Optimal construction of a fund of funds*

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Abstract

We study the problem of diversifying a given initial capital over a finite number of investment funds that follow different trading strategies. The investment funds operate in a market where a finite number of underlying assets may be traded over finite discrete time. We present a numerical procedure for finding a diversification that is optimal in the sense of a given convex risk measure. The procedure is illustrated on an asset-liability management problem where the liabilities correspond to a pension insurance portfolio.

1 Introduction

There exist several instances of portfolio optimization problems where an optimal trading strategy can be characterized in terms of problem data. General dynamic portfolio optimization problems with trading restrictions, liabilities and general return/claim distributions, however, remain unsolved and in practical applications one often has to rely on approximations or heuristics. This is the case e.g. in pricing and hedging of some complex financial instruments as well as in asset-liability management (ALM) of insurance companies.

The main contribution of this paper is to describe a computational procedure for constructing good investment strategies out of a given set of candidate (basis) strategies. The procedure can effectively employ expert guesses of good strategies by adjusting their linear combination to the given objective and underlying probability distribution. The procedure combines simulations with large scale convex optimization and it can be efficiently implemented with modern

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solvers for convex optimization. The resulting strategy is easy to evaluate in simulations.

We illustrate the optimization process on a problem coming from the Finnish pension insurance industry. The liabilities are taken as the claim process associated with current claims portfolio of the private sector occupational pension system and the investment horizon is 82 years. The results reveal a significant improvement over a set of standard investment styles that are often recommended for long term investors. The optimization process is repeated with different model specifications in order to illustrate how the optimized strategy adapts to liabilities and to user-specified risk tolerances.

The rest of this paper is organized as follows. We begin by reviewing some well-known parametric investment strategies in Section 2. Section 3 states the optimization problem and Section 4 outlines the numerical procedure for its solution. The application to pension fund management is reported in Section 5. The market model used in the case study is described in the Appendix.

2 Basic investment strategies

We study dynamic trading over finite discrete time $t = 0, \dots, T$ from the perspective of an investor who has *initial capital* w_0 and liabilities characterized by their *claim process* $c = (c_t)_{t=1}^T$. Here c_t denotes the claim the investor has to pay at time t. The claim process c is allowed to be random and to take both positive and negative values so it can be used to model uncertain expenses as well as income.

The underlying financial market is modeled by a finite set J of securities that can be traded at every $t = 0, \dots, T$. The return on asset $j \in J$ over holding period $[t-1, t]$ will be denoted by $R_{t,j}$. The interpretation is that if $h_{t-1,j}$ units of cash is invested in asset $j \in J$ at time $t-1$, the investment will be worth $R_{t,j} h_{t-1,j}$ at time t. The return processes $R_j = (R_{t,j})_{t=1}^T$ are assumed to be positive but otherwise their joint distribution with the claim process c is arbitrary.

Several rules have been proposed for updating an investment portfolio in an uncertain dynamic environment. The simplest are the *buy and hold* (BH) strategies where an initial investment portfolio is held over time without updates. For nonzero claim process c, however, BH strategies may be infeasible. A natural modification is to liquidate each asset in the proportion of the initial investments to cover the claims. The resulting strategy consists of investing

$$h_{t,j} = \begin{cases} \pi_j w_0 & t = 0, \\ R_{t,j} h_{t-1,j} - \pi_j c_t & t = 1, \dots, T, \end{cases}$$

units of cash in asset $j \in J$ at the beginning of the holding period starting at time t. Here π_j is the proportion invested in asset $j \in J$ at time $t = 0$. Such strategies will be "self-financing" in the sense that they allow for paying out the claims without need for extra capital after time $t = 0$. If the claim process c is null, the BH strategy requires no transactions after time $t = 0$.

Another well-known strategy is the *fixed proportions* (FP) strategy where at each time and state the allocation is rebalanced into predetermined proportions given by a vector $\pi \in \mathbb{R}^J$ whose components sum up to one. In other words,

$$h_t = \pi w_t,$$

where for $t = 1, \dots, T$,

$$w_t = \sum_{j \in J} h_{t-1,j} R_{t,j} - c_t$$

is the *net wealth* of the investor at time t.

A *target date fund* (TDF) is a popular strategy in the pension industry (Bodie and Treussard (2007)). In a TDF, the proportion invested in risky assets is decreased as retirement date approaches. In our multi-asset setting we implement TDFs by adjusting the allocation between two complementary subsets J^r and J^s of the set of all assets J. Here J^s consists of "safe" assets and J^r consists of the rest (the "risky" assets). In a TDF, the *proportional exposure* to J^r at time t is given by

$$e_t = a - bt.$$

The parameter a gives the initial proportion invested in J^r and b specifies how fast the proportion is decreased with time. Nonnegativity of the exposure in the risky assets can be guaranteed by choosing a and b so that

$$a \geq 0 \quad \text{and} \quad a - bT \geq 0.$$

A TDF can be written as

$$h_t = \pi_t w_t$$

where the vector π_t is dynamically adjusted to give the specified proportional exposure:

$$\sum_{j \in J^r} \pi_{t,j} = e_t.$$

To complete the definition, one has to determine how the wealth is allocated within J^r and J^s. We do this according to FP rules.

One of the best known strategies is the *constant proportion portfolio insurance* (CPPI) strategy; see e.g. Black and Jones (1987), Black and Perold (1992) and Perold and Sharpe (1995). In a CPPI, the proportional exposure to the risky assets is given by

$$\begin{aligned} e_t &= \frac{m}{w_t} \max\{w_t - F_t, 0\} \\ &= m \max\{1 - \frac{F_t}{w_t}, 0\}, \end{aligned}$$

where the "floor" F_t represents the value of outstanding claims at time t and the parameter $m \geq 0$ determines the fraction of the "cushion" (wealth over the floor) invested in risky assets. One can limit the maximum proportional exposure to a given upper bound l by defining the exposure as

$$e_t = \min\{m \max\{1 - \frac{F_t}{w_t}, 0\}, l\}.$$

3 The optimization problem

We propose to diversify a given initial capital w_0 among a finite number of investment strategies in order to acchieve a return distribution that better suits the liabilities and risk preferences of the investor. As long as the individual strategies cover the given liabilities (see the previous section) so will the overall strategy obtained with diversification. One is then free to search for an optimal diversification. Appropriately diversifying among parametric classes of investment strategies one may be able to produce new superior strategies which do not belong to the original parametric classes; see Section 5.3.

The problem of diversifying among a finite set $\{h^i \mid i \in I\}$ of strategies can be written as

$$\underset{\alpha \in X}{\text{minimize}} \quad \rho(\sum_{i \in I} \alpha^i w_T^i),$$

where w_T^i is the terminal wealth obtained by following strategy $i \in I$ when starting with initial capital w_0,

$$X = \{\alpha \in \mathbb{R}_+^I \mid \sum_{i \in I} \alpha^i = 1\}$$

and ρ is a *convex risk measure* that quantifies the preferences of the decision maker over random terminal wealth distributions; see e.g. Föllmer and Schied (2004) or Rockafellar (2007).

Several choices of ρ may be considered. We will concentrate on the Conditional Value at Risk ($CV@R$) which is particularly convenient in the optimization context. According to Rockafellar and Uryasev (2000), $CV@R_\delta$ at confidence level δ of a random variable w can be expressed as

$$CV@R_\delta(w) = \inf_\gamma E\left[\frac{1}{1-\delta}\max\{\gamma - w, 0\} - \gamma\right].$$

Moreover, the minimum over γ is achieved by Value at Risk at confidence level δ. The problem of optimal diversification with respect to $CV@R_\delta$ can be written as

$$\underset{\alpha \in X, \gamma}{\text{minimize}} \quad E\left[\frac{1}{1-\delta}\max\{\gamma - \sum_{i \in I} \alpha^i w_T^i, 0\} - \gamma\right]. \tag{1}$$

The problem thus becomes that of minimizing a convex expectation function over a finite number of variables. Mathematically, it is close to the classical problem of maximizing the expected utility in a one period setting and, consequently, similar techniques can be applied for its solution; see e.g. Sharpe (2007).

4 Numerical procedure

In order to solve (1), we will first make a quadrature approximation of the objective; see Pennanen and Koivu (2005), Koivu and Pennanen (2009). That is, we generate a finite number N of return and claim scenarios (R^k, c^k), $k = 1, \dots, N$ over the planning horizon $t = 0, \dots, T$ and approximate the expectation by

$$\frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{1-\delta}\max\{\gamma - \sum_{i\in I}\alpha^i w_T^{i,k}, 0\} - \gamma\right],$$

where $w_T^{i,k}$ is the terminal wealth along scenario k obtained with strategy h^i. Here and in what follows, R^k denotes a realization of the $|J|$-dimensional process $(R_t)_{t=1}^T$ where $R_t = (R_{t,j})_{j\in J}$. The computation of $w_T^{i,k}$ is straightforward: given a realization (R^k, c^k) and a strategy h^i, the corresponding wealth process $w^{i,k} = (w_t^{i,k})_{t=0}^T$ is given recursively by

$$w_t^{i,k} = \begin{cases} w_0 & \text{for } t = 0, \\ \sum_{j\in J} R_{t,j}^k h_{t-1,j}^{i,k} - c_t^k & \text{for } t > 0. \end{cases}$$

Algorithmically, the solution procedure can be summarized as follows.

1. Generate N scenarios of asset returns R_t and claims c_t over $t = 1, \dots, T$.
2. Evaluate each basis strategy $i \in I$ along each of the scenarios $k = 1, \dots, N$ and record the corresponding terminal wealth $w_T^{i,k}$.
3. Solve the optimization problem

$$\underset{\alpha\in X,\gamma}{\text{minimize}} \quad \frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{1-\delta}\max\{\gamma - \sum_{i\in I}\alpha^i w_T^{i,k}, 0\} - \gamma\right] \tag{2}$$

   for the optimal diversification weights α^i.

There are several possibilities for solving (2). We follow Rockafellar and Uryasev (2000) and reformulate (2) as the linear programming problem

$$\begin{aligned} \underset{\alpha\in\mathbb{R}^I,\gamma\in\mathbb{R},s\in\mathbb{R}^N}{\text{minimize}} \quad & \frac{1}{N}\sum_{k=1}^{N}\left(\frac{1}{1-\delta}s^k - \gamma\right) \\ \text{subject to} \quad & s^k \geq \gamma - \sum_{i\in I}\alpha^i w_T^{i,k} \quad k = 1, \dots, N, \\ & \sum_{i\in I}\alpha^i = 1, \\ & \alpha^i, s^k \geq 0. \end{aligned}$$

This LP has $|I| + N + 1$ variables, where $|I|$ is the number of funds and N is the number of scenarios in the quadrature approximation of the expectation. Modern commercial solvers are able to solve LP problems with millions of variables and constraints.

5 Case study: pension fund management

Consider a closed pension fund whose aim is to cover its accrued pension liabilities with given initial capital. The pension claims are of the defined benefit type and they depend on the wage and consumer price indices. According to the current Finnish mortality tables, all the liabilities will be amortized in 82 years. The following section describes the stochastic return/claim process $(R, c) = (R_t, c_t)_{t=1}^T$ and Section 5.2 lists the basic strategies that will be used in the numerical study in Section 5.3.

5.1 Assets and liabilities

The set J of primitive assets consists of

1. Euro area money market,
2. Euro area government bonds,
3. Euro area equity,
4. US equity,
5. Euro area real estate.

These are the assets in which the individual funds described in Section 2 invest. The above asset classes may also be viewed as investment funds themselves. For the money market fund, the return over a holding period of Δt is determined by the short rate Y_1,

$$R_{t,1} = e^{\Delta t Y_{t-1,1}},$$

The short rate will be modeled as a strictly positive stochastic process which will imply that $R_1 > 1$. The return of the government bond fund will be given by the formula

$$R_{t,2} = \Delta t Y_{t-1,2} + \left(\frac{1 + Y_{t,2}}{1 + Y_{t-1,2}} \right)^{-D},$$

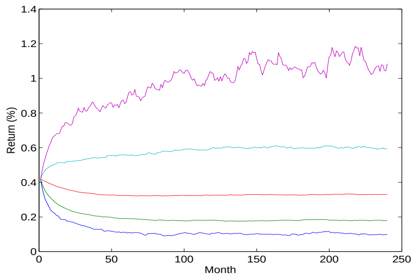
where $Y_{t,2}$ is the average yield to maturity of the bond fund at time t and D is the modified duration of the fund. The total returns of the equity and real estate funds are given in terms of their total return indices S_j,

$$R_{t,j} = \frac{S_{t,j}}{S_{t-1,j}}, \quad j = 3, 4, 5.$$

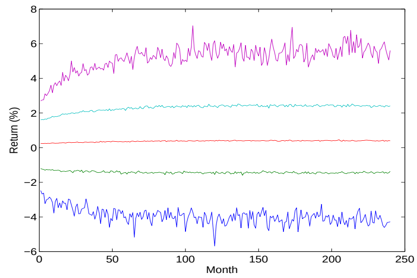
The pension fund's liabilities consist of the accrued benefits of the plan members. The population of the pension plan is distributed into different cohorts based on the members' age and gender. The fraction of retirees in each cohort increases with age and reaches 100% by the age of 68. The youngest cohort is 18 years of age and all the members are assumed to die by the age of 100. The defined benefit pensions depend on stochastic wage and consumer price indices.

We will model the evolution of the short rate, the bond yield, the total return indices of equities as well as the wage and consumer price indices with a Vector Equilibrium Correction-model (Engle and Granger (1987)) augmented with GARCH innovations. A detailed description of the model together with the estimated model parameters is given in the Appendix.

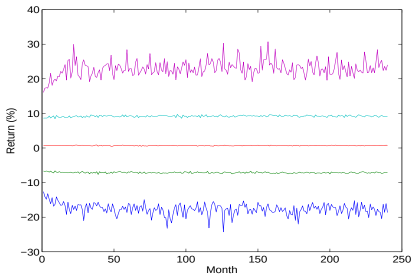
Figure 1 displays the 0.1%, 5%, 50% (median), 95% and the 99.9% percentiles of the simulated asset return distributions over the first twenty years of the 82 year investment horizon. Figure 2 displays the development of the median and the 95% confidence interval of the yearly pension claims over the 82 year horizon.



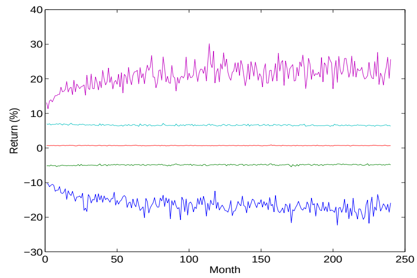
(a) Money market fund



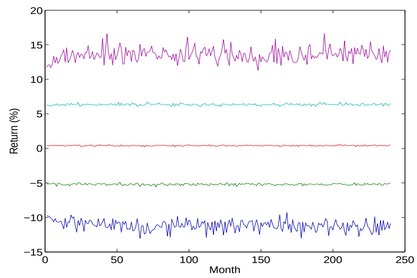
(b) Bond fund



(c) Euro area equity fund



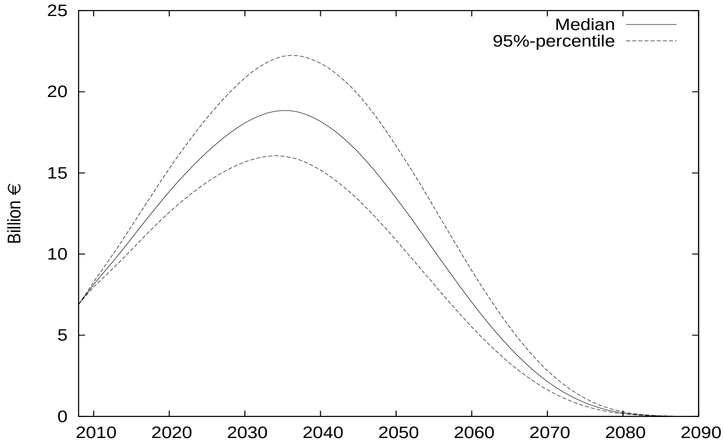
(d) US equity fund



(e) Euro area real estate fund

Figure 1: Evolution of the 0.1%, 5%, 50%, 95% and 99.9% percentiles of monthly asset return distributions over twenty years.

Figure 2: Median and 95% confidence interval of the projected pension expenditure c over the 82 year horizon.



5.2 The investment funds

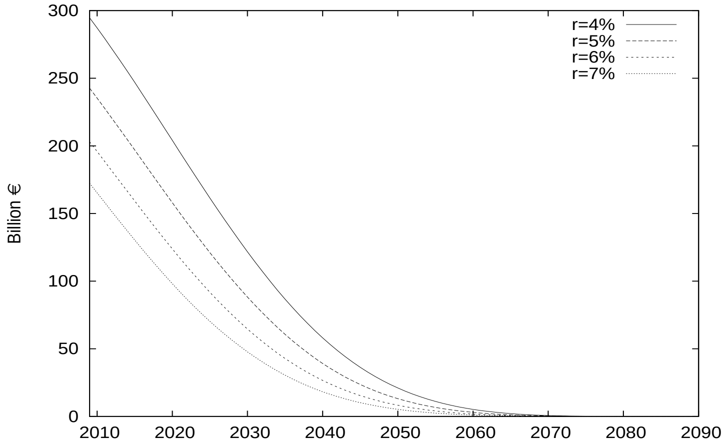
We will diversify a given initial capital among different investment funds as described in Section 3. The considered funds follow the trading rules listed in Section 2 with varying parameters. The set J^s of "safe assets" consists of the money market and bond investments.

We take five buy and hold strategies each of which invest all in a single asset. More general BH strategies can be generated by diversifying among such simple BH strategies. We use 11 FP strategies with varying parameters π. In TDF and CPPI strategies, we always use fixed proportion allocations within the safe assets J^s and the risky assets J^r. We use 20 TDF strategies with varying values for a and b. In the case of CPPI strategies, we define the floor through

$$\begin{aligned} F_T &= 0, \\ F_t &= (1+r)F_{t-1} - \bar{c}_t \quad t = 0, \dots, T, \end{aligned}$$

where r is a deterministic discount factor and $\bar{c}_t$ is the median of claim amount at time t; see Figure 3. This corresponds to the traditional actuarial definition of "technical reserves" for an insurance portfolio. We generate 40 CPPI strategies with varying values for the multiplier m and the discount factor r in the definition of the floor.

Figure 3: Development of the floor F with different discount factors r over the 82 year horizon.



5.3 Results

We computed an optimal diversification over the above funds assuming an initial capital of 225 billion euros and a confidence level δ of 97.5% in the definition of the optimization problem (2). We constructed the corresponding linear programming problem with 20000 scenarios as described in Section 4. The resulting LP consisted of 20072 variables and 20001 constraints. The LP was solved with MOSEK interior point solver and AMD 3GHz processor in approximately 30 seconds.

The optimal solution is given in Table 1 with the characteristics of the funds in the optimal diversification. The $CV@R_{97.5\%}$ of the optimally constructed fund of funds is 251. The last column of Table 1 gives the $CV@R$ numbers obtained with the individual funds in the optimal fund of funds. The constructed fund of funds clearly improves upon them. The best $CV@R_{97.5\%}$ value among all individual funds is 1020, which means that the best individual fund is roughly 300% riskier than the optimal diversification. Surprisingly, this fund is not included in the optimal fund of funds. All the $CV@R$-values were computed on an independent set of 100000 scenarios.

The optimal allocation in terms of the primitive assets at time $t = 0$ is given in Figure 4. Figure 5(a) gives the proportion of risky assets at the beginning of year 20 as a function of total wealth. While the level of wealth clearly affects the optimal portfolio, there is significant variation that is not explained by it. This just illustrates the fact that the optimal strategy may be a complicated function of the underlying risk factors. For comparison, Figure 5(b) plots an analogous "exposure diagram" for the CPPI-strategy that got the highest weight in the optimal fund of funds. In a CPPI strategy, the exposure to the risky assets is a well-defined function of the level of wealth.

Table 1: Optimized fund of funds. The first column gives the optimal weights of the funds in the optimized fund of funds. The second and the third columns give type and parameters, respectively, of the corresponding strategy in the notation of Sections 2 and 5.3. The last column gives the $CV@R_{97.5\%}$ for each individual fund in billions of euros.

Weight (%)	Type	Parameters	$CV@R_{97.5\%}$ (billion €)
66.5	BH	Bonds	1569
2.9	BH	Euro Equity	6567
10.4	BH	US Equity	5041
2.2	FP	$m = 0.8$	3324
3.9	CPPI	$m = 1, l = 100\%, r = 4\%$	1420
9.9	CPPI	$m = 2, l = 100\%, r = 4\%$	1907
4.2	CPPI	$m = 2, l = 100\%, r = 5\%$	2417

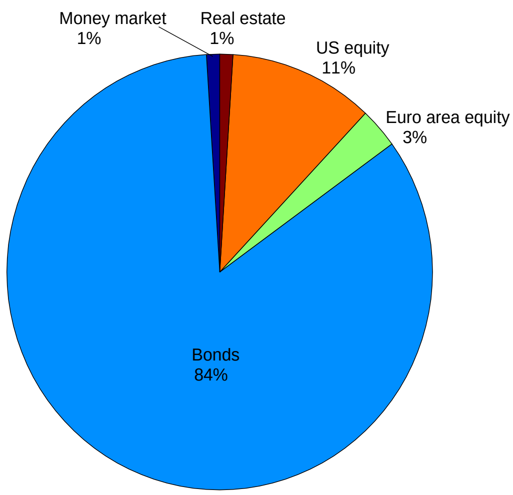
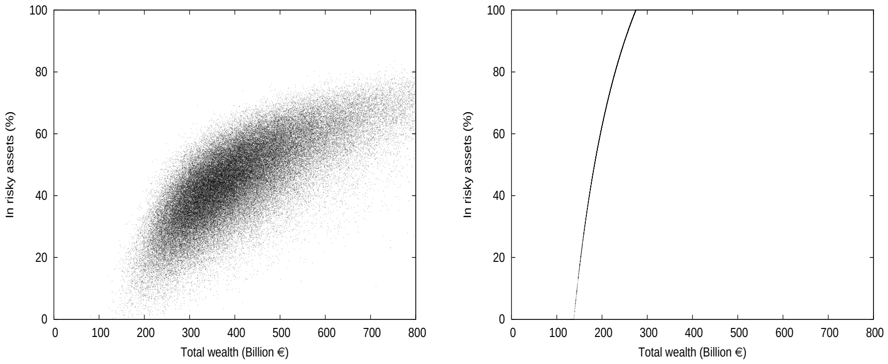


Figure 4: Optimal allocation in terms of the primitive assets at time $t = 0$.

To gain insight on how risk preferences affect the optimal asset allocation, we computed the optimal fund of funds for five different confidence levels $\delta \in \{80\%, 90\%, 95\%, 97.5\%, 99\%\}$. The optimal allocations in terms of the primitive assets at time $t = 0$ are given in Figure 6(a). Expectedly, the proportion of equities increases as the confidence level (i.e. the "risk aversion") is lowered.

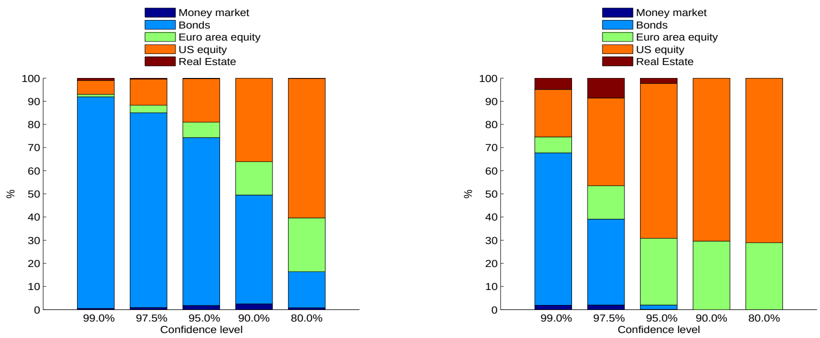
To illustrate the effect of liabilities on the optimal asset allocation we re-



(a) Optimized strategy (b) CPPI

Figure 5: Proportion of the “risky assets” in year $t = 20$ as a function of total wealth. Figure (a) depicts the optimized strategy while (b) corresponds to the CPPI strategy with the largest weight in the optimal fund of funds.

computed the optimal diversifications without liabilities. Accordingly, the basis strategies were constructed by setting $(c_t)_{t=1}^T = 0$ in the specifications of Section 2. The optimal allocations in terms of the primitive assets at time $t = 0$ are given in Figure 6(b). Comparing the optimal allocations in Figures 6(a) and 6(b), it is clear that the liabilities play an essential role in the determination of an optimal allocation. In particular, for a highly risk averse investor, bonds seem to provide the best hedge for the long term liabilities among the considered asset classes.



(a) Asset weights with liabilities (b) Asset weights without liabilities ($c = 0$)

Figure 6: Optimal initial allocations in the primitive assets with varying confidence levels δ.

6 Conclusions

This paper applied the strategic optimization technique developed in Koivu and Pennanen (2009) to a long term asset liability management (ALM) problem. The ALM problem was formulated as that of diversifying a given initial capital optimally over a finite number of investment funds with varying investment styles. The funds follow given parametric investment strategies that are constructed so that they allow for required claim payments. The optimality criterion was taken to be the Conditional Value at Risk with a given confidence level but other choices of risk measures can be used as well.

The proposed optimization framework was applied to an asset liability management problem coming from pension insurance industry. The optimized fund of funds outperformed the best individual investment strategy by a wide margin. The promising results open ample possibilities for future research. An interesting application would be to use the approach in pricing of insurance liabilities. The ability to adjust the hedging strategy to given insurance portfolio and risk preferences is essential in incomplete markets; see Hilli et al. (2009). The approach can also be extended to accommodate for trading restrictions and liquidity costs.

A The time series model

As described in Section 5.1, the returns on the primitive assets as well as the pension claims are expressed in terms of seven economic factors; short term (money market) interest rate (Y_1), euro area government bond yield (Y_2), euro area total return equity index (S_3), US total return equity index S_4, euro area total return real estate index (S_5), Finnish wage index (W) and euro area consumer price index (C). We will model the evolution of the stochastic factors with a Vector Equilibrium Correction-model (Engle and Granger (1987)) augmented with GARCH innovations. To guarantee the positivity of the processes Y_1, Y_2, S_3, S_4, S_5, W and C we will model their natural logarithms as real-valued stochastics processes. More precisely, we will assume that the 7-dimensional process

$$\xi_t = \begin{bmatrix} \ln Y_{t,1} \\ \ln Y_{t,2} \\ \ln S_{t,3} \\ \ln S_{t,4} \\ \ln S_{t,5} \\ \ln W_t \\ \ln C_t \end{bmatrix}$$

follows a VEqC-GARCH process

$$\Delta\xi_t - \delta = \mu_t + \sigma_t \varepsilon_t, \tag{3}$$

where

$$\mu_t = A(\Delta\xi_{t-1} - \delta) + \alpha(\beta^T\xi_{t-1} - \gamma) \tag{4}$$

and

$$\sigma_t^2 = C\sigma_{t-1}\varepsilon_{t-1}(C\sigma_{t-1}\varepsilon_{t-1})^T + D\sigma_{t-l}^2 D^T + \Omega. \tag{5}$$

The matrix A in (4) captures the autoregressive behavior of the time series, the second term takes into account the long-term behavior of ξ_t around statistical equilibria described by the system of linear equations $\beta^T\xi = \gamma$, where β is a matrix and γ is a vector of appropriate dimensions. The vector δ gives the long term average drift of the process ξ. The instantaneous volatility matrix σ_t is modelled by the multivariate GARCH-model (5), where C, D and Ω are parameter matrices of appropriate dimension.

In its most general form, the above model has a very high number of free parameters. To simplify the estimation procedure and to obtain a parsimonious model, we will assume that the matrices A, C and D are diagonal and fix the matrix β as

$$\beta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

The specification of the matrix β implies that the government bond yield and the spread between the bond yield and the short rate are mean reverting processes.

We take the parameter vectors δ and γ as *user specified parameters* and set their values to

$$\delta = 10^{-3} \begin{bmatrix} 0 & 0 & 7.5 & 7.5 & 5.0 & 2.0 & 3.0 \end{bmatrix}^T,$$

$$\gamma = \begin{bmatrix} \ln(5) \\ \ln(5/4) \end{bmatrix}.$$

The vector δ specifies the long term *median* values of the equity and real estate returns as well as the growth rates of consumer prices and wages. On the other hand, the vector γ specifies the long term median values of the government bond yield, the spread between the bond yield and short rate. The chosen value of γ implies that the median values of the short rate $Y_{t,1}$ and the bond yield $Y_{t,2}$ will equal 4 and 5, respectively.

The remaining parameters were estimated using monthly data between January 1991 and July 2008 by applying an estimation procedure where all insignificant parameters were deleted one by one until all remaining parameters were significant at a 5% confidence level. The data used in estimation is summarized in Table 2 and the estimated parameter matrices are given below.

$$A = 10^{-2} \begin{bmatrix} 41.995 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 14.807 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 96.233 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 93.422 \end{bmatrix}$$

Table 2: Data series used in the estimation

Stochastic factor	Historical time series
Y_1	Three month EURIBOR (FIBOR prior to EURIBOR)
Y_2	Yield of a German government bond portfolio with an average modified duration of five years
S_3	MSCI Euro area total return equity index
S_4	MSCI US total return equity index
S_5	EPRA/NAREIT Eurozone total return real estate index
W	Seasonally adjusted Finnish wage index (Statistics Finland)
C	Seasonally adjusted Eurozone consumer price index (Eurostat)

$$\alpha = 10^{-2}\begin{bmatrix} 0 & -2.119 & 0 & 0 & 0 & 0 & 0 \\ 1.514 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$C = 10^{-2}\begin{bmatrix} 25.788 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 29.816 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 41.952 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 38.588 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 28.071 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 31.8125 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D = 10^{-2}\begin{bmatrix} 88.301 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 91.236 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 86.412 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 91.373 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 94.117 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 81.056 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Omega = 10^{-6}\begin{bmatrix} 202.241 & 71.004 & -0.460 & 0.723 & -1.622 & -0.015 & -0.105 \\ 71.004 & 170.507 & 30.889 & 9.200 & -3.682 & 0.134 & -0.277 \\ -0.460 & 30.889 & 202.430 & 53.547 & 54.036 & 0.021 & 0.199 \\ 0.723 & 9.200 & 53.547 & 25.330 & 14.050 & 0.003 & 0.021 \\ -1.622 & -3.682 & 54.036 & 14.050 & 44.769 & -0.094 & 0.179 \\ -0.015 & 0.134 & 0.021 & 0.003 & -0.094 & 0.010 & 0.019 \\ -0.105 & -0.277 & 0.199 & 0.021 & 0.179 & 0.019 & 0.198 \end{bmatrix}.$$

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