Duality and optimality conditions in stochastic optimization and mathematical finance

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Abstract

This article studies convex duality in stochastic optimization over finite discrete-time. The first part of the paper gives general conditions that yield explicit expressions for the dual objective in many applications in operations research and mathematical finance. The second part derives optimality conditions by combining general saddle-point conditions from convex duality with the dual representations obtained in the first part of the paper. Several applications to stochastic optimization and mathematical finance are given.

Key words. Stochastic optimization, convex duality, optimality conditions

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t=0}^T$ of sub σ-algebras of $\mathcal{F}$ and consider the dynamic stochastic optimization problem

$$\text{minimize} \quad Ef(x,u) := \int f(x(\omega), u(\omega), \omega) dP(\omega) \quad \text{over } x \in \mathcal{N} \qquad (\mathcal{P}_u)$$

parameterized by a measurable function $u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^m)$. Here and in what follows,

$$\mathcal{N} := \{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^{n_t})\},$$

for given integers n_t and f is *convex normal integrand* on $\mathbb{R}^n \times \mathbb{R}^m \times \Omega$, where $n := n_0 + \ldots + n_T$ (that is, the epigraphical mapping $\omega \mapsto \{(x,u,\alpha) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mid f(x,u,\omega) \leq \alpha\}$ is closed convex-valued and measurable; see e.g. [25, Chapter 14]). The variable $x \in \mathcal{N}$ is interpreted as a decision strategy where x_t

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is the decision taken at time t. Throughout this paper, we define the expectation of a measurable function ϕ as $+\infty$ unless the positive part ϕ^+ is integrable (In particular, the sum of extended real numbers is defined as $+\infty$ if any of the terms equals $+\infty$). Since, by [25, Corollary 14.34], normality of f implies that it is $\mathcal{B}(\mathbb{R}^n \times \mathbb{R}^m) \otimes \mathcal{F}$-measurable, the function Ef is thus well-defined extended real-valued function on $\mathcal{N} \times L^0(\Omega, \mathcal{F}, P; \mathbb{R}^m)$. We will assume throughout that the function $f(\cdot, \cdot, \omega)$ is *proper*, *lower semicontinuous* and *convex* for every $\omega \in \Omega$.

It was shown in [10] that, when applied to $(\mathcal{P}_u)$, the conjugate duality framework of Rockafellar [20] allows for a unified treatment of many well known duality frameworks in stochastic optimization and mathematical finance. An important step in the analysis is to derive dual expressions for the optimal value function

$$\varphi(u) := \inf_{x \in \mathcal{N}} Ef(x, u)$$

over an appropriate subspace of L^0. Since φ is convex (see e.g. [20, Theorem 1]), the absence of a duality gap is equivalent to the closedness of the value function. Pennanen and Perkkiö [13] and more recently Perkkiö [15] gave conditions that guarantee that φ is closed and that the optimum in $(\mathcal{P}_u)$ is attained for every $u \in L^0$. The given conditions provide far reaching generalizations of well-known no-arbitrage conditions used in financial mathematics.

The present paper makes two contributions to the duality theory for $(\mathcal{P}_u)$. First, we extend the general duality framework of [10] by allowing more general dualizing parameters and by relaxing the time-separability property of the Lagrangian. We show that, under suitable conditions, the expression in [10, Theorem 2.2] is still valid in this extended setting. This also provides a correction to [10, Theorem 2.2] which omitted certain integrability conditions that are needed in general; see [11]. Second, we give optimality conditions for the optimal solutions of $(\mathcal{P}_u)$. Again, we follow the general conjugate duality framework of [20] by specializing the saddle-point conditions to the present setting. The main difficulty here is that, in general, the space $\mathcal{N}$ does not have a proper topological dual so we cannot write the generalized Karush-Kuhn-Tucker condition in terms of subgradients. Nevertheless, the dual representations obtained in the first part of the paper allow us to write the saddle-point conditions more explicitly in many interesting applications.

In the case of perfectly liquid financial markets, we recover well-known optimality conditions in terms of martingale measures. For Kabanov's currency market model with transaction costs [5], we obtain optimality conditions in terms of dual variables that extend the notion of a "consistent price system" to possibly nonconical market models. We treat problems of convex optimal control under the generalized framework of Bolza much as in [24]. Our formulation and its embedding in the conjugate duality framework of [20] is slightly different from that in [24], however, so direct comparisons are not possible. Our formulation is motivated by applications in mathematical finance. In particular, the optimality conditions for the currency market model are derived by specializing those obtained for the problem of Bolza.

2 Duality

From now on, we will assume that the parameter u belongs to a decomposable space $\mathcal{U} \subset L^0$ which is in separating duality with another decomposable space $\mathcal{Y} \subset L^0$ under the bilinear form

$$\langle u, y \rangle = E(u \cdot y).$$

Recall that $\mathcal{U}$ is *decomposable* if

$$\mathbb{1}_A u + \mathbb{1}_{\Omega \setminus A} u' \in \mathcal{U}$$

whenever $A \in \mathcal{F}$, $u \in \mathcal{U}$ and $u' \in L^\infty$; see e.g. [21]. Examples of such dual pairs include the Lebesgue spaces $\mathcal{U} = L^p$ and $\mathcal{Y} = L^q$ and decomposable pairs of Orlicz spaces; see Section 3. The *conjugate* of $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$ is the extended real-valued convex function on $\mathcal{Y}$ defined by

$$\varphi^*(y) = \sup_{u \in \mathcal{U}} \{\langle u, y \rangle - \varphi(u)\}.$$

Here and in what follows, $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$. If φ is closed[1] with respect to the weak topology induced on $\mathcal{U}$ by $\mathcal{Y}$, the biconjugate theorem (see e.g. [20, Theorem 5]) gives the *dual representation*

$$\varphi(u) = \sup_{y \in \mathcal{Y}} \{\langle u, y \rangle - \varphi^*(y)\}.$$

This simple identity is behind many well known duality relations in operations research and mathematical finance. It was shown in [13] that appropriate generalizations of the *no-arbitrage* condition from mathematical finance guarantee the closedness of φ. Recently, the conditions were extended in [15] to allow for more general objectives.

In general, it may be difficult to derive more explicit expressions for φ^*. Following [20], one can always write the conjugate as

$$\varphi^*(y) = -\inf_{x \in \mathcal{N}} L(x, y),$$

where the *Lagrangian* $L : \mathcal{N} \times \mathcal{Y} \to \overline{\mathbb{R}}$ is defined by

$$L(x, y) = \inf_{u \in \mathcal{U}} \{Ef(x, u) - \langle u, y \rangle\}.$$

We will show that, under appropriate conditions, the second infimum in

$$\varphi^*(y) = -\inf_{x \in \mathcal{N}} \inf_{u \in \mathcal{U}} \{Ef(x, u) - \langle u, y \rangle\}$$

can be taken scenariowise while the first infimum may be restricted to the space $\mathcal{N}^\infty$ of essentially bounded strategies. Both infima are easily calculated in many interesting applications; see [10] and the examples below.

[1] Recall that a convex function is *closed* if it is lower semicontinuous and either proper or a constant. A function is *proper* if it never takes the value $-\infty$ and it is finite at some point.

Accordingly, we define the *Lagrangian integrand* on $\mathbb{R}^n \times \mathbb{R}^m \times \Omega$ by

$$l(x,y,\omega) := \inf_{u\in\mathbb{R}^m} \{f(x,u,\omega) - u\cdot y\}.$$

We will also need the pointwise conjugate of f:

$$\begin{aligned} f^*(v,y,\omega) :&= \sup_{x\in\mathbb{R}^n, u\in\mathbb{R}^m} \{x\cdot v + u\cdot y - f(x,u,\omega)\} \\ &= \sup_{x\in\mathbb{R}^n} \{x\cdot v - l(x,y,\omega)\}. \end{aligned}$$

By [25, Theorem 14.50], the pointwise conjugate of a normal integrand is also a normal integrand. Clearly, l is upper semicontinuous in the second argument since it is the pointwise infimum of continuous functions of y. Similarly,

$$\underline{l}(x,y,\omega) := \sup_{v\in\mathbb{R}^n} \{x\cdot v - f^*(v,y,\omega)\}$$

is lower semicontinuous in the first argument. In fact, $\underline{l}(\cdot,y,\omega)$ is the biconjugate of $l(\cdot,y,\omega)$ while $-l(x,\cdot,\omega)$ is the biconjugate of $-\underline{l}(x,\cdot,\omega)$; see [19, Theorem 34.2]. The function $(x,\omega)\mapsto \underline{l}(x,y(\omega),\omega)$ is a normal integrand for any $y\in\mathcal{Y}$ while $(y,\omega)\mapsto -l(x(\omega),y,\omega)$ is a normal integrand for any $x\in\mathcal{N}$ and thus, the integral functionals $E\underline{l}$ and El are well-defined on $\mathcal{N}\times\mathcal{Y}$ (recall our convention of defining an integral as $+\infty$ unless the positive part of the integrand is integrable). Indeed, we have $l(x(\omega),y,\omega) = -h^*(y,\omega)$ for $h(u,\omega) := f(x(\omega),u,\omega)$, where h is a normal integrand, by [25, Proposition 14.45(c)]. Similarly for $\underline{l}$.

Restricting strategies to the space $\mathcal{N}^\infty \subset \mathcal{N}$ of essentially bounded strategies gives rise to the auxiliary value function

$$\tilde{\varphi}(u) = \inf_{x\in\mathcal{N}^\infty} Ef(x,u).$$

Under the conditions of Theorem 2 below, the conjugates of $\tilde{\varphi}$ and φ coincide, or in other words, closures of $\tilde{\varphi}$ and φ are equal. The following lemma from [15] will play an important role. We denote

$$\mathcal{N}^\perp := \{v\in L^1(\Omega,\mathcal{F},\mathbb{R}^n) \,|\, E(x\cdot v) = 0\ \forall x\in\mathcal{N}^\infty\}.$$

Lemma 1. *Let $x\in\mathcal{N}$ and $v\in\mathcal{N}^\perp$. If $[x\cdot v]^+\in L^1$, then $E(x\cdot v)=0$.*

We will use the notation

$$\operatorname{dom}_1 Ef := \{x\in\mathcal{N} \,|\, \exists u\in\mathcal{U}:\ Ef(x,u) < +\infty\}.$$

Recall that *algebraic closure*, $\operatorname{acl} C$, of a set C is the set of points x such that $(x,z]\subset C$ for some $z\in C$. Clearly, $C\subseteq \operatorname{acl} C$ (simply because $(x,x]=\emptyset\subset C$ for all $x\in C$) while in a topological vector space, $\operatorname{acl} C\subseteq \operatorname{cl} C$. The relative interior of C will be denoted by $\operatorname{rint} C$. Given a measurable set-valued mapping $D:\Omega\rightrightarrows\mathbb{R}^n$, we will use the notation

$$\mathcal{N}^\infty(D) := \{x\in\mathcal{N}^\infty \,|\, x\in D\ P\text{-a.s.}\}.$$

Theorem 2. *Let $y \in \mathcal{Y}$. If $\operatorname{dom} El(\cdot, y) \cap \mathcal{N}^\infty \subseteq \operatorname{acl}(\operatorname{dom}_1 Ef \cap \mathcal{N}^\infty)$, then*

$$\tilde{\varphi}^*(y) = -\inf_{x \in \mathcal{N}^\infty} El(x, y).$$

If $\operatorname{dom} El(\cdot, y) \cap \mathcal{N}^\infty \subseteq \operatorname{acl}(\operatorname{dom}_1 Ef \cap \mathcal{N}^\infty(\operatorname{rint} \operatorname{dom}_1 f))$, then

$$\tilde{\varphi}^*(y) = -\inf_{x \in \mathcal{N}^\infty} E\underline{l}(x, y).$$

We always have

$$\tilde{\varphi}^*(y) \leq \varphi^*(y) \leq \inf_{v \in \mathcal{N}^\perp} Ef^*(v, y).$$

In particular, if there exists $v \in \mathcal{N}^\perp$ such that $\tilde{\varphi}^*(y) = Ef^*(v, y)$, then

$$\tilde{\varphi}^*(y) = \varphi^*(y).$$

Proof. By the interchange rule ([25, Theorem 14.60]), $L(x, y) = El(x, y)$ for $x \in \operatorname{dom}_1 Ef$ and thus,

$$\begin{aligned} -\tilde{\varphi}^*(y) &= \inf_{x \in \mathcal{N}^\infty} L(x, y) = \inf_{x \in \mathcal{N}^\infty \cap \operatorname{dom}_1 Ef} L(x, y) \\ &= \inf_{x \in \mathcal{N}^\infty \cap \operatorname{dom}_1 Ef} El(x, y) \geq \inf_{x \in \mathcal{N}^\infty} El(x, y). \end{aligned}$$

The converse holds trivially if $\operatorname{dom} El(\cdot, y) \cap \mathcal{N}^\infty = \emptyset$. Otherwise, let $a \in \mathbb{R}$ and $x \in \mathcal{N}^\infty$ be such that $El(x, y) < a$. By the first assumption, there exists an $x' \in \operatorname{dom}_1 Ef \cap \mathcal{N}^\infty$ such that $(1 - \lambda)x + \lambda x' \in \operatorname{dom}_1 Ef$ for all $\lambda \in (0, 1]$. By convexity, $L((1 - \lambda)x + \lambda x', y) = El((1 - \lambda)x + \lambda x', y) < a$ for λ small enough and thus

$$-\tilde{\varphi}^*(y) = \inf_{x \in \mathcal{N}^\infty} El(x, y).$$

To prove the second claim, we repeat the line segment argument above with $x' \in \operatorname{dom}_1 Ef \cap \mathcal{N}^\infty(\operatorname{rint} \operatorname{dom}_1 f)$. By [19, Theorem 34.3], $x \in \operatorname{cl} \operatorname{dom}_1 f$ almost surely so, by [19, Theorem 6.1], $(x, x') \subset \operatorname{rint} \operatorname{dom}_1 f$ almost surely as well. Thus, the infimum in

$$-\tilde{\varphi}^*(y) = \inf_{x \in \mathcal{N}^\infty} El(x, y)$$

can be restricted to those x for which $x \in \operatorname{rint} \operatorname{dom}_1 f$. Then, by [19, Theorem 34.2], we may replace l by $\underline{l}$ without affecting the infimum.

As to the last claim, the Fenchel inequality gives

$$f(x, u) + f^*(v, y) \geq x \cdot v + u \cdot y.$$

Therefore, for $(x, u) \in \operatorname{dom} Ef$ and $v \in \mathcal{N}^\perp$ with $Ef^*(v, y) < \infty$, we have $E(x \cdot v) = 0$ by Lemma 1, so we get

$$\varphi^*(y) = \sup_{x \in \mathcal{N}, u \in \mathcal{U}} E[u \cdot y - f(x, u)] \leq \inf_{v \in \mathcal{N}^\perp} Ef^*(v, y).$$

Since $\tilde{\varphi} \geq \varphi$, have $\tilde{\varphi}^* \leq \varphi^*$. This completes the proof of the inequalities. The last statement concerning the equality clearly follows from the inequalities. $\square$

Theorem 2 gives conditions under which the conjugate of the value function of $(\mathcal{P}_u)$ can be expressed as

$$\varphi^*(y) = -\inf_{x \in \mathcal{N}^\infty} El(x, y). \tag{1}$$

In all the applications below, the first condition holds without the algebraic closure on $\operatorname{dom}_1 Ef \cap \mathcal{N}^\infty$. In the deterministic case, the first condition is automatically satisfied since then, $\operatorname{dom} l(\cdot, y) \subseteq \operatorname{cl} \operatorname{dom}_1 f$ for all y, by [19, Theorem 34.3], while $\mathcal{N}^\infty = \mathbb{R}^n$ where the algebraic closure of a convex set coincides with its topological closure. The algebraic closure can be replaced by the topological closure also when $El(\cdot, y)$ is upper semicontinuous on the closure of $\operatorname{dom}_1 Ef \cap \mathcal{N}^\infty$. At the moment, it is an open question whether the algebraic closure could be replaced by a topological closure in general.

The second claim of Theorem 2 gives conditions that allow one to replace l by $\underline{l}$ in the above expression. One can then resort to the theory of normal integrands when calculating the infimum. It was shown in [10] that this yields many well-known dual expressions in operations research and mathematical finance. Unfortunately, the proof of the above expression in [10, Theorem 2.2] was incorrect and the given conditions are not sufficient in general; see [11]. The following example illustrates what can go wrong if the condition on the domains is omitted. Here and in what follows, δ_C denotes the indicator function of a set C: $\delta_C(x)$ equals 0 or $+\infty$ depending on whether $x \in C$ or not.

Example 3. *Let $\mathcal{F}_0$ be trivial, $n_0 = 1$, $\mathcal{U} = L^\infty$, $\mathcal{Y} = L^1$, $\beta \in L^2$ be such that β^+ and β^- are unbounded, and let*

$$f(x, u, \omega) = \delta_{\mathbb{R}_-}(\beta(\omega)x_0 + u)$$

so that $\operatorname{dom} Ef = \{x \in \mathcal{N} \mid x_0 = 0\} \times L^\infty_-$, $\tilde{\varphi}^* = \delta_{L^1_+}$ and

$$l(x, y, \omega) = y\beta(\omega)x_0 - \delta_{\mathbb{R}_+}(y).$$

For $y = \beta^+$, we get $\inf_{x \in \mathcal{N}^\infty} El(x, y) = -\infty$ while $\tilde{\varphi}^*(y) = 0$. Here $\operatorname{dom} El(\cdot, y) = \mathcal{N}$, so the first condition in Theorem 2 is violated.

The last part of Theorem 2 gives a sufficient condition for $\varphi^* = \tilde{\varphi}^*$, which together with the first part of the theorem gives (1). We do not require the time-separability structure on the Lagrangian integrand assumed in [10]. Instead, we assume the existence of a $v \in \mathcal{N}^\perp$ such that $\tilde{\varphi}(y) = Ef^*(v, y)$. Such a v can be seen as a "shadow price of information" for the minimization problem

$$\text{minimize} \quad El(x, y) \quad \text{over}\, x \in \mathcal{N}^\infty;$$

see [22, 23, 1, 14]. Sufficient conditions for the existence of v in the discrete-time setting are given in [22, 14]. In the applications below, the existence of v is obtained by more direct arguments.

The following describes a general situation where all the assumptions of Theorem 2 are satisfied.

Example 4. *Consider the problem*

$$\begin{array}{ll} \text{minimize} & Ef_0(x) \quad \text{over } x \in \mathcal{N} \\ \text{subject to} & f_j(x) \le 0 \ P\text{-a.s.}, \ j = 1, \dots, m, \end{array}$$

where f_j are normal integrands. The problem fits the general framework with

$$f(x, u, \omega) = \begin{cases} f_0(x, \omega) & \text{if } f_j(x, \omega) + u_j \le 0 \text{ for } j = 1, \dots, m, \\ +\infty & \text{otherwise.} \end{cases}$$

This model was studied in Rockafellar and Wets [23] who gave optimality conditions in terms of dual variables. We will return to optimality conditions in the next section. For now, we note that if $f_j(x) \in \mathcal{U}$ for all $x \in L^\infty$ and $j = 0, \dots, m$, then the assumptions of Theorem 2 are satisfied.

The condition implies $\mathrm{dom}_1\, Ef \cap \mathcal{N}^\infty = \mathcal{N}^\infty$ *so the first two conditions in Theorem 2 hold. The Lagrangian integrand can be written as*

$$l(x, y, \omega) = \begin{cases} f_0(x, \omega) + \sum_{j=1}^m y_j f_j(x, \omega) & \text{if } y \in \mathbb{R}^m_+, \\ -\infty & \text{otherwise.} \end{cases}$$

By [20, Theorem 22], the function $El(\cdot, y)$ is Mackey-continuous (with respect to the usual pairing of L^∞ and L^1) so, by convexity, the function

$$\phi_y(\tilde{x}) := \inf_{x \in \mathcal{N}^\infty} El(x + \tilde{x}, y)$$

is Mackey-continuous as well; see e.g. [20, Theorem 8]. By [20, Theorem 11], this implies that ϕ_y has a subgradient at the origin, i.e. a $v \in L^1$ such that

$$El(x + \tilde{x}, y) \ge \phi_y(0) + E(\tilde{x} \cdot v) \qquad \forall \tilde{x} \in L^\infty, \ x \in \mathcal{N}^\infty$$

or equivalently

$$El(\tilde{x}, y) \ge \phi_y(0) + E((\tilde{x} - x) \cdot v) \qquad \forall \tilde{x} \in L^\infty, \ x \in \mathcal{N}^\infty.$$

This implies $v \in \mathcal{N}^\perp$ and $\inf_{\tilde{x} \in L^\infty} E[l(\tilde{x}, y) - \tilde{x} \cdot v] = \phi_y(0)$. By the interchange rule ([25, Theorem 14.60]), this can be written as $-Ef^*(v, y) = \phi_y(0) = -\tilde{\varphi}^*(y)$, so the last condition of Theorem 2 holds.

Section 4 of [9] studied financial models for pricing and hedging of portfolio-valued contingent claims (claims with physical delivery) along the lines of Kabanov [5]; see Example 13 below. The following example is concerned with the more classical financial model with claims with cash-delivery.

Example 5. *Consider the problem*

$$\text{minimize} \quad EV\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\right) \quad \text{over} \quad x \in \mathcal{N}, \tag{ALM}$$

where V is a convex normal integrand on $\mathbb{R}\times\Omega$ such that $V(\cdot,\omega)$ is nondecreasing nonconstant and $V(0,\omega)=0$. This models the optimal investment problem of an agent with financial liabilities $u\in\mathcal{U}$ and a "disutility function" V. The $\mathcal{F}_t$-measurable vector s_t gives the unit prices of "risky" assets at time t and the vector x_t the units held over $(t,t+1]$; see e.g. Rásonyi and Stettner [17] and the references therein.

Assume that, for every $x\in\mathcal{N}^\infty$, there is a $u\in\mathcal{U}$ such that $EV(u-\sum_{t=0}^{T-1}x_t\cdot\Delta s_{t+1})<\infty$, then the closure of the value function φ of (ALM) *has the dual representation*

$$(\operatorname{cl}\varphi)(u)=\sup_{y\in\mathcal{Q}}E[uy-V^*(y)],$$

where $\mathcal{Q}$ is the set of positive multiples of martingale densities $y\in\mathcal{Y}$, i.e. densities dQ/dP of probability measures $Q\ll P$ under which the price process s is a martingale.

Indeed, (ALM) *fits the general model $(\mathcal{P}_u)$ with*

$$\begin{aligned} f(x,u,\omega)&=V\left(u-\sum_{t=0}^{T-1}x_t\cdot\Delta s_{t+1}(\omega),\omega\right),\\ l(x,y,\omega)&=-V^*(y,\omega)-y\sum_{t=0}^{T-1}x_t\cdot\Delta s_{t+1}(\omega),\\ f^*(v,y,\omega)&=\begin{cases}V^*(y,\omega) & \text{if } v_t=-y\Delta s_{t+1}(\omega) \text{ for } t<T \text{ and } v_T=0,\\ +\infty & \text{otherwise.}\end{cases}\end{aligned}$$

The integrability condition implies $\operatorname{dom}_1 Ef\cap\mathcal{N}^\infty=\mathcal{N}^\infty$ so the first two conditions of Theorem 2 hold. Thus,

$$-\tilde{\varphi}^*(y)=\inf_{x\in\mathcal{N}^\infty}El(x,y)=-EV^*(y)+\inf_{x\in\mathcal{N}^\infty}E\left[-\sum_{t=0}^{T-1}x_t\cdot(y\Delta s_{t+1})\right].$$

By Fenchel inequality,

$$uy-\sum_{t=0}^{T-1}x_t\cdot(y\Delta s_{t+1})\le V\left(u-\sum_{t=0}^{T-1}x_t\cdot\Delta s_{t+1}\right)+V^*(y)\quad P\text{-a.s.}$$

where for every $x\in\mathcal{N}^\infty$ and $y\in\operatorname{dom}EV^*$, the right side is integrable for some $u\in\mathcal{U}$. Thus, $-\sum_{t=0}^{T-1}x_t\cdot(y\Delta s_{t+1})$ is integrable for every $x\in\mathcal{N}^\infty$ and $y\in\operatorname{dom}EV^*$. Therefore the last infimum equals $-\infty$ unless $E_t(y\Delta s_{t+1})=0$ for every $t<T$, i.e. unless $y\in\mathcal{Q}$. Moreover, if $y\in\operatorname{dom}\tilde{\varphi}^*$ then $Ef^*(v,y)=\tilde{\varphi}^*(y)$ holds with $v_t=y\Delta s_{t+1}$ for $t<T$ and $v_T=0$, so the last condition of Theorem 2 holds.

The following example addresses a parameterized version of the generalized problem of Bolza studied in Rockafellar and Wets [24].

Example 6. *Consider the problem*

$$\underset{x\in\mathcal{N}}{\text{minimize}} \quad E\sum_{t=0}^{T} K_t(x_t, \Delta x_t + u_t), \tag{2}$$

where $n_t = d$, $\Delta x_t := x_t - x_{t-1}$, $x_{-1} := 0$ *and each* K_t *is an* $\mathcal{F}_t$*-measurable normal integrand on* $\mathbb{R}^d \times \mathbb{R}^d \times \Omega$.

We assume that, for every $x \in \mathcal{N}^\infty$ *with* $x_t \in \operatorname{cl}\operatorname{dom}_1 K_t$ *for all* t*, there exists* $\bar{x} \in \mathcal{N}^\infty$ *with* $\bar{x}_t \in \operatorname{rint}\operatorname{dom}_1 K_t$ *for all* t *and* $(\bar{x}, x) \subset \operatorname{dom}_1 Ef$*, and that* ${}^a u \in \mathcal{U}$ *and* ${}^a y \in \mathcal{Y}$ *for every* $u \in \mathcal{U}$ *and* $y \in \mathcal{Y}$*, where* ${}^a u_t = E_t u_t$*. Then the closure of the value function* φ *of* (2) *has the dual representation*

$$(\operatorname{cl}\varphi)(u) = \sup_{y\in\mathcal{Y}\cap\mathcal{N}} E\sum_{t=0}^{T}[u_t\cdot y_t - K_t^*(E_t\Delta y_{t+1}, y_t)]$$

for every adapted $u \in \mathcal{U}$.

Indeed, the problem fits our general framework with

$$\begin{aligned}
f(x,u,\omega) &= \sum_{t=0}^{T} K_t(x_t, \Delta x_t + u_t, \omega),\\
l(x,y,\omega) &= \sum_{t=0}^{T} \left[-x_t\cdot\Delta y_{t+1} + H_t(x_t, y_t, \omega)\right],\\
f^*(v,y,\omega) &= \sum_{t=0}^{T} K_t^*(v_t + \Delta y_{t+1}, y_t),
\end{aligned}$$

where $u = (u_0, \dots, u_T)$ *with* $u_t \in \mathbb{R}^d$ *and*

$$H_t(x_t, y_t, \omega) := \inf_{u_t\in\mathbb{R}^d}\{K_t(x_t, u_t, \omega) - u_t\cdot y_t\}$$

is the associated Hamiltonian*.*

The domain condition in Theorem 2 is satisfied, so

$$-\tilde{\varphi}^*(y) = \inf_{x\in\mathcal{N}^\infty} E\underline{l}(x,y) = \inf_{x\in\mathcal{N}^\infty} E\sum_{t=0}^{T}\left[-x_t\cdot\Delta y_{t+1} + \underline{H}_t(x_t, y_t)\right],$$

where $\underline{H}_t(x_t, y_t, \omega) = \sup_{v_t}\{v_t\cdot x_t - K_t^*(v_t, y_t, \omega)\}$*. Thus, by the interchange rule [25, Theorem 14.50],*

$$\begin{aligned}
-\tilde{\varphi}^*(y) &= -\inf_{x\in\mathcal{N}^\infty} E\sum_{t=0}^{T}[-x_t\cdot E_t\Delta y_{t+1} + \underline{H}_t(x_t, y_t, \omega)]\\
&= -E\sum_{t=0}^{T} K_t^*(E_t\Delta y_{t+1}, y_t)
\end{aligned}$$

for adapted y*. Moreover, with* $v_t := E_t\Delta y_{t+1} - \Delta y_{t+1}$ *we get* $\tilde{\varphi}^*(y) = Ef^*(v,y)$*, where* $v \in \mathcal{N}^\perp$*. The last condition in Theorem 2 thus holds. For any* y*, Jensen's inequality gives*

$$
\begin{aligned}
-\tilde{\varphi}^*(y) &= \inf_{x\in\mathcal{N}^\infty} El(x,y) \\
&\le \inf_{x\in\mathcal{N}^\infty} E\sum_{t=0}^{T}[-x_t\cdot E_t\Delta y_{t+1} + H_t(x_t, E_t y_t)] \\
&= -\tilde{\varphi}^*({}^a y).
\end{aligned}
$$

Therefore, for adapted u*, we get*

$$
\operatorname{cl}\varphi(u) = \sup_{y\in\mathcal{Y}\cap\mathcal{N}} E\sum_{t=0}^{T}[u_t\cdot y_t - K_t^*(E_t\Delta y_{t+1}, y_t)].
$$

The dual representation of the value function in Example 6 was claimed to hold in [9] under the assumption that the Hamiltonian is lsc in x. The claim is, however, false in general since it omitted the domain condition in Theorem 2. The integrability condition posed in Example 6 not only provides a sufficient condition for that, but it also makes the lower semicontinuity of the Hamiltonian a redundant assumption.

The following example shows how the equality $\varphi^* = \tilde{\varphi}^*$ may fail to hold even when the first condition of Theorem 2 is satisfied.

Example 7. *Let* $T = 1$*,* $n = 2$*,* $\mathcal{F}_0$ *is the trivial* σ*-algebra, and*

$$
f(x,u,\omega) = |x_0 - 1| + \delta_{\mathbb{R}_-}(\alpha(\omega)|x_0| - x_1) + \frac{1}{2}|u|^2,
$$

where $\alpha \in L^0(\mathcal{F}_1)$ *is positive and unbounded. It is easily checked that with* $\mathcal{U} = \mathcal{Y} = L^2$ *the first condition of Theorem 2 holds but* $\tilde{\varphi}(u) = 1 + \frac{1}{2}\|u\|^2$ *while* $\varphi(u) = \frac{1}{2}\|u\|^2$*, so in particular, the last condition of Theorem 2 cannot hold.*

3 Optimality conditions

The previous section as well as the articles [10, 13] were concerned with dual representations of the value function φ. Continuing in the general conjugate duality framework of Rockafellar [20], this section derives optimality conditions for $(\mathcal{P}_u)$ by assuming the existence of a subgradient of φ at u. Besides optimality conditions, this assumption implies the lower semicontinuity of φ at u (with respect to the weak topology induced on $\mathcal{U}$ by $\mathcal{Y}$) and thus, the absence of a duality gap as well. Whereas in the above reference, the topology of convergence in measure in $\mathcal{N}$ played an important role, below, topologies on $\mathcal{N}$ are irrelevant.

Recall that a $y \in \mathcal{Y}$ is a *subgradient* of φ at $u \in \mathcal{U}$ if

$$
\varphi(u') \ge \varphi(u) + \langle u' - u, y\rangle \quad \forall u' \in \mathcal{U}.
$$

The set of all such y is called the *subdifferential* of φ at u and it is denoted by $\partial\varphi(u)$. If $\partial\varphi(u) \neq \emptyset$, then φ is lower semicontinuous at u and

$$\varphi(u) = \langle u, y\rangle - \varphi^*(y)$$

for every $y \in \partial\varphi(u)$. By [20, Theorem 11], $\partial\varphi(u) \neq \emptyset$, in particular, when φ is continuous at u.

We assume from now on that Ef is closed in u and that φ is proper.

Theorem 8. *Let $x \in \mathcal{N}$ be feasible and $y \in \mathcal{Y}$. If $y \in \partial\varphi(u)$ and $v \in \mathcal{N}^\perp$ are such that $\varphi^*(y) = Ef^*(v, y)$, then an x solves $(\mathcal{P}_u)$ if and only if*

$$(v, y) \in \partial f(x, u)$$

P-almost surely, or equivalently, if

$$v \in \partial_x l(x, y) \quad \text{and} \quad u \in \partial_y[-l](x, y).$$

Conversely, if x and y satisfy $(v, y) \in \partial f(x, u)$ almost surely for some $v \in \mathcal{N}^\perp$ with $Ef^*(v, y) < \infty$, then x solves $(\mathcal{P}_u)$, $y \in \partial\varphi(u)$, and $\varphi^*(y) = Ef^*(v, y)$.

Proof. Assume that $f^*(v, y)$ is integrable. The subdifferential condition is equivalent to having equality in the Fenchel inequality

$$f(x, u) + f^*(v, y) \geq x \cdot v + u \cdot y.$$

By Lemma 1, $E[x \cdot v] = 0$, so the inequality is satisfied as an equality if and only if

$$E[f(x, u) + f^*(v, y)] = E[u \cdot y].$$

Since we always have both $\varphi(u) \leq Ef(x, u)$ and $\varphi^*(y) \leq Ef^*(v, y)$ (by Theorem 2), this equality is equivalent to having

$$\begin{aligned} \varphi(u) + \varphi^*(y) &= E[u \cdot y], \\ Ef(x, u) &= \varphi(u), \\ Ef^*(v, y) &= \varphi^*(y), \end{aligned}$$

which are equivalent to x solving $(\mathcal{P}_u)$ and $y \in \partial\varphi(u)$ with $\varphi^*(y) = Ef^*(v, y)$. By [19, Theorem 37.5], the almost sure condition $(v, y) \in \partial f(x, u)$ is equivalent to

$$v \in \partial_x l(x, y) \quad \text{and} \quad u \in \partial_y[-l](x, y)$$

P-almost surely. $\square$

Example 9. *In Example 4, the optimality conditions of Theorem 8 mean that*

$$\begin{aligned} f_j(x) + u_j &\leq 0, \\ x \in \operatorname*{argmin}_{z \in \mathbb{R}^n}\{f_0(z) &+ \sum_{j=1}^m y_j f_j(z) - z \cdot v\}, \\ y_j f_j(z) &= 0 \quad j = 1, \dots, m, \\ y_j &\geq 0 \end{aligned}$$

P-almost surely. These are the optimality conditions derived in [23], where sufficient conditions were given for the existence of an optimal $x \in \mathcal{N}^\infty$ and the corresponding dual variables $y \in \mathcal{Y}$ and $v \in \mathcal{N}^\perp$ (in our notation); see [23, Theorem 1].

We will now describe another general setup which covers many interesting applications and where the subdifferentiability condition $\partial\varphi(u) \neq \emptyset$ in Theorem 8 is satisfied. This framework is motivated by Biagini [2], where similar arguments were applied to optimal investment in the continuous-time setting. The idea is simply to require stronger continuity properties on φ in order to get the existence of dual variables directly in $\mathcal{Y}$ (without going through the more exotic space $(L^\infty)^*$ as in [23]; see also [14]).

A topological vector space is said to be *barreled* if every closed convex absorbing set is a neighborhood of the origin. By [20, Corollary 8B], a lower semicontinuous convex function on a barreled space is continuous throughout the algebraic interior (core) of its domain. On the other hand, by [20, Theorem 11], continuity implies subdifferentiability. Fréchet spaces and, in particular, Banach spaces are barreled. We will say that $\mathcal{U}$ is *barreled* if it is barreled with respect to a topology compatible with the pairing with $\mathcal{Y}$.

Example 10. *Let $\theta : \mathbb{R} \to \mathbb{R}$ be a finite* Young-function*, i.e., finite nonconstant convex and even function with $\theta(0) = 0$. The associated* Orlicz space *is defined by*

$$L^\theta(\Omega, \mathcal{F}, P; \mathbb{R}^m) = \{u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^m) \,|\, \exists \alpha > 0 : \; E\theta(|u|/\alpha) < \infty\}$$

and the Morse space *(aka Orlicz-heart) by*

$$M^\theta(\Omega, \mathcal{F}, P; \mathbb{R}^m) = \{u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^m) \,|\, E\theta(|u|/\alpha) < \infty \;\forall \alpha > 0\}.$$

By [16, Theorems 3.3.10 and 3.4.3], both are Banach-spaces under the norm

$$\|u\|_\theta := \inf\{\alpha > 0 \,|\, E\theta(|u|/\alpha) \leq 1\}$$

and, by [16, Theorem 4.1.7], the norm dual of M^θ may be identified with L^{θ^*} through the bilinear form

$$\langle u, y\rangle = E(u \cdot y).$$

It is also easily verified that both spaces are decomposable. We are thus in the general setting of Section 2.

The following applies Theorem 8 to optimal investment in perfectly liquid financial markets.

Example 11. *Consider Example 5 and assume that $\mathcal{U}$ is barreled, EV is finite on $\mathcal{U}$ and EV^* is proper in $\mathcal{Y}$. Then an $x \in \mathcal{N}$ solves* (ALM) *if and only if it is feasible and there exists $y \in \mathcal{Q}$ such that*

$$y \in \partial V(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}) \quad P\text{-a.s.}$$

The assumptions holds in particular if $\mathcal{U} = M^\theta$ and $\mathcal{Y} = L^{\theta^*}$ for a finite Young-function θ such that $V(u,\omega) \le \theta(u) + u \cdot y(\omega) + \gamma(\omega)$ for some $y \in L^{\theta^*}$ and $\gamma \in L^1$. This idea was was developed in [3] in the context of utility maximization problems in continuous time.

Proof. By [20, Theorem 21], EV is lower semicontinuous so by [20, Corollary 8B], it is continuous. Since

$$\varphi(u) \le Ef(0,u) = EV(u),$$

[20, Theorem 8] implies that φ is continuous and thus subdifferentiable throughout $\mathcal{U}$, by [20, Theorem 11]. Moreover, $\varphi^*(y) = Ef^*(v,y)$ for every $y \in \operatorname{dom}\varphi^*$ and $v \in \mathcal{N}^\perp$ given by $v_t = y\Delta s_{t+1}$. The assumptions of Theorem 8 are thus satisfied. The subdifferential conditions for the Lagrangian integrand l can now be written as

$$v_t = -y\Delta s_{t+1} \quad P\text{-a.s. for } t < T \text{ and } v_T = 0,$$

$$u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} \in \partial V^*(y) \quad P\text{-a.s.}$$

Since $v \in \mathcal{N}^\perp$, the former means that $y \in \mathcal{Q}$ while the latter can be written in terms of ∂V as stated; see [20, Theorem 12]. □

The optimality conditions in Example 11 are classical in financial mathematics; for continuous-time models, see e.g. Schachermayer [26] or Biagini and Frittelli [3] and the references therein.

The next example applies Theorem 8 to the problem of Bolza from Example 6. It constructs a subgradient $y \in \partial\varphi(u)$ using Jensen's inequality.

Example 12. *Consider Example 6 and assume that $\mathcal{U}$ is barreled and that there exists a normal integrand θ such that $E\theta$ is finite on $\mathcal{U}$, $E\theta^*$ is proper on $\mathcal{Y}$, and $\varphi(u) \le \theta(u)$ for every $u \in \mathcal{U} \cap \mathcal{N}$. Then an $x \in \mathcal{N}$ solves*

$$\underset{x\in\mathcal{N}}{\text{minimize}} \quad E\sum_{t=0}^{T} K_t(x_t, \Delta x_t + u_t)$$

for a $u \in \mathcal{U} \cap \mathcal{N}$ if and only if x is feasible and there exists $y \in \mathcal{Y} \cap \mathcal{N}$ such that

$$(E_t\Delta y_{t+1}, y_t) \in \partial K_t(x_t, \Delta x_t + u_t)$$

P-almost surely for all t, or equivalently, if

$$E_t\Delta y_{t+1} \in \partial_x H_t(x_t, y_t),$$
$$\Delta x_t + u_t \in \partial_y[-H_t](x_t, y_t)$$

P-almost surely for all t.

Proof. The space $\mathcal{U} \cap \mathcal{N}$ is a barreled space and its continuous dual may be identified with $\mathcal{Y} \cap \mathcal{N}$. Indeed, by Hahn–Banach, any $y \in (\mathcal{U} \cap \mathcal{N})^*$ may be extended to a $\bar{y} \in \mathcal{Y}$ for which ${}^a\bar{y} \in \mathcal{Y} \cap \mathcal{N}$ coincides with y on $\mathcal{U} \cap \mathcal{N}$. Moreover, for any closed convex absorbing set B in $\mathcal{U} \cap \mathcal{N}$, we have that $\hat{B} = \{u \in \mathcal{U} \mid {}^a u \in B\}$ is a closed convex absorbing set in $\mathcal{U}$, so it is a neighborhood of the origin of $\mathcal{U}$ which implies that $\hat{B} \cap \mathcal{N} \subset B$ is a neighborhood of the origin of $\mathcal{U} \cap \mathcal{N}$.

By [20, Theorem 21], $E\theta$ is lower semicontinuous so by [20, Corollary 8B], it is continuous. Thus, φ is continuous and in particular, subdifferentiable on $\mathcal{U} \cap \mathcal{N}$ (see [20, Theorem 11]) so for every $u \in \mathcal{U} \cap \mathcal{N}$ there is a $y \in \mathcal{Y} \cap \mathcal{N}$ such that

$$\varphi(u') \geq \varphi(u) + \langle u' - u, y \rangle \quad \forall u' \in \mathcal{U} \cap \mathcal{N}.$$

Since each K_t is $\mathcal{F}_t$-measurable, Jensen's inequality gives

$$\varphi(u') \geq \varphi({}^a u') \geq \varphi(u) + \langle {}^a u' - u, y \rangle = \varphi(u) + \langle u' - u, y \rangle \quad \forall\ u' \in \mathcal{U},$$

so $y \in \partial\varphi(u)$ as well. Moreover, as observed in Example 6, we have $Ef^*(v, y) = \varphi^*(y)$ for $v_t = E_t \Delta y_{t+1} - \Delta y_{t+1}$. The subdifferential conditions in Theorem 8 for the Lagrangian integrand l become

$$\begin{aligned} v_t + \Delta y_{t+1} &\in \partial_x H_t(x_t, y_t)\ P\text{-a.s.} \quad \forall t, \\ u_t + \Delta x_t &\in \partial_y[-H_t](x_t, y_t)\ P\text{-a.s.} \quad \forall t. \end{aligned}$$

By [19, Theorem 37.5], these are equivalent to the conditions in terms of K_t. □

The optimality condition in terms of K_t in Example 12 can be viewed as a *stochastic Euler-Lagrange condition* in discrete-time much like that in [24, Theorem 4]. There is a difference, however, in that [24] studied the problem of minimizing $E \sum K_t(x_{t-1}, \Delta x_t)$ and, accordingly, the measurability conditions posed on the dual variables were different as well. The condition in terms of H_t can be viewed as a *stochastic Hamiltonian system* in discrete-time; see [18, Section 9] for deterministic models in continuous-time. Our assumptions on the problem also differ from those made in [24]. Whereas the assumptions of [24] and the line of argument follows that in [23] (see Example 9), our assumption implies the continuity of φ on the adapted subspace $\mathcal{U} \cap \mathcal{N}$.

It is essential that the growth condition in Example 12 is required only for adapted $u \in \mathcal{U}$. Indeed, since x is adapted, it would be much more to ask for the upper bound to hold for nonadapted u. This is the case e.g. in the following example which is a discrete-time variant of the multivariate utility maximization problems of Kabanov and Safarian [6, Section 4.2] and Campi and Owen [4]. While [4] gave sufficient conditions for the existence of dual solutions in the Banach dual of L^∞, our conditions guarantee the existence of dual solutions in the space $\mathcal{Y}$ of measurable functions. Moreover, we generalize the market models of [6, 4] by allowing for portfolio constraints and general convex trading costs like in [8, 12, 13]. It should be noted that Campi and Owen went further to establishing the existence of primal solutions under additional conditions on the utility functions. Under certain conditions (that extend the "asymptotic elasticity" conditions from [7]) primal existence can be established also in the abstract duality framework of Section 2 but that requires a different line of analysis that can be found in [13] and [15].

Given a closed convex set $K \subseteq \mathbb{R}^d$, we denote by $N_K(x)$ the *normal cone* of K at x, i.e. $N_K(x) := \{v \in \mathbb{R}^d \,|\, v \cdot (z - x) \leq 0 \ \forall z \in K\}$ for $x \in K$ and $N_K(x) = \emptyset$ for $x \notin K$.

Example 13. *Given an adapted $\mathbb{R}^d$-valued process $u = (u_t)_{t=0}^T$, consider the problem*

$$\begin{aligned} &\text{minimize} && E\sum_{t=0}^{T} V_t(k_t) \quad \text{over} \quad x, k \in \mathcal{N} \\ &\text{subject to} && \Delta x_t + u_t - k_t \in C_t,\ x_t \in D_t \quad P\text{-a.s. } t = 0, \dots, T, \end{aligned} \tag{OCP}$$

where $x_{-1} := 0$, C_t and D_t are random closed convex $\mathcal{F}_t$-measurable[2] sets and V_t is an $\mathcal{F}_t$-measurable nondecreasing (wrt $\mathbb{R}^d_+$) convex normal integrand. The problem (OCP) *can be interpreted as an optimal investment-consumption problem where x_t is the portfolio of assets an agent holds over $(t, t+1]$ and k_t is an investment he makes at time t. The random vector u_t is interpreted as a portfolio-valued contingent claim the agent has to deliver at time t. The set D_t describes portfolio constraints while C_t is the set of portfolios that are available for free in the financial market at time t. The functions V_t give the agent's "disutility" when delivering a portfolio k_t at time t. Alternativevly, one can interpret $-k_t$ as consumption at time t and $c \mapsto -V_t(-c)$ as a multivariate utility funtion like e.g. in [4]. Allowing for consumption at intermediate dates, not just at maturity T, is essential in the presence of portfolio constraints that may prevent the agent from holding certain portfolios over time.*

We assume that $0 \in C_t$, $0 \in D_t$ almost surely for all t and that $D_T \equiv \{0\}$. The first two conditions just mean that the agent is allowed not to trade at all while the last condition means that the agent has to close his positions at time T. We also assume that $V_t(0) = 0$ and that

$$\{k_t \in \mathbb{R}^d \,|\, V_t^\infty(k_t) \leq 0,\ -k_t \in C_t^\infty\}$$

is a linear space P-almost surely for all t. This last condition means that portfolios which are are freely available in the market and which the agent would not mind adding to his portfolio are actually redundant in the sense that both the agent and the market is completely indifferent to them.

Besides the above economic assumptions, let $\mathcal{U}$, $\mathcal{Y}$ and θ be as in Example 12 and assume that, for every $u \in \mathcal{U} \cap \mathcal{N}$, the optimum value of (OCP) *is less than $E\theta(u)$. If $u \in \mathcal{U} \cap \mathcal{N}$ is such that the optimum value in* (OCP) *is finite, then a pair $x, k \in \mathcal{N}$ solves* (OCP) *if and only if (x, k) is feasible and there exists*

[2] C_t is *$\mathcal{F}_t$-measurable* if $\{\omega \in \Omega \mid C_t(\omega) \cap O \neq \emptyset\} \in \mathcal{F}_t$ for every open O.

$y \in \mathcal{Y} \cap \mathcal{N}$ *such that*

$$
\begin{aligned}
E_t \Delta y_{t+1} &\in N_{D_t}(x_t),\\
y_t &\in N_{C_t}(\Delta x_t + u_t - k_t),\\
k_t &\in \operatorname*{argmin}_{k \in \mathbb{R}^d} \{V_t(k) - k \cdot y_t\}
\end{aligned}
$$

almost surely for all t. Note that, if there are no portfolio constraints, then $N_{D_t}(x_t) = \{0\}$ so y is a martingale. If, in addition, C is conical, then the second inclusion implies that $y_t \in C_t^*$ so y is a "consistent price system" in the sense of [5]. The above conditions in terms of normal cones provide a natural generalization to nonconical market models with portfolio constraints.

Proof. Problem (OCP) fits the format of Example 12 with

$$K_t(x_t, u_t, \omega) = \inf_{k_t \in \mathbb{R}^d} G_t(x_t, u_t, k_t, \omega),$$

where

$$G_t(x_t, u_t, k_t, \omega) = V_t(k_t, \omega) + \delta_{D_t(\omega)}(x_t) + \delta_{C_t(\omega)}(u_t - k_t).$$

Indeed, by [19, Theorem 9.2] and [25, Proposition 14.47], the linearity assumption implies that K_t is a convex normal integrand and the infimum in its definition is attained. By [25, Example 14.62], a pair (x, k) solves (OCP) if and only if x solves (2) and k_t achieves the infimum in the definition of K_t above.

It now suffices to apply the Euler–Lagrange condition in Example 12 and the fact that, by [20, Theorem 24],

$$
\begin{aligned}
\partial K_t(x_t, u_t, \omega) &= \{(v_t, y_t) \mid (v_t, y_t, 0) \in \partial G_t(x_t, u_t, k_t(\omega), \omega)\}\\
&= N_{D_t(\omega)}(x_t) \times \{y_t \mid y_t \in N_{C_t(\omega)}(u_t - k_t(\omega)),\ y_t \in \partial V_t(k_t(\omega))\},
\end{aligned}
$$

where the last line follows from the subdifferential calculus rules in Theorems 23.8 and 23.9 of [19]. □

Remark 14. *The boundedness condition on the optimum value in Example 13 is satisfied, for example, if the functions $EV_t(u_t)$ are finite for all $u \in \mathcal{U} \cap \mathcal{N}$. Such an assumption, however, precludes problems where trading strategies are required to be self-financing so that* $\operatorname{dom} V_t \subseteq \delta_{\mathbb{R}^d_-}$ *for $t < T$. In such situations, it is more reasonable to assume that there exists a normal integrand θ such that $E\theta$ is finite on $\mathcal{U}$, $E\theta^*$ is proper on $\mathcal{Y}$ and*

$$V_T(u_1 + \cdots + u_T) \leq \theta(u) \quad \forall u \in \mathbb{R}^{(T+1)d}.$$

If there are no portfolio constraints for $t < T$, this implies the upper bound in Example 12. Indeed, if $u \in \mathcal{U} \cap \mathcal{N}$ and we define $x, k \in \mathcal{N}$ by $x_t = x_{t-1} - u_t$ and $k_t = 0$ for $t < T$ while $x_T = 0$ and $k_T = u_0 + \ldots + u_T$, we get $\Delta x_t + u_t - k_t = 0 \in C_t$ while the objective becomes $V_T(u_1 + \cdots + u_T)$.

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