

Convex integral functionals of processes of bounded variation

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Abstract

This article characterizes conjugates and subdifferentials of convex integral functionals over the linear space of stochastic processes of essentially bounded variation (BV) when the space is identified with the Banach dual of the space of regular processes. Our proofs are based on new results on the interchange of integration and minimization of integral functionals over BV processes. Under mild conditions, the domain of the conjugate is shown to be contained in the space of semimartingales which leads to several applications in the duality theory in stochastic control and mathematical finance.

Keywords. stochastic process, bounded variation; integral functional; convex duality

AMS subject classification codes. 46N10, 60G07

1 Introduction

This article studies convex integral functionals of the form

$$Ef(x) = E \left[\int_{[0,T]} h_t(x_t) d\mu_t + k_0(x_0) + k_T(x_{T+}) \right]$$

defined on the linear space \mathcal{N}^∞ of adapted left-continuous processes of essentially bounded variation in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. Here μ is a positive atomless optional random measure on $[0, T]$, h is a convex normal integrand on $\Omega \times [0, T] \times \mathbb{R}^d$, and k_0 and k_T are convex normal integrands on $\Omega \times \mathbb{R}^d$. The main result of this paper gives an explicit expression for the

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conjugate of Ef when the space \mathcal{N}^∞ is identified with the Banach dual of *regular processes*, the optional projections of continuous processes with integrable supremum norm.

Under fairly general conditions, the domain of the conjugate of Ef is contained in the space of regular quasimartingales which are semimartingales given by optional projections of continuous processes of integrable variation. This opens up the possibility of treating various problems in stochastic optimal control and mathematical finance with the theory of convex duality and integral functionals. Integral functionals of processes of bounded variation arise, for example, in problems of optimal investment under transaction costs and portfolio constraints. Such problems involve integral functionals of both the investment strategy as well as its derivative which is a random measure. The basic theory of convex integral functionals of random measures have been developed in a companion paper [PP16]. Combining this with the main result of the present paper, allows for a unified treatment not only of optimal investment problems but also of more general problems in singular stochastic control much like [Bis73] unified convex stochastic control problems without singularities. A general class of singular stochastic control problems will be treated in a followup paper.

Our proofs are based on a version of the “interchange rule” that allows for reversing the order of minimization and integration in the optimization of integral functionals. In the case of decomposable spaces of measurable functions, such results go back to the works of [Roc68] and [Val75]. [Roc71] treated the nondecomposable space of continuous functions by embedding it in the space of essentially bounded measurable functions. [BV88, Theorem 1] gives a general interchange rule on spaces that are stable under continuous partitions of unity. This approach was used in [Per14] to study integral functionals of BV functions. In this paper we extend these arguments to the stochastic setting by combining them with interchange rules for Suslin space-valued functions from [Val75].

The rest of this paper is organized as follows. Section 2 studies integral functionals in the deterministic setting over the space X of left-continuous functions of bounded variation. The section is split in two subsections, the first one giving an interchange rule for minimization and integration and the second on conjugates and subdifferentials when X is regarded as the Banach dual of the space of continuous functions. Our main results are given in Section 3 which follows a similar structure in the study of integral functionals over \mathcal{N}^∞ .

2 Integral functionals of BV functions

Given a positive Radon measure μ on $[0, T]$ and h a convex normal integrand on \mathbb{R}^d (see Appendix 4.2), the associated integral functional on the space of measurable \mathbb{R}^d -valued functions is defined by

$$I_h(x) := \int h(x) d\mu := \int_{[0, T]} h_t(x_t) d\mu_t.$$

This section studies I_h on the space X of left-continuous functions $x : \mathbb{R} \rightarrow \mathbb{R}^d$ of bounded variation such that x is constant outside of a fixed time interval $[0, T]$. Throughout this section, we assume that μ is atomless and has support $[0, T]$. Section 2.1 gives sufficient conditions for the interchange of the order of integration and minimization over X . Section 2.2 uses the interchange rule to give an explicit expression for the conjugate of I_h with respect to the pairing of X with the space of \mathbb{R}^d -valued continuous functions.

2.1 Interchange rule

Recall that in a decomposable space \mathcal{X} of measurable \mathbb{R}^d -valued functions on $[0, T]$, one has the interchange rule

$$\inf_{x \in \mathcal{X}} \int h(x) d\mu = \int \inf_{x \in \mathbb{R}^d} h(x) d\mu$$

as soon as the infimum on the left is finite; see [RW98, Theorem 14.60]. For this to hold when, instead of a decomposable space, one minimizes over X , we will need to control the behavior of the set

$$\text{dom } h_t = \{x \in \mathbb{R}^d \mid h_t(x) < \infty\}$$

as a function of t . Recall that a function is left-continuous in the usual sense if and only if it is continuous with respect to the topology τ generated by the left-open intervals $\{(s, t] \mid s < t\}$. Accordingly, a set-valued mapping $S : [0, T] \rightrightarrows \mathbb{R}^d$ is said to be *left-inner semicontinuous* (left-isc) if $\{t \mid S_t \cap A \neq \emptyset\}$ is τ -open for any open $A \subseteq \mathbb{R}^d$.

The following theorem is basically a reformulation of [Per14, Theorem 4]. Given a measurable set-valued mapping S , we will use the notation

$$L^\infty(S) := \{x \in L^\infty \mid x \in S \text{ } \mu\text{-a.e.}\}$$

and equip L^∞ with the usual norm topology. We denote the interior of a set A by $\text{int } A$.

Theorem 1. *Assume that $\text{dom } h$ is left-isc, and that for every $x \in X$,*

$$x \in \text{int } L^\infty(\text{dom } h) \implies x \in \text{dom } I_h \implies x_t \in \text{cl dom } h_t \forall t.$$

If I_h is proper on X and $X \cap \text{int } L^\infty(\text{dom } h) \neq \emptyset$, then

$$\inf_{x \in X} \int h(x) d\mu = \int \inf_{x \in \mathbb{R}^d} h(x) d\mu.$$

Proof. By Theorems 4 and 2 of [Per14], it suffices to show that every left-continuous w with $w_t \in \text{cl dom } h_t$ μ -a.e. satisfies $w_t \in \text{cl dom } h_t$ for all t . To have this, we can follow the arguments in the proof of [Per14, Theorem 4] to get that, for every $\epsilon > 0$ and t , there exists $x \in \text{dom } I_h$ such that $|x_t - w_t| < \epsilon$. By assumption, such x satisfies $x_t \in \text{cl dom } h_t$, and since $\epsilon > 0$ was arbitrary, we have that $w_t \in \text{cl dom } h_t$ as well. \square

When I_h is lsc in L^∞ , the first condition

$$x \in \text{int } L^\infty(\text{dom } h) \implies x \in \text{dom } I_h$$

in Theorem 1 is equivalent to I_h being continuous at every $x \in \text{int } L^\infty(\text{dom } h)$; see [Roc74, Corollary 8B]. The second condition

$$x \in \text{dom } I_h \implies x_t \in \text{cl dom } h_t \quad \forall t$$

in Theorem 1 holds in particular if $t \rightarrow \text{cl dom } h_t$ is *left-continuous* in the sense that it is left-isc and its graph $\{(t, x) \mid x \in \text{cl dom } h_t\}$ is closed in the product of τ -topology on $[0, T]$ and the Euclidean topology on \mathbb{R}^d .

2.2 Duality

The space X may be identified with $\mathbb{R}^d \times M$ where M is the space of \mathbb{R}^d -valued Radon measures on $[0, T]$. Indeed, given $x \in X$ there is a unique \mathbb{R}^d -valued Radon measure Dx on $[0, T]$ such that $x_t = x_0 + Dx([0, t])$ for all $t \in [0, T]$ and $x_t = x_0 + Dx([0, T])$ for $t > T$; see e.g. [Fol99, Theorem 3.29]. The value of $x \in X$ on (T, ∞) will be denoted by x_{T+} .

By the Riesz representation theorem, M may be identified with the Banach dual of the space C of continuous functions on $[0, T]$ when C is equipped with the supremum norm. Indeed, C and M are in separating duality under the bilinear form

$$\langle u, \theta \rangle := \int u d\theta.$$

Similarly, X and $V := \mathbb{R}^d \times C$ are in separating duality under the bilinear form

$$\langle v, x \rangle := v_{-\infty} \cdot x_0 + \int v dx.$$

Here, the notation corresponds to our convention of identifying elements of V with continuous functions on $\{-\infty\} \cup [0, T]$ (Analogously, we may identify X with the space of Radon measures on $\{-\infty\} \cup [0, T]$).

[Roc71] and more recently [Per14, Per17] gave conditions under which the conjugate of an integral functional I_h on C can be expressed as

$$(I_h)^* = J_{h^*},$$

where, for a normal integrand f , the functional $J_f : M \rightarrow \overline{\mathbb{R}}$ is defined by

$$J_f(\theta) = \int f(d\theta^a/d\mu) d\mu + \int f^\infty(d\theta^s/d|\theta^s|) d|\theta^s|,$$

where θ^a and θ^s are the absolutely continuous and the singular part, respectively, of θ with respect to μ , $|\theta^s|$ is the total variation of θ^s , and f^∞ is the normal integrand defined pointwise as the recession function of f_t ; see the appendix.

Theorem 2 below gives an expression for the conjugate and subdifferential of I_h with respect to the pairing of X with V . Given $x \in X$, we denote by $\partial h(x)$

the set-valued mapping $t \mapsto \partial h_t(x_t)$. We also use the notation $\partial^s h := \partial \delta_{\text{cl dom } h}$. We will denote by V_{BV} the linear subspace of V consisting of $v \in V$ that have bounded variation on $[0, T]$. Given $v \in V_{BV}$, we set $dv/d\mu := d(Dv)^a/d\mu$ and $dv/d|Dv^s| := d(Dv)^s/d|(Dv)^s|$.

Theorem 2. *Let $f(x) = I_h(x) + k_0(x_0) + k_T(x_{T+})$, where k_0 and k_T are closed proper convex functions on \mathbb{R}^d and h is a convex normal integrand satisfying the assumptions of Theorem 1. Then f is closed and*

$$f^*(v) = \begin{cases} J_{h^*}(-Dv) + k_0^*(v_{-\infty} - v_0) + k_T^*(v_T) & \text{if } v \in V_{BV}, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, $v \in \partial f(x)$ if and only if

$$\begin{aligned} -dv/d\mu &\in \partial h(x) \quad \mu\text{-a.e.}, \\ -dv/d|Dv^s| &\in \partial^s h(x) \quad |(Dv)^s|\text{-a.e.}, \\ v_{-\infty} - v_0 &\in \partial k_0(x_0), \\ v_T &\in \partial k_T(x_{T+}). \end{aligned}$$

Proof. This is a special case of Theorem 4 below. The conjugate formula also follows as in the proof of [PP14, Theorem 2.2]. \square

3 Integral functionals of BV processes

Let (Ω, \mathcal{F}, P) be a probability space and let $(\mathcal{F}_t)_{t \geq 0}$ be an increasing sequence of σ -algebras on Ω such that $\mathcal{F}_t = \bigcap_{t' > t} \mathcal{F}_{t'}$ and \mathcal{F}_0 contains all the P -null sets. We will denote the linear space of left-continuous adapted processes of *essentially bounded variation* by \mathcal{N}^∞ . That is, $x \in \mathcal{N}^\infty$ if $x \in X$ almost surely, the pathwise total variation of x is essentially bounded and x_t is \mathcal{F}_t -measurable for all $t \in [0, T]$.

Recall that the *predictable* and *optional* σ -algebras on $\Omega \times [0, T]$ are the ones generated by left- and right-continuous, respectively, adapted processes. In particular, the elements of \mathcal{N}^∞ are predictable. We denote by \mathcal{T} the set of *stopping times*, that is, functions $\tau : \Omega \rightarrow [0, T] \cup \{+\infty\}$ such that $\{(\omega, t) \mid \tau(\omega) \leq t\}$ is optional. If a (not necessarily adapted) stochastic process v is \mathcal{T} -integrable in the sense that v_τ is integrable for every $\tau \in \mathcal{T}$, then, e.g., by [HWY92, Theorem 5.1], there exists an optional process ${}^o v$ such that

$$E[v_\tau \mathbb{1}_{\{\tau < \infty\}} \mid \mathcal{F}_\tau] = {}^o v_\tau \mathbb{1}_{\{\tau < \infty\}} \quad P\text{-a.s. for all } \tau \in \mathcal{T},$$

where $\mathcal{F}_\tau := \{A \in \mathcal{F} \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t \forall t \in \mathbb{R}_+\}$. The process ${}^o v$ is called the *optional projection* of v and it is unique a.s.e.¹

Recall that (see e.g. [HWY92, Theorem 3.16]) if τ is a stopping time then $[\tau, \infty) \subset \Omega \times \mathcal{F}$ is optional. A stopping time is said to be *predictable* if $[\tau, \infty)$

¹The abbreviation a.s.e. stands for “ P -almost surely everywhere on $[0, T]$ ”, that is, outside an evanescent set.

predictable. If a (not necessarily adapted) stochastic process v is such that v_τ is integrable for every predictable time $\tau \in \mathcal{T}$, then, e.g., by [HWY92, Theorem 5.2], there exists a predictable process ${}^p v$ such that

$$E[v_\tau \mathbb{1}_{\{\tau < \infty\}} \mid \mathcal{F}_{\tau-}] = {}^p v_\tau \mathbb{1}_{\{\tau < \infty\}} \quad P\text{-a.s. for all predictable } \tau \in \mathcal{T},$$

where $\mathcal{F}_{\tau-} := \mathcal{F}_0 \vee \sigma\{A \cap \{t < \tau\} \mid A \in \mathcal{F}_t, t \in \mathbb{R}_+\}$. The process ${}^p v$ is called the *predictable projection* of v and it is unique a.s.e.

Let μ be a random positive measure on $[0, T]$ and let

$$\mathbb{L}^p := L^p(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]), \eta; \mathbb{R}^d),$$

where the measure η is defined by $\eta(A) := E \int \mathbb{1}_A d\mu$. We will assume throughout that μ is atomless, has full support almost surely and that it is *optional* in the sense that

$$E \int v d\mu = E \int {}^o v d\mu$$

for all bounded v .

Let h be a normal $\mathcal{F} \otimes \mathcal{B}([0, T])$ -integrand on \mathbb{R}^d and define $I_h : X \times \Omega \rightarrow \overline{\mathbb{R}}$ by

$$I_h(x, \omega) := I_{h(\cdot, \omega)}(x),$$

where the right side is defined as in Section 2. We assume throughout that there exist $v \in \mathbb{L}^1(\mathbb{R}^d)$ and nonnegative $\alpha \in \mathbb{L}^1$ such that

$$h(x) \geq x \cdot v - \alpha. \quad (1)$$

The following is proved in the appendix.

Lemma 3. *The function I_h is a normal integrand on X .*

By Lemma 3, the integral functional

$$EI_h(x) := \int_{\Omega} I_h(x(\omega), \omega) dP(\omega)$$

is a well-defined convex function on \mathcal{N}^∞ . Section 3.1 below gives an interchange rule for EI_h and Section 3.3 gives an expression for the conjugate of EI_h with respect to the pairing of \mathcal{N}^∞ with regular processes to be defined in Section 3.2.

3.1 Interchange rule

The following result extends Theorem 1 to the stochastic setting. Given a measurable set-valued mapping S from $\Omega \times [0, T]$ to \mathbb{R}^d , we will use the notation

$$\mathbb{L}^\infty(S) := \{x \in \mathbb{L}^\infty \mid x \in S \text{ } \eta\text{-a.e.}\}$$

and equip \mathbb{L}^∞ with the usual norm topology. We denote the closed unit ball with radius r by \mathbb{B}_r . We say that a normal integrand is *predictable* if its epigraphical mapping is predictable.

Theorem 4. *Let h be a predictable normal integrand that satisfies the assumptions of Theorem 1 almost surely and assume that there exists an $\bar{x} \in \mathcal{N}^\infty$ with $\bar{x} \in \text{int } \mathbb{L}^\infty(\text{dom } h)$ and that for every $x \in \text{int } \mathbb{L}^\infty(\text{dom } h)$ there is an $r > 0$ and $\beta \in \mathbb{L}^1$ with*

$$h_t(x_t + x') \leq \beta_t \quad \forall x' \in \mathbb{B}_r.$$

Then,

$$\inf_{x \in \mathcal{N}^\infty} E \int h(x) d\mu = E \int \inf_{x \in \mathbb{R}^d} h(x) d\mu.$$

Proof. By Lemma 3, I_h is a convex normal integrand on $X \times \Omega$. Applying first the interchange rule for expectation and minimization [PP16, Theorem 1] and then the interchange rule Theorem 1, we get

$$\inf_{x \in L^\infty(X)} E \int h(x) d\mu = E \inf_{x \in X} \int h(x) d\mu = E \int \inf_{x \in \mathbb{R}^d} h(x) d\mu,$$

where $L^\infty(X)$ is the space of (possibly nonadapted) left continuous processes of essentially bounded variation. Since $\mathcal{N}^\infty \subset L^\infty(X)$,

$$\inf_{x \in \mathcal{N}^\infty} EI_h(x) \geq \inf_{x \in L^\infty(X)} EI_h(x) \geq \inf_{x \in L^\infty(X)} EI_h({}^p x),$$

where the second inequality follows from Jensen's inequality for predictable normal integrands; see Lemma 16 in the appendix. By [DM82, Theorem VI.43], ${}^p x$ is left continuous with right limits.

We show next that the last infimum above can be restricted to those $x \in L^\infty(X)$ for which $\mathbb{B}(x_t, \tilde{r}) \subset \text{dom } h_t$ for some $\tilde{r} > 0$. If $x \in \text{dom } EI_h$, then $x \in \text{dom } I_h(x)$ a.s., so $x_t \in \text{cl dom } h_t$ a.s.e. We may assume that \bar{x} satisfies $\bar{x} \in \text{dom } h$, and we may redefine x as \bar{x} on a P -null set so that $x \in \text{cl dom } h$. Defining $x^\nu = \frac{1}{\nu} \bar{x} + (1 - \frac{1}{\nu})x$, we have $x^\nu \in \text{int dom } h$ for all ν and, by convexity, $EI_h(x^\nu) \leq \frac{1}{\nu} EI_h(\bar{x}) + (1 - \frac{1}{\nu}) EI_h(x)$. Moreover, since $\mathbb{B}(\bar{x}_t, r) \subset \text{dom } h_t$ and $x_t \in \text{cl dom } h_t$, we have $\mathbb{B}(x_t^\nu, r/\nu) \subset \text{dom } h_t$ by convexity.

Thus, it suffices to show that, for every $\epsilon > 0$ and $\tilde{x} \in L^\infty(X)$ with $\mathbb{B}(\tilde{x}_t, \tilde{r}) \subset \text{dom } h_t$ for some $\tilde{r} > 0$, there exists $x \in \mathcal{N}^\infty$ such that

$$EI_h({}^p \tilde{x}) > EI_h(x) - \epsilon.$$

Since $\text{dom } h$ is predictable, it follows from Jensen's inequality for set-valued mappings [KP16, Corollary 20] that $\mathbb{B}({}^p \tilde{x}_t, \tilde{r}) \subset \text{dom } h_t$. By [Roc71, Theorem 2], there exists $r' \in (0, \tilde{r})$ such that

$$EI_h({}^p \tilde{x}) > EI_h(x) - \epsilon/2$$

for every x such that $|x_t - {}^p \tilde{x}_t| < r'$ a.s.e. For positive integers ν , we define recursively $\tau^0 = 0$ and $\tau^\nu = \inf\{t \geq \tau^{\nu-1} \mid |{}^p \tilde{x}_t - {}^p \tilde{x}_{\tau^{\nu-1}}| \geq r'/2\}$ so that the process $\hat{x} = \sum_{\nu=0}^{\infty} {}^p \tilde{x}_{\tau^\nu} \mathbb{1}_{(\tau^\nu, \tau^{\nu+1}]}$ is predictable with $|\hat{x}_t - {}^p \tilde{x}_t| < r'$.

For every n , we define a stopping time $\sigma^n = \inf\{t \mid |D\hat{x}|([0, t]) \geq n\}$ and a process $x^n = \mathbb{1}_{[0, \sigma^n]}\hat{x} + \mathbb{1}_{(\sigma^n, T]}\bar{x}$. By construction, $x^n \in \mathcal{N}^\infty$ and $x^n \rightarrow \hat{x}$ a.s.e. Moreover,

$$h(x^n) \leq \max\{\hat{\beta}, \bar{\beta}\},$$

for some positive $\hat{\beta}$ and $\bar{\beta}$ with $E \int \max\{\hat{\beta}, \bar{\beta}\} d\mu < \infty$. Therefore, we may apply Fatou's lemma on $[0, T] \times \Omega$ to obtain

$$EI_h(\hat{x}) \geq EI_h(x^n) - \epsilon/2$$

for n large enough. □

3.2 \mathcal{N}^∞ as a Banach dual

This section presents the basic functional analytic framework for our main result to be given in Section 3.3. In particular, we identify \mathcal{N}^∞ as the topological dual of the Banach space of regular processes. We also recall some basic properties of quasimartingales that feature in the main result.

Let $L^\infty(M)$ be the linear space of *random \mathbb{R}^d -valued Radon measures* with essentially bounded variation and let $\mathcal{M}^\infty \subseteq L^\infty(M)$ be the space of essentially bounded *optional* Radon measures on \mathbb{R}^d , i.e. those $\theta \in L^\infty(M)$ such that

$$E \int v d\theta = E \int {}^o v d\theta \quad \forall v \in L^1(C).$$

Here $L^1(C)$ denotes the Banach space of random continuous functions v with the norm

$$\|v\|_{L^1(C)} := E \sup_{t \in [0, T]} |v_t|.$$

The linear mapping $x \mapsto (x_0, Dx)$ defines an isomorphism from \mathcal{N}^∞ to $\mathbb{R}^d \times \mathcal{M}^\infty$. By Theorem 5 below, \mathcal{M}^∞ may be identified with the dual of the Banach space of regular processes, so \mathcal{N}^∞ is a Banach dual as well.

Recall that a process is *regular* if it is the optional projection of a process from $L^1(C)$; see [Bis78]. We will denote the space of regular processes by \mathcal{R}^1 . The following result, essentially proved already in Bismut [Bis78], is from [PP16].

Theorem 5. *The space \mathcal{R}^1 is a Banach space under the norm*

$$\|v\|_{\mathcal{R}^1} := \sup_{\tau \in \mathcal{T}} E|v_\tau|$$

and its dual may be identified with \mathcal{M}^∞ through the bilinear form

$$\langle v, \theta \rangle_{\mathcal{R}^1} := E \int v d\theta.$$

The dual norm can be expressed as

$$\|\theta\|_{\mathcal{M}^\infty} = \text{ess sup} \|\theta\|_{TV}.$$

Our main result, Theorem 8 below, involves regular processes which are also quasimartingales. Recall that a process v is a *quasimartingale* if it is adapted, right continuous, has $E|v_t| < \infty$ for all t and

$$\text{Var}(v) := \sup_{\pi} \left\{ E \left[\sum_{i \in \pi} |E[v_{t_{i+1}} - v_{t_i} | \mathcal{F}_{t_i}]| \right] \right\} < \infty,$$

where the supremum is taken over all finite partitions π of $[0, T]$. The number $\text{Var}(v)$ is known as the *mean variation* of v . Theorem 6 below says in particular that quasimartingales are the optional projections of *IV processes*, i.e. (not necessarily) adapted processes whose initial value as well as pathwise total variation are integrable.

A process is said to be of *class (D)* if the set $\{v_{\tau} | \tau \in \mathcal{T}\}$ is uniformly integrable. Regular processes are of class (D). The following theorem, where $\mathcal{R}_m^1 \subseteq \mathcal{R}^1$ denotes the linear space of càdlàg martingales, summarizes some basic properties of quasimartingales.

Theorem 6. *A càdlàg process v is a quasimartingale of class (D) if and only if $v = m + a$ for an $m \in \mathcal{R}_m^1$ and a predictable process a of integrable variation with $a_0 = 0$. The decomposition is unique. One then has $\text{Var}(v) = E\|Da\|_{TV}$ and*

$$E \left[\int_{[0, T]} v dx \right] = E \left[v_T \cdot x_{T+} - v_0 \cdot x_0 - \int_{[0, T]} x da \right]$$

for every $x \in \mathcal{N}^{\infty}$. *Quasimartingales of class (D) are the optional projections of IV processes.*

Proof. The first two claims are given Sections 3 and 4 of [DM82, Appendix 2]. The integration by parts formula follows from the first claim and the integration by parts formula for semimartingales (recalling that $x \in \mathcal{N}^{\infty}$ is left-continuous). If $v = m + a$, then it is the optional projection of the IV process $\mathbb{1}_{[0, T]} m_T + a$. Conversely, given an IV process b , we see from the definitions of optional projection and mean variation that ${}^o b$ has a finite mean variation. Moreover, the values of an IV process b are bounded by $b_0 + \|Db\|_{TV}$ so b as well as ${}^o b$ are of class (D). \square

We will denote the linear space of regular quasimartingales by \mathcal{R}_{qm}^1 .

Corollary 7. *On \mathcal{R}^1 , we have $\text{Var} = \sigma_{\mathcal{D}}$, where*

$$\mathcal{D} = \{Dx \in \mathcal{M}^{\infty} \mid x \in \mathcal{N}^{\infty}, |x| \leq 1, x_0 = x_{T+} = 0\}.$$

We have $v \in \mathcal{R}_{qm}^1$ if and only if $v = m + a$ for an $m \in \mathcal{R}_m^1$ and a predictable continuous process a of integrable variation with $a_0 = 0$. Regular quasimartingales are the optional projections of continuous IV process.

Proof. By (3.4) in [DM82, Appendix II], $\text{Var} \leq \sigma_{\mathcal{D}}$. The opposite inequality follows from the expression $\text{Var}(v) = E\|Da\|_{TV}$ and the integration by parts

formula in Theorem 6. Given $v \in \mathcal{R}_{qm}^1$ and its decomposition in Theorem 6, we have $a = v - m \in \mathcal{R}^1$ since $\mathcal{R}_m^1 \subset \mathcal{R}_{qm}^1$. Thus, by [DM82, Remark VI.50(d)], ${}^p a = a_-$ while ${}^p a = a$ since a is predictable. Hence, a is continuous. The last claim follows from the fact that v is the optional projection of $\mathbb{1}m_T + a$. \square

3.3 Conjugates of integral functionals on \mathcal{N}^∞

By Theorem 5, \mathcal{N}^∞ may be identified with the Banach dual of the space $\mathcal{V}^1 := \mathbb{R}^d \times \mathcal{R}^1$ under the bilinear form

$$\langle x, v \rangle_{\mathcal{N}^\infty} = E \left[x_0 \cdot v_{-\infty} + \int v dx \right],$$

where we regard elements of \mathcal{V}^1 as regular processes on $\{-\infty\} \cup [0, T]$. Our main result, Theorem 4 below, gives an explicit expression for the conjugate of an integral functional on \mathcal{N}^∞ with respect to the above pairing.

Let h be a predictable convex normal integrand and define

$$f(x, \omega) = I_h(x, \omega) + k_0(x_0, \omega) + k_T(x_{T+}, \omega),$$

where k_0 and k_T are convex normal \mathcal{F}_0 - and \mathcal{F}_T -integrand on \mathbb{R}^d , respectively. Here all three terms define normal integrands, so f is a normal integrand as well; see [PP16, Lemma 24]. We assume throughout that Ek_0 and Ek_T are proper on $L^\infty(\mathbb{R}^d)$ and that Ek_0^* and Ek_T^* are proper on $L^1(\mathbb{R}^d)$.

Given $v \in \mathcal{R}_{qm}^1$ and its unique decomposition $v = m + a$ in Corollary 7, we denote $Dv := Da$ and continue to use the notation $dv/d\mu := d(Dv)^a/d\mu$ and $dv/d|Dv^s| := d(Dv)^s/d|(Dv)^s|$ from Section 2.2. We also define $\mathcal{V}_{qm}^1 := \mathbb{R}^d \times \mathcal{R}_{qm}^1$.

Theorem 8. *Let h satisfy the assumptions of Theorem 4. Then $Ef : \mathcal{N}^\infty \rightarrow \overline{\mathbb{R}}$ is closed and its conjugate can be expressed as*

$$(Ef)^*(v) = \begin{cases} E[J_{h^*}(-Dv) + k_0^*(v_{-\infty} - v_0) + k_T^*(v_T)] & \text{if } v \in \mathcal{V}_{qm}^1, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, $v \in \partial Ef(x)$ if and only if

$$\begin{aligned} -dv/d\mu &\in \partial h(x) \quad \mu\text{-a.e.}, \\ -dv/d|Dv^s| &\in \partial^s h(x) \quad |(Dv)^s|\text{-a.e.}, \\ v_{-\infty} - v_0 &\in \partial k_0(x_0), \\ v_T &\in \partial k_T(x_{T+}) \end{aligned}$$

almost surely.

Proof. By the assumptions of Theorem 4, there exist α and $r > 0$ such that $Ef(\bar{x} + x) \leq \alpha$ for all $x \in \mathcal{N}^\infty$ with $x_0 = x_{T+} = 0$ and $|x_t(\omega)| \leq r$ a.s.e. Thus,

$$\begin{aligned}
(Ef)^*(v) &= \sup_{x \in \mathcal{N}^\infty} \{\langle x, v \rangle - Ef(x)\} \\
&\geq \sup_{x \in \mathcal{N}^\infty} \{\langle \bar{x} + x, v \rangle - Ef(\bar{x} + x) \mid |x_t| \leq r, x_0 = x_{T+} = 0\} \\
&\geq \sup_{x \in \mathcal{N}^\infty} \{\langle \bar{x} + x, v \rangle - \alpha \mid |x_t| \leq r, x_0 = x_{T+} = 0\} \\
&\geq \langle \bar{x}, v \rangle - \alpha + \sup_{x \in \mathcal{N}^\infty} \{\langle Dx, v \rangle \mid |x_t| \leq r, x_0 = x_{T+} = 0\} \\
&= \langle \bar{x}, v \rangle - \alpha + r \operatorname{Var}(v),
\end{aligned}$$

where the last equality follows from Corollary 7. When $\operatorname{Var}(v) < \infty$, the integration by parts formula in Theorem 6 gives

$$\begin{aligned}
(Ef)^*(v) &= \sup_{x \in \mathcal{N}^\infty} E[\langle v, x \rangle - f(x)] \\
&= \sup_{x \in \mathcal{N}^\infty} E \left[v_T \cdot x_{T+} + (v_{-\infty} - v_0) \cdot x_0 - \int x da - f(x) \right] \\
&= \sup_{x \in \mathcal{N}^\infty} E \left[- \int x da - I_h(x) \right] + Ek_0^*(v_{-\infty} - v_0) + Ek_T^*(v_T).
\end{aligned}$$

Here the last equality follows by first noting that a and μ do not have atoms at the origin and that Ek_0 and Ek_T are proper so that we may take expectation separately from each term, and then by applying the interchange rule for minimization and expectation [RW98, Theorem 14.60]. It thus suffices to show that

$$\sup_{x \in \mathcal{N}^\infty} E \left[- \int x da - I_h(x) \right] = EJ_{h^*}(-Dv).$$

Using [HWY92, Theorem 5.15], there exists a predictable set A such that $E \int 1_{A^c} (dv/d\mu) d\mu = E \int 1_A d|(Dv)^s| = 0$. Defining $\bar{\mu} = |(Dv)^s| + \mu$ and

$$\bar{h}_t(x, \omega) = \begin{cases} h_t(x, \omega) + x \cdot (dv/d\mu)_t(\omega) & \text{if } (\omega, t) \in A, \\ \delta_{\operatorname{cl} \operatorname{dom} h_t(\omega)}(x) + x \cdot (dv/d|(Dv)^s|)_t(\omega) & \text{otherwise,} \end{cases}$$

we have, by the last assumption in Theorem 1, that

$$E \left[I_h(x) + \int x da \right] = E \int \bar{h}(x) d\bar{\mu}.$$

We have

$$\inf_{x \in \mathbb{R}^d} \bar{h}_t(x, \omega) = \begin{cases} -h_t^*(-(dv/d\mu)_t(\omega), \omega) & \text{if } (\omega, t) \in A, \\ -(h_t^*)^\infty(-(dv/d|(Dv)^s|)_t(\omega)) & \text{otherwise} \end{cases}$$

while it is straightforward to verify the assumptions in Theorem 4, so

$$\inf_{x \in \mathcal{N}^\infty} E \int \bar{h}(x) d\bar{\mu} = E \int \inf_{x \in \mathbb{R}^d} \bar{h}(x) d\bar{\mu} = -J_{h^*}(-Dv).$$

To prove the subgradient formula, let $x \in \text{dom } Ef$ and $v \in \mathcal{V}_{qm}^1$. By Fenchel inequality,

$$\begin{aligned} h(v) + h^*(-dv/d\mu) &\geq -x \cdot (dv/d\mu) \quad \mu\text{-a.e.}, \\ (h^*)^\infty(-dv/d|Dv^s|) &\geq -x \cdot (dv/d|Dv^s|) \quad |(Dv)^s|\text{-a.e.}, \\ k_0(x_0) + k_0^*(v_{-\infty} - v_0) &\geq x_0 \cdot (v_{-\infty} - v_0), \\ k_T(x_{T+}) + k_T^*(v_T) &\geq x_{T+} \cdot v_T \end{aligned}$$

almost surely. By the definition of a subgradient, $v \in \partial Ef(x)$ if and only if $Ef(x) + (Ef)^*(v) = \langle v, x \rangle$. Using the first part of the theorem and the integration parts formula in Theorem 6, we see that this is equivalent to having the above inequalities satisfied as equalities which in turn is equivalent to the stated pointwise subdifferential conditions. \square

We say that a normal integrand h is *integrable* if $h(x) \in \mathbb{L}^1$ for every constant process x . For real-valued and integrable h , Theorem 4 takes a simpler form.

Corollary 9. *If h is real-valued and integrable, then the conjugate of $Ef : \mathcal{N}^\infty \rightarrow \overline{\mathbb{R}}$ can be expressed as*

$$(Ef)^*(v) = \begin{cases} E [J_{h^*}(-Dv) + k_0^*(v_{-\infty} - v_0) + k_T^*(v_T)] & \text{if } v \in \mathcal{V}_{qm}^1 \text{ and } Dv \ll \mu, \\ +\infty & \text{otherwise} \end{cases}$$

and $v \in \partial Ef(x)$ if and only if $Dv \ll \mu$ and

$$\begin{aligned} -dv/d\mu &\in \partial h(x) \quad \mu\text{-a.e.}, \\ v_{-\infty} - v_0 &\in \partial k_0(x_0), \\ v_T &\in \partial k_T(x_{T+}) \end{aligned}$$

almost surely.

Proof. Inspection of the proof of [Roc71, Theorem 2] reveals that, integrability of h implies that for every $x \in \mathbb{L}^\infty$ and $r > 0$, there exists $\beta \in \mathbb{L}^1$ such that

$$h(x + x') \leq \beta \quad \forall x' \in \mathbb{B}_r.$$

Thus, since $\text{dom } h_t = \mathbb{R}^d$ for all t , f satisfies the assumptions of Theorem 8 and $\partial^s h_t = \{0\}$ for all t almost surely. \square

In the opposite extreme where, instead of a real-valued function, h is the indicator function of a random set, Theorem 8 takes the following form.

Corollary 10. *Let $h = \delta_S$, where S is a predictable closed convex-valued mapping, and assume that there exists an $\bar{x} \in \text{dom } Ef \cap \mathcal{N}^\infty$ with $\bar{x} \in \text{int } \mathbb{L}^\infty(S)$. Then the conjugate of $Ef : \mathcal{N}^\infty \rightarrow \mathbb{R}$ can be expressed as*

$$(Ef)^*(v) = \begin{cases} E \left[\int \sigma_S(-d(Dv)/d|Dv|)d|Dv| + k_0^*(v_{-\infty} - v_0) + k_T^*(v_T) \right] & \text{if } v \in \mathcal{V}_{qm}^1, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, $v \in \partial Ef(x)$ if and only if

$$\begin{aligned} -d(Dv)/d|Dv| &\in N_S(x) \text{ } |Dv|\text{-a.e.}, \\ v_{-\infty} - v_0 &\in \partial k_0(x_0), \\ v_T &\in \partial k_T(x_{T+}) \end{aligned}$$

almost surely.

Note that when S is the unit ball and $k_0 = k_{T+} = \delta_{\{0\}}$, Corollary 10 gives the expression $\text{Var} = \sigma_{\mathcal{D}}$ from Corollary 7.

Corollary 11. *If $h = 0$, then*

$$(Ef)^*(v) = \begin{cases} E [k_0^*(v_{-\infty} - v_0) + k_T^*(v_T)] & \text{if } v \in \mathcal{R}_m^1, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, $v \in \partial Ef(x)$ if and only if $v \in \mathcal{R}_m^1$ and

$$\begin{aligned} v_{-\infty} - v_0 &\in \partial k_0(x_0), \\ v_T &\in \partial k_T(x_{T+}) \end{aligned}$$

almost surely.

The Mackey topology on a locally convex vector space U , is the convex topology generated by the level sets of support functions of weakly-compact sets on the dual space Y . It is denoted by $\tau(U, Y)$. By the Mackey–Arens theorem, $\tau(U, Y)$ is the strongest locally convex topology under which every continuous linear functional can be represented as $u \mapsto \langle u, y \rangle$ for some $y \in Y$. A version of Alaoglu’s theorem states that if a convex function on U is Mackey continuous, then the level sets of its conjugate are $\sigma(Y, U)$ -compact; see [Roc74, Theorem 10b].

Theorem 12. *If h , k_0 and k_T are real-valued and integrable, then Ef is Mackey-continuous and $(Ef)^*$ has compact level-sets in \mathcal{V}^1 .*

Proof. By [Roc74, Theorem 22], EI_h is $\tau(\mathbb{L}^\infty, \mathbb{L}^1)$ -continuous on \mathbb{L}^∞ while Ek_0 and Ek_T are $\tau(L^\infty, L^1)$ -continuous on L^∞ . By Lemma 15, it suffices to show that the embedding i of \mathcal{N}^∞ is weakly continuous on \mathbb{L}^∞ and that $x \mapsto x_0$ and $x \mapsto x_{T+}$ are weakly continuous from \mathcal{N}^∞ to L^∞ .

Let $w \in \mathbb{L}^1$ and define $z \in L^1(C)$ by $z_t = \int_{[0,t]} w d\mu$. Integration by parts gives

$$\begin{aligned} \langle i(x), w \rangle_{\mathbb{L}^\infty} &= E \int x dz \\ &= E \left[x_{T+} \cdot z_T - \int z dx \right] \\ &= E \left[x_0 \cdot z_T + \int (z_T - z) dx \right] \\ &= E \left[x_0 \cdot E_0 z_T + \int {}^o(z_T - z) dx \right] \\ &= \langle x, v \rangle_{\mathcal{V}^1}, \end{aligned}$$

where $v = (E_0 z_T, {}^o(z_T - z)) \in \mathcal{V}^1$. Thus, the adjoint $i^* : \mathbb{L}^1 \rightarrow \mathcal{V}^1$ of i has full domain and is given by $i^*(w) = (E_0 z_T, {}^o(z_T - z))$. This implies the weak continuity of both i and i^* . The continuity of $x \mapsto x_T$ follows from

$$E(x_T \cdot z) = E \left(x_0 \cdot z + \int z dx \right) = E \left(x_0 \cdot E_0 z + \int {}^o z dx \right)$$

and that of $x \mapsto x_0$ from $E(x_0 \cdot z) = E(x_0 \cdot E_0 z)$. \square

We end this section by studying asymptotic properties of integral functionals and dense subsets of \mathcal{V}^1 . The following result gives an integral representation of the recession function of Ef ; see the appendix.

Theorem 13. *If $Ef : \mathcal{N}^\infty \rightarrow \overline{\mathbb{R}}$ is proper, then*

$$(Ef)^\infty(x) = E [I_{h^\infty}(x) + k_0^\infty(x_0) + k_T^\infty(x_{T+})].$$

Proof. Like in the proof of Theorem 12, we may view Ef as the sum of three functions, each one of which is the composition of a continuous linear mapping and an integral functional. The lower bound (1) implies that EI_h is lsc on \mathbb{L}^∞ , and the properness assumptions of Ek_0^* and Ek_T^* imply that Ek_0 and Ek_T are lsc on L^∞ . It now suffices to apply the last part of [PP16, Theorem 2] to the integral functionals and to use the general facts that the recession function of a sum/composition is the sum/composition of the recession functions whenever the sum/composition is proper. \square

Following [RW98, Section 3D], we say that a normal integrand h is *coercive* if $h^\infty = \delta_{\{0\}}$.

Corollary 14. *If h , k_0 and k_T are coercive, then $\text{dom}(Ef)^*$ is dense in \mathcal{V}^1 . In particular, $\{v \in \mathcal{V}_{qm}^1 \mid Dv \ll \mu\}$ is dense in \mathcal{V}^1 .*

Proof. By Theorem 13, Ef is coercive on \mathcal{N}^∞ . By [Roc66, Theorem 5.B], this is equivalent to $\text{dom}(Ef)^*$ being dense in \mathcal{V}^1 . The last claim follows e.g. by taking h , k_0 and k_T quadratic. \square

The above implies, in particular, that Itô processes are dense in \mathcal{R}^1 .

Remark 1. *Our results are easily specialized to functionals of the form*

$$Ef_0(x) := E[I_h(x) + k_{T+}(x_{T+})]$$

on the space $\mathcal{N}_0^\infty := \{x \in \mathcal{N}^\infty \mid x_0 = 0\}$. Indeed, we may pair \mathcal{N}_0^∞ with \mathcal{R}^1 via

$$\langle x, v \rangle = E \int v dx,$$

the weak topologies of \mathcal{N}_0^∞ and \mathcal{R}^1 are simply the relative topologies weak topologies when \mathcal{N}_0^∞ and \mathcal{R}^1 are viewed as subspaces of \mathcal{N}^∞ and \mathcal{V}^1 , respectively. Setting $k_0 = \delta_0$, we have

$$(Ef_0)^*(v) = (Ef)^*(0, v).$$

4 Appendix

This section recalls some basic definitions and facts from convex duality and the theory of integral functionals.

4.1 Duality

When U is in separating duality with another linear space Y , the *conjugate* of an extended real-valued convex function g on U is the extended real-valued function g^* on Y defined by

$$g^*(y) = \sup_{u \in U} \{ \langle u, y \rangle - g(u) \}.$$

A $y \in Y$ is a *subgradient* of g at u if

$$g(u') \geq g(u) + \langle u' - u, y \rangle \quad \forall u' \in U.$$

The set $\partial g(u)$ of all subgradients is known as the *subdifferential* of g at u . We often use the fact $y \in \partial g(u)$ if and only if

$$g(u) + g^*(y) = \langle u, y \rangle.$$

The *recession function* of a closed proper convex function g is defined by

$$g^\infty(u) = \sup_{\alpha > 0} \frac{g(\bar{u} + \alpha u) - g(\bar{u})}{\alpha},$$

where the supremum is independent of the choice of $\bar{u} \in \text{dom } g$; see [Roc66, Corollary 3C]. By [Roc66, Corollary 3D], $\delta_{\text{dom } g^*}^* = g^\infty$.

Lemma 15. *If $A : X \rightarrow U$ is a weakly continuous linear mapping with respect to the pairings of X with V and U with Y , then A is Mackey-continuous.*

Proof. If $O \subset \tau(U, Y)$ is a neighborhood of the origin, there exists a weakly compact $D \subset Y$ with $\text{lev}_1 \sigma_D \subset O$ so

$$A^{-1}(O) \supset \{x \in \mathcal{N}^\infty \mid \sigma_D(Ax) \leq 1\} = \{x \in \mathcal{N}^\infty \mid \sigma_{A^*D}(x) \leq 1\}.$$

Weak continuity of A implies that it has a weakly continuous adjoint A^* , so $A^*D \subset V$ is weakly compact. □

4.2 Integral functionals

Given a measurable space (Ξ, \mathcal{A}) and a locally convex topological vector space U , a set-valued mapping $S : \Xi \rightrightarrows U$ is *measurable* if the inverse image $S^{-1}(O) := \{\xi \in \Xi \mid S(\xi) \cap O \neq \emptyset\}$ of every open $O \subseteq U$ is in \mathcal{A} . An extended real-valued function $f : U \times \Xi \rightarrow \overline{\mathbb{R}}$ is said to be a *normal \mathcal{A} -integrand* on \mathbb{R}^d if the *epi-graphical mapping*

$$\xi \mapsto \text{epi } f(\cdot, \xi) = \{(u, \alpha) \in U \times \mathbb{R} \mid f(u, \xi) \leq \alpha\}$$

is closed-valued and measurable. A normal integrand f is said to be *convex* if $f(\cdot, \xi)$ is a convex function for every $\xi \in \Xi$. When U is a Suslin space as well as a countable union of Borel sets that are Polish in the relative topology, a normal integrand is $\mathcal{B}(U) \otimes \mathcal{A}$ -measurable (see [PP16]), so $\xi \mapsto f(u(\xi), \xi)$ is \mathcal{A} -measurable whenever $u : \Xi \rightarrow U$ is \mathcal{A} -measurable. Given a nonnegative measure η on (Ξ, \mathcal{A}) , the measurability implies that the associated *integral functional*

$$I_f(u) := \int_{\Xi} f(u(\xi), \xi) d\eta(\xi)$$

is a well-defined extended real-valued function on the space $L^0(\Xi, \mathcal{A}, \eta; U)$ of equivalence classes of U -valued \mathcal{A} -measurable functions. Here and in what follows, we define the integral of a measurable function as $+\infty$ unless the positive part of the function is integrable. The function I_f is called the *integral functional* associated with the normal integrand f . If f is a convex normal integrand, I_f is convex on $L^0(\Xi, \mathcal{A}, \eta; U)$.

Lemma 16 (Jensen's inequality). *Assume that h is an optional convex normal integrand, μ is an optional random measure and that*

$$h(x) \geq x \cdot v - \alpha$$

for some optional v and nonnegative α such that $\int |v| d\mu$ and $\int \alpha d\mu$ are integrable. Then

$$EI_h(x) \geq EI_h({}^o x)$$

for every bounded process x . If h , μ and v are predictable, then

$$EI_h(x) \geq EI_h({}^p x)$$

for every bounded process x .

Proof. We define $\hat{\mu} \ll \mu$ by $d\hat{\mu}/d\mu = \beta := {}^o(1/(1 + \int d\mu))$ so that $\hat{\mu}$ defines an optional bounded measure $\hat{\eta}(A) = E \int \mathbb{1}_A d\hat{\mu}$ on $\Omega \times [0, T]$. Moreover, $E I_h(x) = E \int \hat{h}(x) d\hat{\mu}$, where $\hat{h}(x) = h(x)/\beta$ is an optional convex normal integrand. We have

$$\hat{h}^*(v) = h^*(\beta v)/\beta,$$

so the lower bound implies that $E \int \hat{h}^*(v/\beta) d\hat{\mu}$ is finite. Thus we may apply the interchange of integration and minimization on $(\Omega \times [0, T], \mathcal{O}, \hat{\eta})$ and on $(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]), \hat{\eta})$ (see [RW98, Theorem 14.60]) to get

$$\begin{aligned} E I_h({}^o x) &= E \int \hat{h}({}^o x) d\hat{\mu} \\ &= \sup_{v \in \mathbb{L}^1(\Omega \times [0, T], \mathcal{O}, \hat{\eta})} E \int [{}^o x \cdot v - \hat{h}^*(v)] d\hat{\mu} \\ &= \sup_{v \in \mathbb{L}^1(\Omega \times [0, T], \mathcal{O}, \hat{\eta})} E \int [x \cdot v - \hat{h}^*(v)] d\hat{\mu} \\ &\leq \sup_{v \in \mathbb{L}^1(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]), \hat{\eta})} E \int [x \cdot v - \hat{h}^*(v)] d\hat{\mu} \\ &= E \int \hat{h}(x) d\hat{\mu} \\ &= E I_h(x). \end{aligned}$$

The predictable case is proved similarly. \square

4.3 Proof of Lemma 3

By [PP16, Lemma 22], it suffices to show that $I_h(\cdot, \omega)$ is lsc almost surely and that I_h is $\mathcal{F} \otimes \mathcal{B}(X)$ -measurable.

To show that $I_h(\cdot, \omega)$ is lsc almost surely, we denote

$$L_\omega^\infty := L^\infty([0, T], \mathcal{B}([0, T]), \mu(\omega); \mathbb{R}^d).$$

Since $\mu(\omega)$ is atomless, the embedding of $(X, \sigma(X, V))$ to $(L_\omega^\infty, \sigma(L_\omega^\infty, L_\omega^1))$ is continuous (see the proof of [PP14, Theorem 2.1]) while the lower bound implies that $I_h(\cdot, \omega)$ is $\sigma(L_\omega^\infty, L_\omega^1)$ -lsc, by [Roc76, Theorem 3C].

To prove the measurability, let S be the space of ladcag functions (left-continuous with right limits) from $[0, T]$ to \mathbb{R}^d . Equipped with the Skorokhod topology (with obvious changes of signs since we deal with left-continuous instead of right continuous functions) S is a Polish space; see [HWY92, Theorem 15.17]. Since every sequence converging in S converges pointwise outside a countable set², S satisfies the assumptions of [PP16, Theorem 25], so I_h is $\mathcal{B}(S) \otimes \mathcal{F}$ -measurable. It thus suffices to show that the injection from X to S is measurable.

²A ladcag function has at most a countable set of discontinuities, so this fact follows from the remark on page 452 in [HWY92]

By [Pes95, Theorem 3], $\mathcal{B}(S)$ coincides with the Borel- σ -algebra generated by the topology that S has when equipped with the supremum norm. By [DM82, Theorem VII.65], continuous linear functionals in the weak topology are of the form

$$l(u) := \int u_t da_t + \int u_{t+} da_t^+,$$

where a and a^+ are functions of bounded variation. When $u \in X$, integration by parts gives

$$l(u) = u_{T+}(a_{T+} + a_{T+}^+) - u_0(a_0 + a_0^+) - \int a_t du_t - \int a_{t-}^+ du_t.$$

Since every function of bounded variation is a pointwise limit of a sequence of continuous functions, it is not difficult to verify that l is measurable in $\mathcal{B}(X)$. \square

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