcal{C}^\infty \cap (-\mathcal{C}^\infty) = \{0\}$. Indeed, if $c \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$, there exist $x^1, x^2 \in \mathcal{N}_D$ such that $S_t(\Delta x^1_t) + c_t \leq 0$ and $S_t(\Delta x^2_t) - c_t \leq 0$ and thus,*

$$s_t \cdot \Delta(x^1_t + x^2_t) \leq S_t(\Delta x^1_t) + S_t(\Delta x^2_t) \leq 0,$$

with the first inequality being strict unless $\Delta x^1_t = \Delta x^2_t = 0$. If s is arbitrage-free, we must have $\Delta x^1_t = \Delta x^2_t = 0$ and thus, $c = 0$.

5 Existence of optimal trading strategies

This section gives conditions for the existence of optimal solutions of the optimal investment problem (ALM). This is done by embedding (ALM) in the general stochastic optimization framework of [46]. Besides existence of optimal trading strategies, we find that the value function φ is lower semicontinuous with respect to convergence in measure on the space $\mathcal{M}$ of adapted claims. The lower semicontinuity is important e.g. in deriving dual representations for the optimal value function φ, the reservation value π^0 and the indifference swap rate π.

A function $f : \mathbb{R}^n \times \Omega \to \overline{\mathbb{R}}$ is said to be an $\mathcal{F}$-measurable *normal integrand* if $f(\cdot, \omega)$ is lower semicontinuous for every $\omega \in \Omega$ and the set-valued mapping $\omega \mapsto \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \,|\, f(x, \omega) \leq \alpha\}$ is closed-valued and $\mathcal{F}$-measurable[3]; see [48, Chapter 14] for a general treatment of measurable set-valued mappings and normal integrands. By [48, Corollary 14.34], a normal integrand is always $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}$-measurable, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel sigma algebra on $\mathbb{R}^n$. Conversely, if $\mathcal{F}$ is P-complete, then $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}$-measurability implies that f is a normal integrand as soon as it is lower semicontinuous in the first argument.

We assume from now on that for each t the cost function S_t is an $\mathcal{F}_t$-measurable normal integrand. Given a market model (S, D), we obtain a conical market model (S^∞, D^∞) by defining

$$S_t^\infty(x, \omega) = \sup_{\alpha > 0} \frac{S_t(\alpha x, \omega)}{\alpha},$$
$$D_t^\infty(\omega) = \bigcap_{\alpha > 0} \alpha D_t(\omega).$$

Indeed, by [48, Exercise 14.21], D_t^∞ are $\mathcal{F}_t$-measurable closed convex cones and, by [48, Exercise 14.54], S_t^∞ are $\mathcal{F}_t$-measurable normal integrands sublinear in x. In this conical market model, trades are made at a highest possible unit prices, and constraints for large positions are enforced also for arbitrarily small positions. In models with superlinear trading costs (see Guasoni and Rásonyi [26]), one gets $S_t^\infty(\cdot, \omega) = \delta_{\{0\}}$ which corresponds to the most illiquid market with infinite cost for all nonzero trades. If S itself is sublinear and D is conical, we simply have $(S^\infty, D^\infty) = (S, D)$.

In the following result, we will assume that

$$\mathcal{V}_t(c_t) = \int v_t(c_t(\omega), \omega) dP(\omega),$$

where v_t is a normal integrand on $\mathbb{R} \times \Omega$ such that $v_t(\cdot, \omega)$ is nondecreasing and convex with $v_t(0, \omega) = 0$ for every $\omega \in \Omega$. We define the integral of a measurable function as $+\infty$ unless the positive part of the function is integrable.

[3] A set valued mapping $\omega \mapsto C(\omega)$ is $\mathcal{F}$-measurable if $\{\omega \,|\, C(\omega) \cap O \neq \emptyset\} \in \mathcal{F}$ for every open set O.

Theorem 10 *Assume that* $\{x \in \mathcal{N}_{D^\infty} \,|\, S_t^\infty(\Delta x_t) \le 0\}$ *is a linear space, that* $v_t(\cdot,\omega)$ *are nonconstant and that there is an integrable function* $m \in L^1$ *such that* $v_t \ge m$. *Then the infimum in* (ALM) *is attained for every* $c \in \mathcal{M}$ *and the value function* φ *is proper and lower semicontinuous on* $\mathcal{M}$ *with respect to convergence in measure.*

Proof. When $\mathcal{V}_t(c) = Ev_t(c)$ we can express the value function as

$$\varphi(c) = \inf_{(x,d)\in\mathcal{N}} \int f(x(\omega), d(\omega), c(\omega), \omega) dP(\omega),$$

where $\mathcal{N}$ is the linear space of adapted $\mathbb{R}^J \times \mathbb{R}$-valued processes and f is the extended real-valued function on $(\mathbb{R}^J \times \mathbb{R})^{T+1} \times \mathbb{R}^{T+1} \times \Omega$ defined by

$$f(x,d,c,\omega) = \begin{cases} \sum_{t=0}^T v_t(d_t,\omega) & \text{if } S_t(\Delta x_t,\omega) + c_t \le d_t,\ x_t \in D_t(\omega),\ x_T = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed,

$$\int f(x(\omega), d(\omega), c(\omega), \omega) dP(\omega)$$
$$= \begin{cases} \sum_{t=0}^T \mathcal{V}_t(d_t) & \text{if } S_t(\Delta x_t) + c_t \le d_t \ P\text{-a.s. and } x \in \mathcal{N}_D, \\ +\infty & \text{otherwise} \end{cases}$$

and since $\mathcal{V}_t$ are nondecreasing, the infimum is not affected if we restrict $d_t = S_t(\Delta x_t) + c_t$. We then obtain the optimum value function of (ALM) as claimed.

By [48, 14.32, 14.36, 14.44], f is an $\mathcal{F}$-measurable normal integrand. We are thus in the general setting of [46] so by [46, Theorem 2], φ is lower semicontinuous and the infimum is attained for every $c \in \mathcal{M}$ provided

$$\mathcal{L} := \{(x,d) \in \mathcal{N} \,|\, f^\infty(x,d,0) \le 0 \ P\text{-a.s.}\}$$

is a linear space[4]. Here f^∞ is the extended real-valued function on $(\mathbb{R}^J \times \mathbb{R})^{T+1} \times \mathbb{R}^{T+1} \times \Omega$ defined by

$$f^\infty(x,d,c,\omega) = \sup_{\alpha>0} \frac{f(\alpha x, \alpha d, \alpha c, \omega)}{\alpha}.$$

By [48, 14.54], f^∞ is a normal integrand and, by [47, Theorems 8.7 and 9.3],

$$f^\infty(x,d,c,\omega) = \begin{cases} \sum_{t=0}^T v_t^\infty(d_t,\omega) & \text{if } S_t^\infty(\Delta x_t,\omega) + c_t \le d_t,\ x_t \in D_t^\infty(\omega), \\ +\infty & \text{otherwise.} \end{cases}$$

[4] Theorem 2 of [46] gives lower semicontinuity with respect to a locally convex topology on a space of integrable functions but the proof of [46, Theorem 2] establishes lower semicontinuity with respect to convergence in measure for f with an integrable lower bound.

The lower bound on v_t implies $v_t^\infty \geq 0$. Since $v_t(\cdot, \omega)$ is nondecreasing and nonconstant, we must have $v_t^\infty(d, \omega) > 0$ for $d > 0$; see [47, Corollary 8.6.1]. Thus,

$$\mathcal{L} = \{(x, d) \in \mathcal{N} \times \mathcal{M} \,|\, S_t^\infty(\Delta x_t) \leq d_t \leq 0,\ x_t \in D_t^\infty\ P\text{-a.s.}\}.$$

If $(x, d) \in \mathcal{L}$, the linearity condition gives $S_t^\infty(-\Delta x_t) \leq 0$ and $-x_t \in D_t^\infty$ and then, since $S_t^\infty(\Delta x_t) + S_t^\infty(-\Delta x_t) \geq 0$, by sublinearity of S_t^∞, we get $d = 0$. Thus,

$$\mathcal{L} = \{(x, 0) \in \mathcal{N} \,|\, S_t^\infty(\Delta x_t) \leq 0,\ x_t \in D_t^\infty\}$$

which is linear by assumption. □

The linearity condition in Theorem 10 is a generalization of the no-arbitrage condition in classical perfectly liquid markets. Indeed, when $S_t(x) = s_t \cdot x$ and $D_t \equiv \mathbb{R}^J$, the linearity condition means that any $x \in \mathcal{N}_D$ with $s_t \cdot \Delta x_t \leq 0$ satisfies $s_t \cdot \Delta x_t = 0$. This is exactly the no-arbitrage condition. In the totally risk averse case where $v_t = \delta_{\mathbb{R}_-}$ for every t, we have $\varphi = \delta_{\mathcal{C}}$ and Theorem 10 says that the set $\mathcal{C}$ is closed. In the classical linear model of Example 4, we thus recover the key closedness result of Schachermayer [51, Lemma 2.1]. In nonlinear unconstrained models, the linearity condition corresponds to the *robust no-arbitrage condition* introduced by Schachermayer [52]; see [44, Section 4] for details.

The linearity condition in Theorem 10 may hold even without no-arbitrage conditions. One has $\{x \in \mathcal{N}_{D^\infty} \,|\, S_t^\infty(\Delta x_t) \leq 0\} = \{0\}$, for example, when S is such that $S_t^\infty(x, \omega) > 0$ for all $x \notin \mathbb{R}^J_-$. Such a condition holds e.g. in limit order markets where the limit order books always have finite depth; see e.g. [40]. The linearity condition holds also with linear cost functions $S_t(\cdot, \omega) = s_t(\omega) \cdot x$ provided the price process s is componentwise strictly positive and $D_t^\infty(\omega) \subseteq \mathbb{R}^J_+$ (infinite short selling is prohibited). Theorem 10 generalizes also the existence result for the utility maximization problem studied in Çetin and Rogers [9].

Example 11 *Assume that the cost functions are of the form*

$$S_t(x, \omega) = x^0 + s_t(\omega)\psi(x^1),$$

for a strictly positive adapted process s and an extended real-valued convex function ψ with $\psi(0) = 0$. We have

$$S_t^\infty(x, \omega) = x^0 + s_t(\omega)\psi^\infty(x^1),$$

where ψ^∞ is the recession function of ψ. Under assumption (2.3) of [9], one gets $\psi^\infty = \delta_{\mathbb{R}_-}$ so

$$\{x \in \mathcal{N}_{D^\infty} \,|\, S_t^\infty(\Delta x_t) \leq 0\} = \{x \in \mathcal{N}_{D^\infty} \,|\, \Delta x_t^0 \leq 0,\ \Delta x_t^1 \leq 0\}.$$

Since $x_{-1} = 0$, by definition, and $x_T = 0$ for $x \in \mathcal{N}_{D^\infty}$, this set reduces to the origin so the linearity condition holds. The model of [9] was concerned with optimization of expected utility from terminal wealth, which corresponds to $v_t = \delta_{\mathbb{R}_-}$

for $t < T$ as in Example 1. The utility function in [9] was assumed nonpositive so the lower bound in Theorem 10 holds. We thus obtain the existence of optimal solutions without imposing the extra conditions used in [9].

Nonconstancy of $v_t(\cdot, \omega)$ in Theorem 10 simply means that the investor always prefers more money to less. The lower bound is a more significant restriction since it excludes e.g. the logarithmic utility. In [36] such bounds were avoided but, at present, it is unclear if the lower bound can be relaxed in illiquid markets. However, Guasoni [24, Theorem 5.2] gives the existence of optimal solutions in a continuous time model with a cash-account and proportional transaction costs with similar conditions on the utility function as in [36]. Much like the proof of Theorem 10, his approach was based on the "direct method" rather than duality arguments as e.g. in [36].

6 Weak no-arbitrage conditions

We have not assumed anything like the no-arbitrage condition in our general study of reserving and swap rates. Indeed, Theorems 6 and 8 hold for any market model (S, D) arbitrage-free or not. In the classical linear model, however, we know that superhedging makes sense only if $\pi^0_{\sup}(0) \geq 0$ since otherwise $\pi^0_{\sup}(c) = -\infty$ whenever $\pi^0_{\sup}(c) < \infty$. The condition $\pi^0_{\sup}(0) \geq 0$ has been been referred to as "no-arbitrage of the second type" (Ingersoll [30]), "weak no-arbitrage" (Dermody and Rockafellar [20, 21]) as well as "no strong arbitrage" (LeRoy and Werner [38]). The condition $\pi^0_{\sup}(0) \geq 0$ is also related to the "law of one price". While $\pi^0_{\sup}(0) \geq 0$ means that it is not possible to superhedge the zero claim when starting from strictly negative initial wealth, the law of one price means that it is not possible to *replicate* the zero claim when starting with strictly negative wealth; see e.g. [11].

This section gives minimal consistency conditions for the market model, the disutility functions and the premium processes so that reservation values and indifference swap rates attain finite values. Following [47, page 69], we say that a claim process $c \in \mathcal{M}$ belongs to the *recession cone* of φ if

$$\varphi(\bar{c} + \alpha c) \leq \varphi(\bar{c}) \quad \forall \bar{c} \in \mathcal{M},\ \forall \alpha > 0.$$

An agent can add arbitrary multiples of such claims to his liabilities without worsening the optimum value of (ALM). By Lemma 2, the recession cone of φ contains the set $\mathcal{C}^\infty$ of claim processes that can be superhedged in unlimited amounts.

A necessary condition for the reservation value $\pi^0 : \mathcal{M} \to \overline{\mathbb{R}}$ to be a proper function is that $p^0 = (1, 0, \ldots, 0)$ does not belong to the recession cone of φ since otherwise $\pi^0(c) = -\infty$ for every $c \in \operatorname{dom} \pi^0$. Under the conditions of Theorem 10, this turns out to be sufficient for π^0 to be proper and lower semicontinuous on $\mathcal{M}$.

Proposition 12 *Assume that φ is proper and lower semicontinuous. Then the conditions* *(a) p^0 does not belong to the recession cone of φ,*

(b) $\pi^0(0) > -\infty$,

(c) $\pi^0(c) > -\infty$ for all $c \in \mathcal{M}$,

are equivalent and imply that π^0 is proper and lower semicontinuous on $\mathcal{M}$ and that the infimum

$$\pi^0(c) = \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \leq 0\}$$

is attained for every $c \in \mathcal{M}$.

Proof. We have

$$\pi^0(c) = \inf\{\alpha \,|\, c - \alpha p^0 \in \mathcal{A}\}$$

where $\mathcal{A} = \{c \in \mathcal{M} \,|\, \varphi(c) \leq 0\}$ is closed and convex. By [45, Lemma 3.3]), condition (b) means that p^0 does not belong to the recession cone of $\mathcal{A}$. By [45, Lemma 3.3], (b) is equivalent to (c). The equivalence of (a) and (b) comes from the general fact that the recession cone of a lower semicontinuous convex function coincides with the recession cone of any of its nonempty level sets; see [47, Theorem 8.7] for the proof in the finite-dimensional case (the general case follows by a similar argument from [45, Lemma 3.3]). The rest of the proof is now identical to that of [44, Proposition 2]. □

The recession condition in Proposition 12 holds in particular under the condition $\pi^0(0) \geq 0$ used in Theorem 6. The recession condition can be written as

$$p^0 \notin \bigcap_{\alpha>0} \alpha\mathcal{A},$$

while $\pi^0(0) \geq 0$ means that

$$p^0 \notin \bigcup_{\alpha>0} \alpha\mathcal{A},$$

where $\mathcal{A} = \{c \in \mathcal{M} \,|\, \varphi(c) \leq 0\}$. When $\mathcal{A}$ is a cone, as is the case in a conical market model and the most risk averse agent with $\mathcal{V}_t = \delta_{L^0_-}$, the conditions $\pi^0(0) \geq 0$ and $\pi^0(0) > -\infty$ are equivalent.

It is often natural to assume that $\varphi(c - \alpha p^0) < \varphi(c)$ for all $c \in \operatorname{dom} \varphi$ and $\alpha > 0$, i.e. that receiving strictly positive amounts of cash at time $t = 0$ allows for a strict improvement in the optimal investment problem. This would clearly imply the recession condition in Proposition 12 but it would not be satisfied e.g. in the totally risk averse case where $\varphi = \delta_{\mathcal{C}}$.

For indifference swap rates, similar conclusions can be made. In this case, the value function φ has to be compatible with the premium process $p \in \mathcal{M}$. A minimal requirement is that p does not belong to the recession cone of φ since otherwise $\pi_s(\bar{c}, p; c) = -\infty$ for every $c \in \operatorname{dom} \pi_s(\bar{c}, p; \cdot)$.

Proposition 13 *Assume that φ is proper and lower semicontinuous. Then, for every $\bar{c} \in \operatorname{dom} \varphi$, the conditions* *(a) p does not belong to the recession cone of* φ*,*

(b) $\pi_s(\bar{c}, p; 0) > -\infty$*.*

(c) $\pi_s(\bar{c}, p; c) > -\infty$ *for all* $c \in \mathcal{M}$*,*

are equivalent and imply that $\pi_s(\bar{c}, p; \cdot)$ *is proper and lower semicontinuous on* $\mathcal{M}$ *and that the infimum*

$$\pi_s(\bar{c}, p; c) = \inf\{\alpha \,|\, \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}$$

is attained for every $c \in \mathcal{M}$*.*

Proof. Defining $\mathcal{A}(\bar{c}) = \{c \in \mathcal{M} \,|\, \varphi(\bar{c} + c) \leq \varphi(\bar{c})\}$, we have

$$\pi^0(c) = \inf\{\alpha \,|\, c - \alpha p^0 \in \mathcal{A}(\bar{c})\}.$$

Again, since the recession cone of a lower semicontinuous convex function coincides with the recession cone of any of its nonempty level sets, we have that the recession cone of $\mathcal{A}(\bar{c})$ is independent of $\bar{c} \in \operatorname{dom} \varphi$ and coincides with the recession cone of φ. The rest of the proof is analogous to that of Proposition 12. □

The recession condition in Proposition 13 holds in particular under the condition $\pi_s(\bar{c}, p; 0) \geq 0$ used in Theorem 8. The recession condition can be written as

$$p \notin \bigcap_{\alpha > 0} \alpha \mathcal{A},$$

while $\pi_s(\bar{c}, p; 0) \geq 0$ means that

$$p \notin \bigcup_{\alpha > 0} \alpha \mathcal{A}(\bar{c}),$$

where $\mathcal{A}(\bar{c}) = \{c \in \mathcal{M} \,|\, \varphi(\bar{c} + c) \leq \varphi(\bar{c})\}$. While the condition $\pi_s(\bar{c}, p; 0) \geq 0$ depends on the agent's current financial position $\bar{c}$, the recession condition does not.

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