

HEDGING OF CLAIMS WITH PHYSICAL DELIVERY UNDER CONVEX TRANSACTION COSTS

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Abstract. We study superhedging of contingent claims with physical delivery in a discrete-time market model with convex transaction costs. Our model extends Kabanov’s currency market model by allowing for nonlinear illiquidity effects. We show that an appropriate generalization of Schachermayer’s robust no arbitrage condition implies that the set of claims hedgeable with zero cost is closed in probability. Combined with classical techniques of convex analysis, the closedness yields a dual characterization of premium processes that are sufficient to superhedge a given claim process. We also extend the fundamental theorem of asset pricing for general conical models.

Key words. superhedging, physical delivery, illiquidity, transaction costs, convex duality

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1. Introduction. This paper studies superhedging of contingent claims with physical delivery in markets with temporary illiquidity effects. Our market model is a generalization of the currency market model of Kabanov [12]. In Kabanov’s model price dynamics and transaction costs are modeled implicitly by *solvency cones*, i.e. sets of portfolios which can be transformed into the zero portfolio by self-financing transactions at a given time and state. An essential difference between Kabanov’s model and more traditional models of mathematical finance (including e.g. the transaction cost models of Jouini and Kallal [11], Cvitanić and Karatzas [6] and Kaval and Molchanov [18]) is that Kabanov’s model focuses on contingent claims with *physical delivery*, i.e. claims whose payouts are given in terms of portfolios of assets instead of a single reference asset like cash. Accordingly, notions of arbitrage as well as the corresponding dual variables are defined in terms of vector-valued processes; see e.g. [15], [31].

Astic and Touzi [2] extended Kabanov’s model by allowing general convex solvency regions in the case of finite probability spaces. Nonconical solvency regions allow for the modeling of temporary illiquidity effects where marginal trading costs may depend on the magnitude of a trade as e.g. in Çetin, Jarrow and Protter [3], Çetin and Rogers [4], Çetin, Soner and Touzi [5], Rogers and Singh [27], Çetin and Rogers [4] or Pennanen [19, 20]. These models cover nonlinear illiquidity effects but they assume that agents have no market power in the sense that their trades do not affect the costs of subsequent trades; see [4] for further motivation of this assumption. Temporal illiquidity effects act essentially as nonlinear transaction costs. Moreover, most modern stock exchanges are organized so that the costs are convex with respect to transacted amounts; see [19].

This paper studies general convex solvency regions in general probability spaces in finite discrete time. Using some classical techniques of convex analysis we extend the approaches developed in [15] and [31] for the models with proportional transaction costs. We give a sufficient condition for the closedness of the set of contingent claims with physical delivery that can be hedged with zero investment under convex

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transaction costs. Classical separation arguments then yield dual characterizations of superhedging conditions much as in Pennanen [21] in the case of claims with cash delivery. In the conical case, our sufficient condition coincides with the *robust no arbitrage condition* and the dual variables become *consistent price systems* in the sense of Schachermayer [31]. We also give a version of the “fundamental theorem of asset pricing” for general conical models. Even in the conical case our results improve on the existing ones since we do not assume polyhedrality of the solvency cones.

The rest of this paper is organized as follows. Section 2 describes the market model, and in Section 3 the robust no scalable arbitrage property is introduced. The relevance of this property becomes clear in Theorem 3.3, that provides a basis for the main results in Sections 4 and 5. Section 4 combines Theorem 3.3 with some classical techniques of convex analysis to derive dual characterizations of superhedging conditions. Section 5 generalizes the fundamental theorem of asset pricing to the general conical case. The proof of Theorem 3.3 is contained in Section 6.

2. The market model. We consider a financial market in which d securities can be traded over finite discrete time $t = 0, \dots, T$. The information evolves according to a filtration $(\mathcal{F}_t)_{t=0}^T$ on a probability space (Ω, \mathcal{F}, P) .

For each t and ω we denote by $C_t(\omega) \subset \mathbb{R}^d$ the set of portfolios that are freely available in the market. We assume that for each t the set-valued mapping $C_t : \Omega \rightrightarrows \mathbb{R}^d$ is \mathcal{F}_t -measurable in the sense that

$$C_t^{-1}(U) := \{\omega \in \Omega \mid C_t(\omega) \cap U \neq \emptyset\} \in \mathcal{F}_t$$

for every open set $U \subset \mathbb{R}^d$.

DEFINITION 2.1. A market model is an $(\mathcal{F}_t)_{t=0}^T$ -adapted sequence $C = (C_t)_{t=0}^T$ of closed-valued mappings $C_t : \Omega \rightrightarrows \mathbb{R}^d$ with $\mathbb{R}_+^d \subset C_t(\omega)$ for every t and ω . A market model C is convex, conical, polyhedral, ... if $C_t(\omega)$ has the corresponding property for every t and ω .

Traditionally, portfolios in financial market models have been defined in terms of a reference asset such as cash or some other numéraire; see Example 2.3 below. This is natural when studying financial contracts with cash payments only. Treating all assets symmetrically as in Definition 2.1 was initiated by Kabanov [12].

EXAMPLE 2.2 (Currency markets with proportional transaction costs). If $(s_t)_{t=0}^T$ is an adapted price process with values in \mathbb{R}_+^d and $(\Lambda_t)_{t=0}^T$ an adapted $\mathbb{R}_+^{d \times d}$ -valued process of transaction cost coefficients, the solvency regions (in physical units) were defined in Kabanov [12] as

$$\hat{K}_t := \{x \in \mathbb{R}^d \mid \exists a \in \mathbb{R}_+^{d \times d} : s_t^i x^i + \sum_{j=1}^d (a^{ji} - (1 + \lambda_t^{ij}) a^{ij}) \geq 0, 1 \leq i \leq d\}.$$

A portfolio x belongs to the solvency region \hat{K}_t iff, after some possible transfers $(a^{ji})_{1 \leq i, j \leq d}$, it has only nonnegative components. Thus the solvency region describes the set of all portfolios with “positive” values.

One can also define solvency regions directly in terms of bid-ask spreads as in Schachermayer [31]. If $(\Pi_t)_{t=0}^T$ is an adapted sequence of bid-ask matrices, then

$$\hat{K}_t = \{x \in \mathbb{R}^d \mid \exists a \in \mathbb{R}_+^{d \times d} : x^i + \sum_{j=1}^d (a^{ji} - \pi_t^{ij} a^{ij}) \geq 0, 1 \leq i \leq d\}.$$

For each ω and t the set $\hat{K}_t(\omega)$ is a polyhedral cone and

$$C_t(\omega) := -\hat{K}_t(\omega)$$

defines a conical market model in the sense of Definition 2.1.

EXAMPLE 2.3 (Illiquid markets with cash). A convex cost process is a sequence $S = (S_t)_{t=0}^T$ of extended real-valued functions on $\mathbb{R}^d \times \Omega$ such that for all t the function S_t is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$ -measurable and for each ω the function $S_t(\cdot, \omega)$ is lower semicontinuous, convex and vanishes at 0; see [20, 21]. The quantity $S_t(x, \omega)$ denotes the cost (in cash) of buying a portfolio x at time t and scenario ω . If S is a convex cost process, then

$$C_t(\omega) = \{x \in \mathbb{R}^d \mid S_t(x, \omega) \leq 0\}, \quad t = 0, \dots, T$$

defines a convex market model provided $S_t(\cdot, \omega)$ are componentwise nondecreasing for every t and ω . The main difference between the model studied in [20, 21] and the one studied here is that those papers studied contingent claims with cash-delivery in markets with portfolio constraints possibly on all traded assets, including cash. The present paper studies contingent claims with physical delivery but allows for free transfer of all positions through time.

Models with convex cost processes include, in particular, classical frictionless markets with a cash account (where $S_t((x^0, x^1), \omega) = x^0 + s_t(\omega) \cdot x^1$ for an adapted \mathbb{R}^d -valued price process s) as well as models with bid-ask spreads or proportional transaction costs as e.g. in Jouini and Kallal [11]. Convex cost processes also allow for modeling of illiquidity effects as e.g. in Çetin and Rogers [4] where $d = 2$ and $S_t((x^0, x^1), \omega) = x^0 + s_t(\omega)\varphi(x^1)$ for a strictly positive adapted process $(s_t)_{t=0}^T$ and an increasing convex function $\varphi : \mathbb{R} \rightarrow (-\infty, \infty]$ that describes the costs of illiquidity.

A convex cost process can be identified with the liquidation function P_t in Astic and Touzi [2] through $S_t(x, \omega) = -P_t(-x, \omega)$. Convex cost processes are also related to the supply curve introduced in Çetin et al. [3]. A supply curve $s_t(x, \omega)$ gives a price per unit when buying x units of the risky asset so that the total cost is $S_t(x, \omega) = s_t(x, \omega)x$. Instead of convexity of S , [3] assumed that the supply curve is smooth in x ; see also Example 2.2 in [2].

EXAMPLE 2.4 (Currency markets with illiquidity costs). In order to model non-proportional illiquidity effects in a currency market model as in Example 2.2 one can replace a bid-ask matrix $(\Pi_t)_{t=0}^T$ by a matrix of convex cost processes $S^{ij} = (S_t^{ij})_{t=0}^T$ ($1 \leq i, j \leq d$) on \mathbb{R}_+ . Here $S^{ij}(x, \omega)$ denotes the number of units of asset i for which one can buy x units of asset j . If $(S_t^{ij})_{t=0}^T$, $i, j = 1, \dots, d$ are convex cost processes on \mathbb{R}_+ in the sense of Example 2.3, then

$$C_t(\omega) = \{x \in \mathbb{R}^d \mid \exists a \in \mathbb{R}_+^{d \times d} : x^i \leq \sum_{j=1}^d (a^{ji} - S_t^{ij}(a^{ij}, \omega)), 1 \leq i \leq d\},$$

for $t = 0, \dots, T$ defines a convex market model. Markets with proportional transaction costs correspond to $S^{ij}(x, \omega) = \pi^{ij}(\omega)x$.

3. No arbitrage criteria. An \mathbb{R}^d -valued adapted process $x = (x_t)_{t=0}^T$ is a self-financing portfolio process in a market model $C = (C_t)_{t=0}^T$ if

$$\Delta x_t := x_t - x_{t-1} \in C_t \quad P\text{-a.s.}$$

for every $t = 0, \dots, T$, i.e. the increments Δx_t are freely available in the market. Here and in what follows, we always define $x_{-1} = 0$ and denote by $L^0(C_t, \mathcal{F}_t)$ the set of all \mathcal{F}_t -measurable selectors of a set-valued mapping $C_t : \Omega \rightrightarrows \mathbb{R}^d$, i.e. the set of all \mathcal{F}_t -measurable random vectors x such that $x \in C_t$ almost surely. In particular $L^0(\mathbb{R}^d, \mathcal{F}_t)$ denotes the set of all \mathcal{F}_t -measurable random vectors $x : \Omega \rightarrow \mathbb{R}^d$.

We say that a market model C has the *no-arbitrage* property if

$$A_T(C) \cap L^0(\mathbb{R}_+^d, \mathcal{F}) = \{0\}, \quad (3.1)$$

where $A_T(C)$ denotes the set in L^0 formed by final values x_T of all self-financing portfolio processes $x = (x_t)_{t=0}^T$. Since $x_T = \sum_{t=0}^T (x_t - x_{t-1})$, we have the expression

$$A_T(C) = L^0(C_0, \mathcal{F}_0) + \dots + L^0(C_T, \mathcal{F}_T).$$

Condition (3.1) was introduced in Kabanov and Stricker [16] for conical models under the name “weak no-arbitrage property” in a formally different way. But it is equivalent to (3.1) if C_t contains \mathbb{R}_-^d for all t as noted in Lemma 3.5. of Kabanov [13].

In classical market models the no-arbitrage condition implies the closedness of the set of contingent claims with cash-delivery that can be superhedged at zero cost, a result which is of vital importance in deriving dual characterizations of superhedging and absence of arbitrage. However, as shown in Schachermayer [31], in a market with proportional transaction costs as in Example 2.2, the no-arbitrage property (3.1) does not, in general, imply the closedness of the set $A_T(C)$. Schachermayer [31] also showed, in the case of the conical model of Example 2.2, that $A_T(C)$ is closed is probability if C satisfies the *robust no-arbitrage* condition which can be defined in the general conical case as follows.

Given a market model C let $C_t^0(\omega)$ be the largest linear subspace contained in $C_t(\omega)$. Using the terminology of Kabanov, Rásonyi and Stricker [15] we say that C is *dominated* by another market model \tilde{C} if

$$C_t \subset \tilde{C}_t, \quad \text{and} \quad C_t \setminus C_t^0 \subset \text{ri} \tilde{C}_t \quad \text{for all} \quad t = 0, \dots, T,$$

where $\text{ri} \tilde{C}_t$ denotes the relative interior of a convex set \tilde{C}_t , cf. [24, Chapter 6] for the definition. Since in our setting $\mathbb{R}_-^d \subseteq C_t \subseteq \tilde{C}_t$, the relative interior of \tilde{C}_t coincides with the interior.

DEFINITION 3.1. *A conical market model has the robust no arbitrage property if it is dominated by another conical market model that has the no arbitrage property.*

When moving to general convex market models it is not immediately clear how the condition of robust no-arbitrage should be extended in order to have the closedness of $A_T(C)$. Indeed, in general convex models even the traditional notion of arbitrage has two natural extensions, one being the possibility of making something out of nothing the other one being the possibility of making arbitrarily much out of nothing; see [20]. These correspond to the notions of the *tangent cone* and the *recession cone* from convex analysis. It turns out to be the latter one which is more relevant for closedness of $A_T(C)$. This was noted in [21, Theorem 18] for the case of claims with cash-delivery under convex constraints and is, in fact, suggested already by classical closedness criteria in convex analysis; see [24, Chapter 9].

Given a convex market model C , let

$$C_t^\infty(\omega) = \{x \in \mathbb{R}^d \mid C_t(\omega) + \alpha x \subset C_t(\omega) \ \forall \alpha > 0\}.$$

This is a closed convex cone known as the *recession cone* of $C_t(\omega)$; see [24, Chapter 8]. The recession cone describes the asymptotic behavior of a convex set infinitely far from the origin. Since $C_t(\omega)$ is a closed convex set containing \mathbb{R}_-^d we have, by [24, Theorem 8.1, Theorem 8.2, Corollary 8.3.2, Theorem 8.3], that C^∞ is a closed convex cone containing \mathbb{R}_-^d and

$$C_t^\infty(\omega) = \bigcap_{\alpha > 0} \alpha C_t(\omega)$$

and

$$C_t^\infty(\omega) = \{x \in \mathbb{R}^d \mid \exists x^n \in C_t(\omega), \alpha^n \searrow 0, \text{ with } \alpha^n x^n \rightarrow x\}. \quad (3.2)$$

By [26, Exercise 14.21], the set-valued mappings $\omega \mapsto C_t^\infty(\omega)$ are \mathcal{F}_t -measurable so they define a convex conical market model in the sense of Definition 2.1.

DEFINITION 3.2. *A convex market model has the robust no scalable arbitrage property if C^∞ has the robust no arbitrage property.*

The term “scalable arbitrage” refers to arbitrage opportunities that may be scaled by arbitrarily large positive numbers to yield arbitrarily “large” arbitrage opportunities; see [20]. Such scalable arbitrage opportunities can be related to the market model C^∞ much as in [20, Proposition 17].

We are now ready to state our main result the proof of which can be found in the last section.

THEOREM 3.3. *If C is a convex market model with the robust no scalable arbitrage property then $A_T(C)$ is closed in probability.*

REMARK 3.4. *If C is conical, we have $C_t^\infty(\omega) = C_t(\omega)$ and Theorem 3.3 coincides with [15, Lemma 2] which extends [31, Theorem 2.1] to general conical models.*

REMARK 3.5. *For general convex models $C_t^\infty(\omega) \subseteq C_t(\omega)$ and the condition in Theorem 3.3 may be satisfied even if C fails the no-arbitrage condition. Consider for example a deterministic model where Ω is a singleton and $C_t(\omega) = \mathbb{R}_-^d + \mathbb{B}$ for every t . Here \mathbb{B} denotes the unit ball of \mathbb{R}^d . We get,*

$$A_T(C) = L^0(C_0, \mathcal{F}_0) + \dots + L^0(C_T, \mathcal{F}_T) = \mathbb{R}_-^d + (T+1)\mathbb{B}$$

so C does not have the no-arbitrage property. On the other hand, $C_t^\infty(\omega) = \mathbb{R}_-^d$ is dominated by $\tilde{C}_t(\omega) = \{x \in \mathbb{R}^d \mid \sum_{i=1}^d x^i \leq 0\}$ which does have the no-arbitrage property. Indeed, $A_T(\tilde{C}) = \{x \in \mathbb{R}^d \mid \sum_{i=1}^d x^i \leq 0\}$ so $A_T(\tilde{C}) \cap L^0(\mathbb{R}_+^d, \mathcal{F}) = \{0\}$.

The following illustrates the robust no scalable arbitrage condition in a model with a cash account.

EXAMPLE 3.6 (The model of Çetin and Rogers [4]). *As observed in Example 2.3, the model of Çetin and Rogers [4] corresponds to the two-dimensional model*

$$C_t(\omega) = \{(x^0, x^1) \in \mathbb{R}^2 \mid x^0 + s_t(\omega)\varphi(x^1) \leq 0\}, \quad t = 0, \dots, T, \quad (3.3)$$

for a strictly positive adapted marginal price process $(s_t)_{t=0}^T$ and an increasing convex function $\varphi: \mathbb{R} \rightarrow (-\infty, \infty]$ with $\varphi(0) = 0$. In this case, the recession cones become

$$\begin{aligned} C_t^\infty(\omega) &= \{(x^0, x^1) \mid \alpha x^0 + s_t(\omega)\varphi(\alpha x^1) \leq 0 \forall \alpha > 0\} \\ &= \{(x^0, x^1) \mid x^0 + s_t(\omega)\varphi^\infty(x^1) \leq 0\}, \end{aligned}$$

where $\varphi^\infty(x) = \sup_{\alpha>0} \alpha^{-1} \varphi(\alpha x)$ is the recession function of φ ; see [24, Corollary 8.5.2]. By [24, Theorem 13.3], we have the expression

$$\varphi^\infty(x) = \sup_{y \in I} xy,$$

where I is the range of the gradient mapping $x \mapsto \varphi'(x)$; see [24, page 227].

If we assume, as in [4], that $\inf_x \varphi'(x) = 0$, $\sup_x \varphi'(x) = \infty$, we have

$$\varphi^\infty(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ +\infty & \text{if } x > 0, \end{cases}$$

and thus $C_t^\infty(\omega) = \mathbb{R}_-^2$ almost surely for all t . In this case the robust no scalable arbitrage condition holds with an arbitrary strictly positive marginal price process $(s_t)_{t=0}^T$ whether it satisfies the classical no arbitrage condition or not. In [4] the above growth properties of φ were used to prove the existence of optimal strategies in utility maximization problems and it was noted that the usual no arbitrage condition on the price process $(s_t)_{t=0}^T$ played no role in the proof. The nonexistence of arbitrage opportunities (not just the scalable ones), is related to a stronger no-arbitrage conditions as formulated e.g. in [2] or [20].

More precise characterizations of the robust no scalable arbitrage condition for models with cash account will be given at the end of Section 5.

4. Superhedging. A contingent claim with *physical delivery* is a security that, at some future time, gives its owner a random portfolio of securities (instead of a single security like in the case of cash-delivery). A *contingent claim process with physical delivery* $c = (c_t)_{t=0}^T$ is a security that, at each time $t = 0, \dots, T$, gives its owner an \mathcal{F}_t -measurable random portfolio $c_t \in \mathbb{R}^d$. The linear space of \mathbb{R}^d -valued adapted process will be denoted by \mathcal{A} , i.e.

$$\mathcal{A} = L^0(\mathbb{R}^d, \mathcal{F}_0) \times \dots \times L^0(\mathbb{R}^d, \mathcal{F}_T).$$

We call the elements of \mathcal{A} portfolio processes and denote by \mathcal{A}_+ the set of all non-negative portfolio processes:

$$\mathcal{A}_+ = L^0(\mathbb{R}_+^d, \mathcal{F}_0) \times \dots \times L^0(\mathbb{R}_+^d, \mathcal{F}_T).$$

Given a market model C , we will say that a process $p \in \mathcal{A}$ is a *superhedging premium process* for a claim process $c \in \mathcal{A}$ if there is a portfolio process $x \in \mathcal{A}$ such that $x_T = 0$ and

$$x_t - x_{t-1} + c_t - p_t \in C_t$$

almost surely for every $t = 0, \dots, T$. The requirement that $x_T = 0$ means that everything is liquidated at the terminal date. One could relax this condition to $x_T \geq 0$ but since $\mathbb{R}_-^d \subset C_T$ it would amount to the same thing.

We use claim and premium processes (rather than claims and premiums (prices) with pay-outs only at the end and the beginning) in the present paper mainly for mathematical convenience. However, when moving to market models with portfolio constraints it is essential to distinguish between payments at different points in time, and then claim processes become the natural object of study; see [20, 21]. Claim and premium processes are common in various insurance applications where payments are made e.g. annually.

The superhedging condition can be written as

$$c - p \in A(C),$$

where

$$A(C) = \{c \in \mathcal{A} \mid \exists x \in \mathcal{A} : x_t - x_{t-1} + c_t \in C_t, x_T = 0\},$$

is the set of all claim processes with physical delivery that can be superhedged without any investment. It is easily checked that $A(C)$ is convex subset (a convex cone) of \mathcal{A} if C is convex (conical) market model.

LEMMA 4.1. *Let C be a convex market model.*

1. *The sets $A(C)$ and $A_T(C)$ are related through*

$$\begin{aligned} A_T(C) &= \{c_T \mid (0, \dots, 0, c_T) \in A(C)\}, \\ A(C) &= \{(c_0, \dots, c_T) \mid \sum_{t=0}^T c_t \in A_T(C)\}. \end{aligned}$$

2. *The set $A(C)$ is closed in probability if and only if $A_T(C)$ is closed in probability.*

3. *C has the no-arbitrage property if and only if $A(C) \cap \mathcal{A}_+ = \{0\}$.*

Proof. It suffices to prove the first part since the other two follow from that. The first equation is immediate. As to the second, we have $c \in A(C)$ iff there is an $x \in \mathcal{A}$ such that $x_T = 0$ and $x_t - x_{t-1} + c_t \in C_t$. Defining $\tilde{x}_{-1} = 0$ and $\tilde{x}_t = \tilde{x}_{t-1} + x_t - x_{t-1} + c_t$ we get that $\tilde{x} \in \mathcal{A}$ is self-financing and $\tilde{x}_T = \sum_{t=0}^T c_t$. This just means that $\sum_{t=0}^T c_t \in A_T(C)$. \square

EXAMPLE 4.2. *The classical case of a single premium payment at the beginning and single claim payment at maturity corresponds to $p = (p_0, 0, \dots, 0)$ and $c = (0, \dots, 0, c_T)$. In this case, Lemma 4.1 says that p is a superhedging premium for c if and only if*

$$c_T - p_0 \in A_T(C). \tag{4.1}$$

Dual characterizations of the set of all initial endowments satisfying condition (4.1) have been given in the conical case as in Example 2.2 in [16], [7], [14], [22] and [31]. In conical models the superhedging endowments are characterized in terms of the same dual elements that characterize the no-arbitrage condition. When moving to nonconical models, a larger class of dual variables is needed in order to capture the structure of the sets; see [21] for the case of claims with cash delivery.

By Lemma 4.1, the set $A(C)$ is closed if and only if $A_T(C)$ is closed. Combining Theorem 3.3 with classical techniques of convex analysis, we can derive dual characterizations of superhedging premium process in terms of martingales much as in [21] in the case of claims with cash delivery. Consider the Banach space

$$\mathcal{A}^1 := L^1(\Omega, \mathcal{F}_0, P; \mathbb{R}^d) \times \dots \times L^1(\Omega, \mathcal{F}_T, P; \mathbb{R}^d)$$

and its dual

$$\mathcal{A}^\infty := L^\infty(\Omega, \mathcal{F}_0, P; \mathbb{R}^d) \times \dots \times L^\infty(\Omega, \mathcal{F}_T, P; \mathbb{R}^d).$$

Note that both \mathcal{A}^1 and \mathcal{A}^∞ are linear subspaces of \mathcal{A} . One can then use the classical bipolar theorem to characterize the superhedging condition in terms of the *support function* $\sigma_{A^1(C)} : \mathcal{A}^\infty \rightarrow \overline{\mathbb{R}}$ of the set $A^1(C) := A(C) \cap \mathcal{A}^1$ of integrable claim processes. The support function is given by

$$\sigma_{A^1(C)}(y) = \sup_{c \in A^1(C)} E \sum_{t=0}^T c_t \cdot y_t, \quad y \in \mathcal{A}^\infty.$$

The lemma below expresses $\sigma_{A^1(C)}$ in terms of the support functions of the random sets $C_t(\omega)$

$$\sigma_{C_t(\omega)}(z) = \sup_{x \in C_t(\omega)} x \cdot z, \quad z \in \mathbb{R}^d.$$

By [26, Example 14.51] the function $\sigma_{C_t} : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ is an \mathcal{F}_t -measurable convex normal integrand (see [26, Definition 14.27]). This implies in particular that $\sigma_{C_t(\omega)}(y_t)$ is an \mathcal{F}_t -measurable function whenever y_t is \mathcal{F}_t -measurable. The following corresponds to [20, Lemma 28].

LEMMA 4.3. *Let C be a convex market model and let $y \in \mathcal{A}^\infty$. Then*

$$\sigma_{A^1(C)}(y) = \begin{cases} E \sum_{t=0}^T \sigma_{C_t}(y_t) & \text{if } y \text{ is a nonnegative martingale,} \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. In the following, we define the expectation of an arbitrary random variable φ by setting it equal to $-\infty$ if the negative part of φ is not integrable (the remaining cases being defined unambiguously as real numbers or as $+\infty$). This allows us, in particular, to apply the results of [26, Section 14.F] on extended real-valued integral functionals.

On one hand,

$$\begin{aligned} \sigma_{A^1(C)}(y) &= \sup \left\{ E \sum_{t=0}^T c_t \cdot y_t \mid x \in \mathcal{A}, c \in \mathcal{A}^1 : x_t - x_{t-1} + c_t \in C_t \right\} \\ &\leq \sup \left\{ E \sum_{t=0}^T c_t \cdot y_t \mid x, c \in \mathcal{A} : x_t - x_{t-1} + c_t \in C_t \right\} \\ &= \sup \left\{ E \sum_{t=0}^T (w_t - x_t + x_{t-1}) \cdot y_t \mid x, w \in \mathcal{A} : w_t \in C_t \right\} \\ &\leq \sup \left\{ E \sum_{t=0}^T [\sigma_{C_t}(y_t) + (x_{t-1} - x_t) \cdot y_t] \mid x \in \mathcal{A} \right\} \\ &= E \sum_{t=0}^T \sigma_{C_t}(y_t) + \sup_{x \in \mathcal{A}} E \sum_{t=0}^T (x_{t-1} - x_t) \cdot y_t \\ &= E \sum_{t=0}^T \sigma_{C_t}(y_t) + \sup_{x \in \mathcal{A}} E \sum_{t=0}^{T-1} x_t \cdot (y_{t+1} - y_t), \end{aligned}$$

where the last term vanishes if y is a martingale and equals $+\infty$ otherwise. On the

other hand,

$$\begin{aligned}
\sigma_{A^1(C)}(y) &= \sup\{E \sum_{t=0}^T c_t \cdot y_t \mid x \in \mathcal{A}, c \in A^1 : x_t - x_{t-1} + c_t \in C_t\} \\
&\geq \sup\{E \sum_{t=0}^T c_t \cdot y_t \mid x, c \in A^1 : x_t - x_{t-1} + c_t \in C_t\} \\
&= \sup\{E \sum_{t=0}^T (w_t - x_t + x_{t-1}) \cdot y_t \mid x, w \in A^1 : w_t \in C_t\} \\
&= \sup\{E \sum_{t=0}^T w_t \cdot y_t \mid w \in A^1 : w_t \in C_t\} + \sup_{x \in A^1} E \sum_{t=0}^T (x_{t-1} - x_t) \cdot y_t \\
&= \sum_{t=0}^T \sup\{E w_t \cdot y_t \mid w_t \in L^1(\Omega, \mathcal{F}_t, P; \mathbb{R}^d) : w_t \in C_t\} \\
&\quad + \sup_{x \in A^1} E \sum_{t=0}^{T-1} x_t \cdot (y_{t+1} - y_t).
\end{aligned}$$

Here again the last term vanishes if y is a martingale and equals $+\infty$ otherwise. Applying [26, Theorem 14.60] to the function

$$f(w, \omega) = \begin{cases} -w \cdot y_t(\omega) & \text{if } w \in C_t(\omega), \\ +\infty & \text{otherwise} \end{cases}$$

(which is a normal integrand by [26, 14.32]), we get

$$\sup\{E w_t \cdot y_t \mid w_t \in L^1(\Omega, \mathcal{F}_t, P; \mathbb{R}^d) : w_t \in C_t\} = E \sigma_{C_t}(y_t).$$

Thus,

$$\sigma_{A^1(C)}(y) = \begin{cases} E \sum_{t=0}^T \sigma_{C_t}(y_t) & \text{if } y \text{ is a martingale,} \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, since $\mathbb{R}_-^d \subseteq C_t$ for all t , we have $\sigma_{C_t}(y_t) = \infty$ on the set $\{y_t \notin \mathbb{R}_+^d\}$. This completes the proof. \square

THEOREM 4.4. *Assume that the convex market model C has the robust no scalable arbitrage property and let $c, p \in \mathcal{A}$ be such that $c - p \in A^1$. Then the following are equivalent*

- (i) p is a superhedging premium process for c .
- (ii) $E \sum_{t=0}^T (c_t - p_t) \cdot y_t \leq 1$ for every bounded nonnegative martingale $y = (y_t)_{t=0}^T$ such that $E \sum_{t=0}^T \sigma_{C_t}(y_t) \leq 1$.
- (iii) $E \sum_{t=0}^T (c_t - p_t) \cdot y_t \leq E \sum_{t=0}^T \sigma_{C_t}(y_t)$ for every bounded nonnegative martingale $y = (y_t)_{t=0}^T$.

Proof. By definition, p is a superhedging premium for c if and only if $c - p \in A(C)$. Since $c - p \in A^1$, this can be written as $c - p \in A^1(C)$, where $A^1(C) = A(C) \cap A^1$ is a closed subset of A^1 , by Theorem 3.3 and Lemma 4.1. Since $A^1(C)$ is also a convex set containing the origin, the bipolar theorem (see e.g. [1, Theorem 5.91]) implies that

$c - p \in A^1(C)$ iff

$$E \sum_{t=0}^T (c_t - p_t) \cdot y_t \leq 1 \quad \forall y \in A^1(C)^\circ,$$

where

$$A^1(C)^\circ = \{y \in \mathcal{A}^\infty \mid \sigma_{A^1(C)}(y) \leq 1\}.$$

Thus the equivalence of (i) and (ii) and the proof of (i) \Rightarrow (iii) follow from Lemma 4.3. And obviously (iii) implies (ii). \square

As noted in Remark 3.5, arbitrage opportunities may very well exist under the conditions of Theorem 3.3. They do not interfere with the hedging description in Theorem 4.4.

If C is conical, we have

$$\sigma_{C_t(\omega)}(y) = \begin{cases} 0 & \text{if } y \in C_t^*(\omega), \\ +\infty & \text{otherwise} \end{cases} \quad (4.2)$$

where the closed convex cone

$$C_t^*(\omega) := \{y \in \mathbb{R}^d \mid x \cdot y \leq 0 \quad \forall x \in C_t(\omega)\}$$

is known as the *polar* of $C_t(\omega)$; see [24]. We then have $E \sum_{t=0}^T \sigma_{C_t}(y_t) < \infty$ for a $y \in \mathcal{A}^\infty$ if and only if $y_t \in C_t^*$ almost surely for all t . Thus, in the conical case, superhedging premiums can be characterized in terms the following dual elements introduced in Kabanov [12].

DEFINITION 4.5. *An adapted \mathbb{R}^d -valued non-zero process $y = (y_t)_{t=0}^T$ is called a consistent (resp. strictly consistent) price system for a conical model C if y is a martingale such that $y_t \in C_t^*$ (resp. y has strictly positive components and $y_t \in \text{ri } C_t^*$) almost surely for all $t = 0, \dots, T$.*

Note that the condition $y_t \in \text{ri } C_t^*$ in general does not imply the strict positivity of y_t .

Applying Theorem 4.4 in the conical case and making use of (4.1) and (4.2), we obtain the following.

COROLLARY 4.6. *Assume that C is a conical market model with the robust no arbitrage property. Assume further that \mathcal{F}_0 is a trivial σ -algebra. Let $c_T \in L^1(\Omega, \mathcal{F}_T, P; \mathbb{R}^d)$ and $p_0 \in \mathbb{R}^d$. Then the following are equivalent.*

- (i) $p = (p_0, 0, \dots, 0)$ is a superhedging premium for $c = (0, \dots, 0, c_T)$.
- (ii) $E(c_T \cdot y_T) \leq p_0 \cdot y_0$ for every bounded consistent price system $y = (y_t)_{t=0}^T$.

We illustrate Theorem 4.4 and Corollary 4.6 with the following example.

EXAMPLE 4.7 (The model of Çetin and Rogers [4]). *Consider again the model of Çetin and Rogers [4] described in Example 3.6. Given $z \in \mathbb{R}^2$, we have*

$$\begin{aligned} \sigma_{C_t(\omega)}(z) &= \sup_{x \in \mathbb{R}^2} \{x^0 z^0 + x^1 z^1 \mid x^0 + s_t(\omega) \varphi(x^1) \leq 0\} \\ &= \begin{cases} \sup_{x^1 \in \mathbb{R}} \{-z^0 s_t(\omega) \varphi(x^1) + x^1 z^1\} & \text{if } z^0 \geq 0 \\ +\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} z^0 s_t(\omega) \varphi^*\left(\frac{z^1}{z^0 s_t(\omega)}\right) & \text{if } z^0 > 0 \\ 0 & \text{if } z^0 = z^1 = 0 \\ +\infty & \text{otherwise,} \end{cases} \end{aligned} \quad (4.3)$$

where $\varphi^*(z) = \sup_x \{xz - \varphi(x)\}$ is the convex conjugate of φ .

In the frictionless case where $\varphi(x) = x$, we have $\varphi^*(z) = 0$ if $z = 1$ and $\varphi^*(z) = +\infty$ otherwise so that

$$\sigma_{C_t(\omega)}(z) = \begin{cases} 0 & \text{if } z^0 \geq 0 \text{ and } z^1 = s_t(\omega)z_0, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easily checked that since s is strictly positive, the robust no scalable arbitrage condition is equivalent to the classical no arbitrage condition on s . The consistent price systems are the nonnegative martingales $y = (y^0, y^1)$ such that $y^1 = sy^0$. A nonzero consistent price system defines a density of an absolutely continuous martingale measure for the price process s via $d_T := y_T^0/y_0^0$. Vice versa, every density process $d = (d_t)_{t=0}^T$ of an absolutely continuous martingale measure for s defines a consistent price system y by

$$y_t := (d_t, d_t s_t), \quad t = 0, \dots, T.$$

The hedging condition (ii) of Corollary 4.6 can then be written as

$$E_{P^*} [c_T^0 + c_T^1 s_T] \leq p_0^0 + p_0^1 s_0 \quad \forall P^* \in \tilde{\mathcal{P}}, \quad (4.4)$$

where $\tilde{\mathcal{P}}$ is the set of absolutely continuous martingale measures. In particular, if $c_T^1 = 0$, $p_0^1 = 0$ and if s satisfies the no arbitrage condition, we have that p_0^0 units of cash at time 0 is sufficient to superhedge c_T^0 units of cash at time T if and only if

$$\sup_{P^* \in \tilde{\mathcal{P}}} E_{P^*} [c_T^0] \leq p_0^0,$$

in accordance with classical results for claims with cash-delivery; see e.g. [9, Remark 1.35].

Now assume that there are nontrivial illiquidity costs in the sense that $\varphi(x) \geq x$ for every $x \in \mathbb{R}$ and $\varphi(x) > x$ for some $x \in \mathbb{R}$. Assume also that the model satisfies the robust no scalable arbitrage condition. This happens e.g. when

$$\varphi(x) = \frac{e^{\alpha x} - 1}{\alpha}$$

as in Çetin and Rogers [4, Section 6]. Increasing the illiquidity costs shrinks the set of claims that can be superhedged with a given premium so that (4.4) is no longer a sufficient condition for superhedging. In other words, the superhedging premium under nontrivial illiquidity costs will be in general higher than in the corresponding frictionless model.

To illustrate this, consider the claim $c = (0, \dots, 0, c_T)$ with $c_T := (bs_T, -b)$ for $b \in \mathbb{R}$. Condition (4.4) is clearly satisfied with $p_0 = 0$ so c_T can be hedged without any costs in the frictionless model provided it has the no arbitrage property. The existence of an $x \in \mathbb{R}$ such that $\varphi(x) > x$, on the other hand, implies that the recession function $\varphi^\infty(x) = \sup_{\alpha \geq 0} \alpha^{-1} \varphi(\alpha x)$ dominates the identity nontrivially. By [24, Theorem 13.3], this means that there is an $a > 0$, $a \neq 1$ such that $\varphi^*(a) < \infty$. Consider the martingale v defined by $v_t := (d_t/a, d_t s_t)$, where (d_t) is the density process of an absolutely continuous martingale measure for the price process s . Using (4.3), we get

$$E\sigma_{C_t}(v_t) = \frac{\varphi^*(a)}{a} E[d_t s_t] = \frac{\varphi^*(a)}{a} E[d_0 s_0].$$

Thus, for some $b \in \mathbb{R}$,

$$E[v_T c_T] = b \left(\frac{1}{a} - 1 \right) E[d_T s_T] > \sum_{t=0}^T E \sigma_{C_t}(v_t),$$

so that, by Theorem 4.4, the zero premium is not sufficient to hedge c_T under the illiquidity costs given by φ .

Characterizations similar to Corollary 4.6 were given in [16], [7], [14], [22] and [31] under less restrictive integrability conditions on c_T . In our case the integrability condition in Theorem 4.4 can be relaxed in the following way. If the process $c - p$ is not P -integrable, we can always find a probability measure $\tilde{P} \approx P$ with bounded density such that $c_t - p_t \in L^1(\Omega, \mathcal{F}_t, \tilde{P}, \mathbb{R}^d)$ for all t , e.g.

$$\frac{d\tilde{P}}{dP} := \frac{a}{1 + \sum_{t=0}^T |c_t - p_t|},$$

where a is a normalizing constant. Then Theorem 4.4 holds with \tilde{P} instead of P and we obtain the following corollary.

COROLLARY 4.8. *Assume that C is a conical market model with the robust no arbitrage property and let $c, p \in \mathcal{A}$. Let further $\tilde{P} \approx P$ be a probability measure with bounded density process $z = (z_t)_{t=0}^T$ such that $c_t - p_t \in L^1(\Omega, \mathcal{F}_t, \tilde{P}, \mathbb{R}^d)$ for all t . Then the following are equivalent*

- (i) p is a superhedging premium process for c .
- (ii) $E \sum_{t=0}^T (c_t - p_t) \cdot y_t \leq 1$ for every bounded nonnegative P -martingale $(y_t)_{t=0}^T$ such that $E \sum_{t=0}^T \sigma_{C_t}(y_t) \leq 1$ and such that $y_t/z_t := (y_t^1/z_t, \dots, y_t^d/z_t)$ is almost surely bounded for all t .

Proof. Theorem 4.4 applied with \tilde{P} instead of P yields the equivalence of the following

- (i) p is a superhedging premium process for c .
- (ii) $E \sum_{t=0}^T (c_t - p_t) \cdot z_t \tilde{y}_t \leq 1$ for every nonnegative bounded \tilde{P} -martingale $\tilde{y} = (\tilde{y}_t)_{t=0}^T$ such that $E \sum_{t=0}^T z_t \sigma_{C_t}(\tilde{y}_t) \leq 1$.

Note further that $z \sigma_{C_t}(y) = \sigma_{C_t}(zy)$ for all $y \in \mathbb{R}^d, z \in \mathbb{R}^+$, and that \tilde{y} is a bounded \tilde{P} -martingale iff $z\tilde{y}$ is a bounded P -martingale. Thus (ii) holds for all nonnegative bounded P -martingales y such that $\frac{1}{z}y$ is almost surely bounded. \square

5. Fundamental theorem of asset pricing. Fundamental theorem of asset pricing describes absence of arbitrage by existence of certain pricing functionals. In the classical frictionless model with a cash account these pricing functionals can be identified with densities of equivalent martingale measures. Under proportional transaction costs the robust no-arbitrage condition is equivalent to existence of strictly consistent price systems. This result was proved in [16], [14], [22], [31], [15] for conical *polyhedral* market models, i.e. where each $C_t(\omega)$ is a cone spanned by a finite number of vectors. Non-polyhedral conical models are considered in [23] and in [30] under the assumption of efficient friction, i.e. the cones $C_t(\omega)$ are assumed to be pointed. In [25, Section 3.2] Rokhlin shows without assuming polyhedrality of the cones or efficient friction, that the existence of a strictly consistent price system is equivalent to the following condition:

$$\text{If } \sum_{t=0}^T x_t = 0 \text{ where } x_t \in L^0(C_t, \mathcal{F}_t), \text{ then } x_t \in L^0(C_t^0, \mathcal{F}_t), \quad t = 0, \dots, T. \quad (5.1)$$

However, to show that condition (5.1) is equivalent to the robust no-arbitrage condition one needs [15, Lemma 5, Corollary 1], where polyhedrality of the cones is used; see [29, Theorem 2].

In this section we derive a version of the fundamental theorem of asset pricing for general conical models. This is achieved through the following lemma which allows the application of strict separation arguments much as in [31]. It is similar to Proposition A.5 in [31] but does not rely on polyhedrality of C .

LEMMA 5.1. *If K is a conical market model that has the robust no arbitrage property, then it is dominated by another conical market model \hat{K} which still has the robust no arbitrage property.*

Proof. If K has the robust no arbitrage property, there is an arbitrage free model \tilde{K} such that $K_t \setminus K_t^0 \subset \text{int } \tilde{K}_t$, or equivalently, $\tilde{K}_t^* \setminus \{0\} \subset \text{ri } K_t^*$. Let $e \in \mathbb{R}^d$ be the vector with all components equal to one. Since $-e \in \text{int } K_t(\omega)$, we have

$$K_t^*(\omega) = \bigcup_{\alpha \geq 0} \alpha G_t(\omega) \quad \text{and} \quad \tilde{K}_t^*(\omega) = \bigcup_{\alpha \geq 0} \alpha \tilde{G}_t(\omega),$$

where $G_t(\omega) = \{v \in \mathbb{R}^d \mid v \in K_t^*(\omega), e \cdot v = 1\}$ and $\tilde{G}_t(\omega) = \{v \in \mathbb{R}^d \mid v \in \tilde{K}_t^*(\omega), e \cdot v = 1\}$. Since $\tilde{K}_t^* \setminus \{0\} \subset \text{ri } K_t^*$ we have

$$\tilde{G}_t(\omega) \subset \text{ri } G_t(\omega). \tag{5.2}$$

Let $\hat{G}_t(\omega) := \frac{1}{2}G_t(\omega) + \frac{1}{2}\tilde{G}_t(\omega)$, and let $\hat{K}_t(\omega)$ be the polar of the cone

$$\hat{K}_t^*(\omega) = \bigcup_{\alpha \geq 0} \alpha \hat{G}_t(\omega).$$

It suffices to show that \hat{K} is a market model that dominates K and is dominated by \tilde{K} . By construction, $\hat{K}_t(\omega)$ is closed and convex and, by Proposition 14.11 and Exercise 14.12 of [26], \hat{K}_t is \mathcal{F}_t -measurable, so \hat{K} is indeed a market model. The dominance relations are equivalent to the conditions $\hat{G}_t(\omega) \subset \text{ri } G_t(\omega)$ and $\tilde{G}_t(\omega) \subset \text{ri } \hat{G}_t(\omega)$. By (5.2) and [24, Theorem 6.1],

$$\hat{G}_t(\omega) \subset \frac{1}{2}G_t(\omega) + \frac{1}{2}\text{ri } G_t(\omega) = \text{ri } G_t(\omega).$$

To verify the second condition, we first note that (5.2) implies

$$\begin{aligned} \tilde{G}_t(\omega) &= \frac{1}{2}\tilde{G}_t(\omega) + \frac{1}{2}\tilde{G}_t(\omega) \\ &\subset \frac{1}{2}\text{ri } G_t(\omega) + \frac{1}{2}\tilde{G}_t(\omega), \end{aligned}$$

where the last set is relatively open. We then get

$$\begin{aligned} \tilde{G}_t(\omega) &\subset \text{ri} \left[\frac{1}{2}\text{ri } G_t(\omega) + \frac{1}{2}\tilde{G}_t(\omega) \right] \\ &= \frac{1}{2}\text{ri } G_t(\omega) + \frac{1}{2}\text{ri } \tilde{G}_t(\omega) \\ &= \text{ri } \hat{G}_t(\omega), \end{aligned}$$

where the equalities hold by [24, Corollary 6.6.2]. \square

Equipped with Theorem 3.3 and Lemma 5.1 it is easy to extend the proof of [31, Theorem 1.7] to get the following.

THEOREM 5.2. *A conical market model K has the robust no arbitrage property if and only if there exists a strictly consistent price system y for K . Moreover, the price system y can be chosen bounded.*

Proof. Lemma 5.1 implies the existence of another conical market model \hat{K} such that $\hat{K}_t^* \setminus \{0\} \subset \text{ri}(K_t)^*$. By Theorem 3.3 and Lemma 4.1, the set $A(\hat{K})$ is closed with respect to convergence in measure. Thus $A^1(\hat{K}) = A(\hat{K}) \cap \mathcal{A}^1$ is a convex cone in \mathcal{A}^1 which is closed in the norm topology. Moreover, the no-arbitrage property implies $A^1(\hat{K}) \cap \mathcal{A}_+^1 = \{0\}$, where \mathcal{A}_+^1 denotes the \mathbb{R}_+^d -valued processes in \mathcal{A}^1 .

Since the Banach space \mathcal{A}^1 is weakly compactly generated as a finite product of L^1 -spaces, the Kreps-Yan theorem holds true on \mathcal{A}^1 , see [28], Theorem 1 and the discussion after it. Thus there exists a $y \in \mathcal{A}^\infty$ such that

$$E \sum_{t=0}^T y_t \cdot c_t \leq 0 \quad \forall c \in A^1(\hat{K}) \quad (5.3)$$

and

$$E \sum_{t=0}^T y_t \cdot c_t > 0 \quad \forall c \in \mathcal{A}_+^1 \setminus \{0\}. \quad (5.4)$$

Condition (5.3) can be written as $\sigma_{A^1(\hat{K})}(y) \leq 0$ which, by Lemma 4.3, means that y is a nonnegative martingale with $\sigma_{\hat{K}_t(\omega)}(y_t(\omega)) \leq 0$ almost surely for every t . Since $\hat{K}_t(\omega)$ is a cone, we have $y_t(\omega) \in \hat{K}_t(\omega)^*$. Condition (5.4) means that y is componentwise strictly positive so $y_t \in \hat{K}_t^* \setminus \{0\} \subset \text{ri} K_t^*$ and y is a strictly consistent price system for K .

To prove the converse let y be a strictly consistent price system for K and define a conical model

$$\tilde{K}_t := \{x \in \mathbb{R}^d \mid y_t \cdot x \leq 0\}, \quad t = 0, \dots, T.$$

Since $y \in \text{ri} K$ the model \tilde{K} dominates K . It suffices to show that \tilde{K} satisfies the no arbitrage condition. To this end let $c \in A(\tilde{K}) \cap \mathcal{A}_+$ and let $x \in \mathcal{A}$ be a self financing strategy that hedges c . Then we have

$$y_t \cdot (x_t - x_{t-1} + c_t) \leq 0 \quad \text{for all } t = 0, \dots, T$$

and hence

$$0 \leq \sum_{t=0}^T y_t \cdot c_t \leq - \sum_{t=0}^T y_t \cdot (x_t - x_{t-1}) = \sum_{t=0}^T x_{t-1} \cdot (y_t - y_{t-1}).$$

Since y is a martingale and x is adapted, the process

$$M_s := \sum_{t=0}^s x_{t-1} \cdot (y_t - y_{t-1}), \quad s = 0, \dots, T$$

is a local martingale by Theorem 1 in Jacod and Shirirjaev [10]. Moreover, M is a martingale by [10, Theorem 2] since $M_T \geq 0$ almost surely. Since $M_0 = 0$ we obtain

$$E \left[\sum_{t=0}^T y_t \cdot c_t \right] = 0$$

and hence $\sum_{t=0}^T y_t \cdot c_t = 0$ almost surely. Since y has strictly positive components and $c \in \mathcal{A}_+$ this implies $c_t = 0$ P -a.s. for all t . Thus the no arbitrage condition holds for \tilde{K} . \square

Using Theorem 5.2 we can restate Corollary 4.6 in terms of strictly consistent price systems.

COROLLARY 5.3. *Assume that C is a conical market model with the robust no arbitrage property. Assume further that \mathcal{F}_0 is a trivial σ -algebra. Let $c_T \in L^1(\Omega, \mathcal{F}_T, P, \mathbb{R}^d)$ and $p_0 \in \mathbb{R}^d$. Then the following are equivalent.*

- (i) $p = (p_0, 0, \dots, 0)$ is a superhedging premium for $c = (0, \dots, 0, c_T)$.
- (ii) $E(c_T \cdot y_T) \leq p_0 \cdot y_0$ for every bounded strictly consistent price system $y = (y_t)_{t=0}^T$.

Proof. By Corollary 4.6, it suffices to show that (ii) implies (i). Theorem 5.2 says that there exists a strictly consistent price system y^* for C . Then for $\varepsilon \in (0, 1]$ and for any consistent price system y the process $y^\varepsilon := \varepsilon y^* + (1 - \varepsilon)y$ defines a strictly consistent price system and for ε small enough $E(c_T \cdot y_T^\varepsilon) > p_0 \cdot y_0^\varepsilon$ if $E(c_T \cdot y_T) > p_0 \cdot y_0$. \square

EXAMPLE 5.4. *Consider the frictionless model C with a cash account defined by*

$$C_t(\omega) := \{(x^0, x^1) \in \mathbb{R}^{d+1} \mid x^0 + x^1 \cdot s_t(\omega) \leq 0\}, \quad t = 0, \dots, T,$$

where $(s_t)_{t=0}^T$ is a nonnegative d -dimensional adapted price process. If s is strictly positive, the robust no scalable arbitrage property coincides with the classical no arbitrage property. Theorem 5.2 implies then the existence of a strictly consistent price system y for C . Much as in Example 4.7, we can identify every strictly consistent price system with a density of an equivalent martingale measure for the price process s , and we can write the superhedging condition (ii) of Corollary 5.3 as

$$E_{P^*} [c_T^0 + c_T^1 \cdot s_T] \leq p_0^0 + p_0^1 \cdot s_0 \quad \forall P^* \in \mathcal{P},$$

where \mathcal{P} is the set of equivalent martingale measures. In particular, if $c_T^1 = 0$ and $p^1 = 0$, the condition can be written as

$$\sup_{P^* \in \mathcal{P}} E_{P^*} [c_T^0] \leq p_0^0,$$

in accordance with the classical result for claims with cash-delivery.

For general convex models, Theorem 5.2 can be stated in the following form.

THEOREM 5.5. *A convex market model C has the robust no scalable arbitrage property if and only if there exists a strictly positive martingale y such that $y_t \in \text{ri dom } \sigma_{C_t}$ almost surely for all t .*

Proof. Applying Theorem 5.2 to C^∞ , we get that C^∞ has the robust no-arbitrage property iff there exists a strictly consistent price system for C^∞ . The claim follows by noting that $\text{ri}(C_t^\infty)^* = \text{ri dom } \sigma_{C_t}$, by [24, Theorem 6.3], the bipolar theorem and [24, Corollary 14.2.1]. \square

COROLLARY 5.6. *Let S be a convex cost process on $\mathbb{R}^{d-1} \times \Omega$ in the sense of Example 2.3, such that $S_t(\cdot, \omega)$ is nondecreasing on \mathbb{R}^{d-1} for all t and ω , and let S^* denote the convex conjugate of S . The model*

$$C_t(\omega) = \{(x^0, x^1) \in \mathbb{R}^d \mid x^0 + S_t(x^1, \omega) \leq 0\}, \quad t = 0, \dots, T,$$

satisfies the robust no scalable arbitrage condition if and only if there exists a strictly positive adapted process s with $s_t \in \text{ri dom } S_t^*$ almost surely for all t such that one of the following equivalent conditions holds:

i) the frictionless model

$$\tilde{C}_t(\omega) := \{(x^0, x^1) \in \mathbb{R}^d \mid x^0 + s_t(\omega) \cdot x^1 \leq 0\}, \quad t = 0, \dots, T$$

is arbitrage free;

ii) there exists a probability measure P^* equivalent to P such that s is a martingale under P^* .

Proof. We get

$$\begin{aligned} \sigma_{C_t(\omega)}(z) &= \sup_{x^0, x^1} \{z^0 x^0 + z^1 \cdot x^1 \mid x^0 + S_t(x^1, \omega) \leq 0\} \\ &= \begin{cases} \sup_{x^1} \{z^1 \cdot x^1 - z^0 S_t(x^1, \omega)\} & \text{if } z^0 \geq 0, \\ +\infty & \end{cases} \\ &= \begin{cases} z^0 S_t^*(z^1/z^0, \omega) & \text{if } z^0 > 0, \\ 0 & \text{if } z^0 = 0 \text{ and } z^1 = 0, \\ +\infty & \end{cases} \end{aligned}$$

and thus, $\text{ri dom } \sigma_{C_t(\omega)} = \{(z^0, z^1) \mid z^0 > 0, z^1/z^0 \in \text{ri dom } S_t^*(\omega)\}$. By Theorem 5.5, the model C has the robust no scalable arbitrage property if and only if there exists a strictly positive martingale (y^0, y^1) such that $y_t^1/y_t^0 \in \text{ri dom } S_t^*$ almost surely for all t . The latter condition means that there is a strictly positive adapted process s with $s_t \in \text{ri dom } S_t^*$ such that $y^1 = y^0 s$. In particular s is a martingale under the equivalent martingale measure defined by $dP^*/dP := y_T^0/y_0^0$. Equivalence of i) and ii) follows from Theorem 5.2 as in Example 5.4. \square

REMARK 5.7. *By Theorem 3.3 and Lemma 4.1, the no arbitrage criterion in the above corollary guarantees that the set $A(C) \subset \mathcal{A}$ is closed in probability. This implies, in particular, the closedness of the set*

$$\begin{aligned} A^0(C) &:= \{c^0 \in \mathcal{M} \mid (c^0, 0) \in A(C)\} \\ &= \{c^0 \in \mathcal{M} \mid \exists x \in \mathcal{A} : \Delta x_t^0 + c_t^0 + S_t(\Delta x_t^1) \leq 0 \forall t, x_T = 0\}, \end{aligned}$$

where \mathcal{M} denotes the space of adapted real-valued processes. The set $A^0(C)$ consists of the claim processes with cash delivery that can be superhedged with zero cost. An alternative condition for the closedness of $A^0(C)$ was given in [21, Section 6]. The condition from [21] does not require the cost process S to be nondecreasing and it applies under portfolio constraints of the form $(x_t^0, x_t^1) \in \{0\} \times D_t(\omega)$ where D_t is an \mathcal{F}_t -measurable set-valued mapping with values in closed convex subsets of \mathbb{R}^{d-1} .

The equivalence of conditions i) and ii) in Corollary 5.6 is reminiscent of [20, Theorem 5.4] and, in the case of proportional transaction costs, also of Jouini and Kallal [11, Theorem 3.2.i]. If S^∞ is finite-valued then, by [20, Theorem 5.4], the condition

$$\text{cl } A^0(C^\infty) \cap \mathcal{M}_+ = \{0\},$$

where cl denotes the closure with respect to convergence in measure and \mathcal{M}_+ denotes the nonnegative elements of \mathcal{M} , implies the existence of an adapted process s and a measure P^* equivalent to P such that s is a martingale under P^* and $s_t \in \text{cl dom } S_t^*$ almost surely for all t .

6. Proof of Theorem 3.3. The proof of Theorem 3.3 requires some preparation. First we will use projection techniques similar to those in [31] to extract self-financing portfolio processes ending with 0 in the model C^∞ . For a convex market model C we consider the set

$$\mathcal{N}(C) := \{x \in \mathcal{A} \mid \Delta x_t \in C_t \text{ } P\text{-a.s. } \forall t = 0, \dots, T, \ x_T = 0\}.$$

The next lemma is a version of Lemma 5 in [15], see also Lemma 2.6 in [31].

LEMMA 6.1. *If C is a convex market model that has the robust no scalable arbitrage property then $\mathcal{N}(C^\infty) = \mathcal{N}(C^0)$.*

Proof. We have $\mathcal{N}(C^\infty) \supset \mathcal{N}(C^0)$ simply because $C_t^\infty \supset C_t^0$ almost surely for every t , so it suffices to prove the reverse. Let $x \in \mathcal{N}(C^\infty)$ and assume that $\Delta x_t \in C_t^\infty \setminus C_t^0$ on some set with positive probability for some $t = 0, \dots, T$. This contradicts the robust no scalable arbitrage property. Indeed, if \tilde{C} is a market model such that $C_t^\infty \setminus C_t^0 \subset \text{int } \tilde{C}_t$, we have $\Delta x_t \in \text{int } \tilde{C}_t$ on a nontrivial set, and then for any $e \in \mathbb{R}_+^d \setminus \{0\}$

$$\varepsilon_t(\omega) = \sup\{\varepsilon \in \mathbb{R} \mid \Delta x_t(\omega) + \varepsilon e \in \tilde{C}_t(\omega)\}$$

defines an \mathcal{F}_t -measurable \mathbb{R}_+^d -valued random variable (see e.g. [26, Theorem 14.37]) which does not vanish almost surely. By Lemma 4.1, this would mean that \tilde{C} violates the no-arbitrage condition. \square

For each $t \in \{0, \dots, T\}$ we denote by \mathcal{N}_t the set of all \mathcal{F}_t -measurable random vectors that may be extended to a portfolio in $\mathcal{N}(C^0)$, i.e.

$$\mathcal{N}_t := \{x_t \in L^0(\mathbb{R}^d, \mathcal{F}_t) \mid \exists x_{t+1}, \dots, x_T \text{ s.t. } (0, \dots, 0, x_t, \dots, x_T) \in \mathcal{N}(C^0)\}.$$

LEMMA 6.2. *Let C be a convex market model. Then for each $t \in \{0, \dots, T\}$ there is an \mathcal{F}_t -measurable map $N_t : \Omega \rightrightarrows \mathbb{R}^d$ whose values are linear subspaces of \mathbb{R}^d and $\mathcal{N}_t = L^0(N_t, \mathcal{F}_t)$.*

Proof. We consider first the larger set

$$\begin{aligned} \mathcal{M}_t = \{x_t \in L^0(\mathbb{R}^d, \mathcal{F}_t) \mid \exists x_{t+1}, \dots, x_T \text{ s.t. } \Delta x_s \in L^0(C_s^0, \mathcal{F}_s) \\ \forall s = t+1, \dots, T \text{ and } x_T = 0\} \end{aligned}$$

and show that there is an \mathcal{F}_t -measurable map $M_t : \Omega \rightrightarrows \mathbb{R}^d$ whose values are linear subspaces of \mathbb{R}^d and $\mathcal{M}_t = L^0(M_t, \mathcal{F}_t)$. Then the \mathcal{F}_t -measurable map $N_t : \Omega \rightrightarrows \mathbb{R}^d$ with $N_t(\omega) := M_t(\omega) \cap C_t^0(\omega)$ has the desired properties.

In order to obtain the maps M_t we argue by induction on T . For $T = 0$ we have $\mathcal{M}_T = M_T = \{0\}$. Now assume that the claim holds for any T -step model and consider the $T + 1$ -step model $(C_t)_{t=0}^T$. By the induction hypothesis there exist \mathcal{F}_s -measurable maps M_s whose values are linear subspaces of \mathbb{R}^d such that $\mathcal{M}_s = L^0(M_s, \mathcal{F}_s)$ for each $s \in \{1, \dots, T\}$. Note that $x_0 \in \mathcal{M}_0$ iff $x_0 \in M_1 + C_1^0$ almost surely. Indeed, $x_0 \in \mathcal{M}_0$ iff there exists an $x_1 \in \mathcal{M}_1$ such that $x_1 - x_0 \in C_1^0$ almost surely. The mapping $L(\omega) := M_1(\omega) + C_1^0(\omega)$ is \mathcal{F}_1 -measurable (see [26, Proposition 14.11(c)]) and its values are linear subspaces of \mathbb{R}^d and in particular closed. By theorem on page 135 of [32], there exists the largest closed \mathcal{F}_0 -measurable set-valued map M_0 such that $M_0 \subseteq L$ almost surely and $L^0(M_0, \mathcal{F}_0) = L^0(L, \mathcal{F}_0) = \mathcal{M}_0$. Clearly $M_0(\omega)$ is a linear subspace of \mathbb{R}^d for each ω . \square

For each ω we denote by $N_t^\perp(\omega)$ the orthogonal complement of $N_t(\omega)$ in \mathbb{R}^d . Then $N_t^\perp : \Omega \rightrightarrows \mathbb{R}^d$ is an \mathcal{F}_t -measurable (see [26, 14.12(e)]) map whose values are linear subspaces of \mathbb{R}^d .

LEMMA 6.3. *Let C be a convex market model and let $c \in A(C)$. Then there exists a process $x^\perp \in \mathcal{A}$ such that $x_t^\perp \in N_t^\perp$, $\Delta x_t^\perp + c_t \in C_t$ for all $t = 0, \dots, T$ and $x_T^\perp = 0$.*

Proof. We will prove the existence of the process x^\perp by induction on T . For $T = 0$ we have $x_T^\perp := x_T = 0$. Assume that the claim holds for any T -step model and consider the $T + 1$ -step model $(C_t)_{t=0}^T$. Let $c \in A(C)$ and let $x \in \mathcal{A}$ be such that $\Delta x_t + c_t \in C_t$ for all $t = 0, \dots, T$ and $x_T = 0$. We denote by x_0^0 the \mathcal{F}_0 -measurable projection of x_0 on N_0 and by $(x_1^0, \dots, x_{T-1}^0, 0)$ an extension of x_0^0 to a self-financing portfolio process in the model C^0 (the existence of such an extension is given by Lemma 6.2). Then

$$y_t := x_t - x_t^0, \quad t = 0, \dots, T-1, \quad y_T := 0$$

defines an adapted process y with $y_0 = x_0 - x_0^0 \in N_0^\perp$ almost surely. Moreover, y hedges c . Indeed, since $\Delta x_t^0 \in C_t^0$ we have

$$\Delta y_t + c_t = \Delta x_t + c_t - \Delta x_t^0 \in C_t - C_t^0 = C_t \quad P\text{-a.s.}$$

for all $t = 0, \dots, T$.

The process $(y_1, \dots, y_{T-1}, 0)$ hedges the claim $(c_1 - y_0, c_2, \dots, c_T)$ in the T -step model $(C_t)_{t=1}^T$. Thus by the induction hypothesis there is a process $y^\perp = (y_1^\perp, \dots, y_{T-1}^\perp, 0)$ such that $y_t^\perp \in N_t^\perp$ almost surely and y^\perp hedges $(c_1 - y_0, c_2, \dots, c_T)$. Then the process $x^\perp := (y_0, y_1^\perp, \dots, y_{T-1}^\perp, 0)$ has the required properties. \square

The following lemma was used first in Kabanov and Stricker [17], it is also documented e.g. in [31], [15], [9] and [8]. We refer to these works for the proof.

LEMMA 6.4. *Let $(x^n)_{n=1}^\infty$ be a sequence of random vectors in $L^0(\mathbb{R}^d, \mathcal{F})$ such that $\liminf_n |x^n| < \infty$ P -almost surely. Then there exists an \mathcal{F} -measurable increasing \mathbb{N} -valued random sequence $(\tau^n)_{n=1}^\infty$ such that $(x^{\tau^n})_{n=1}^\infty$ converges almost surely to some $x \in L^0(\mathbb{R}^d, \mathcal{F})$.*

Proof of Theorem 3.3. By Lemma 4.1, the closedness of $A_T(C)$ is equivalent to the closedness of $A(C)$. By Lemma 6.3, we may assume that $x_t \in N_t^\perp$ for all $t = 0, \dots, T-1$ in the definition of $A(C)$.

In order to prove the closedness of $A(C)$ we will prove the following more precise statement: Let $(c^n)_{n=1}^\infty$ be a sequence of claim processes in $A(C)$ such that $c_t^n \rightarrow c_t$ for all t in measure and let $(x^n)_{n=1}^\infty$ be any sequence in \mathcal{A} such that $\Delta x_t^n + c_t^n \in C_t$, $x_t^n \in N_t^\perp$ for all t and $x_T^n = 0$. Then there exists an \mathcal{F} -measurable random subsequence $(\tau^n)_{n=1}^\infty$ of \mathbb{N} and an $x \in \mathcal{A}$ such that $x_t^{\tau^n} \rightarrow x_t$ almost surely, $\Delta x_t + c_t \in C_t$, $x_t \in N_t^\perp$ for all t and $x_T = 0$. In particular, $c \in A(C)$.

The proof will follow by induction on T . For $T = 0$ the statement is obvious since the set $L^0(C_0, \mathcal{F}_0)$ is closed in $L^0(\mathbb{R}^d, \mathcal{F}_0)$ and any hedging strategy is identical 0. Now assume that the statement holds for any T -step model and consider a $T + 1$ -step model $C = (C_t)_{t=0}^T$. Let $c^n \in A(C)$, $n = 0, 1, \dots$ be such that $c_t^n \rightarrow c_t \in \mathcal{A}$ for all t in measure. By passing to a subsequence if necessary, we may assume that $c_t^n \rightarrow c_t$ for all t almost surely. Let $x^n \in \mathcal{A}$, $n = 0, 1, \dots$ be such that $\Delta x_t^n + c_t^n \in C_t$, $x_t^n \in N_t^\perp$ for all t and $x_T^n = 0$.

Case 1: the sequence $(x_0^n)_{n=1}^\infty$ is almost surely bounded. In this case, we can apply Lemma 6.4 to find an \mathcal{F}_0 -measurable random sequence $(\sigma^n)_{n=1}^\infty$ such that $x_0^{\sigma^n}$ converges to some $x_0 \in L^0(\mathbb{R}^d, \mathcal{F}_0)$ almost surely. Then $x_0 \in L^0(N_0^\perp, \mathcal{F}_0)$ since each $N_0^\perp(\omega)$ is a closed subspace of \mathbb{R}^d . Moreover, the sequence of claim processes

$$\tilde{c}^n := (c_1^{\sigma^n} - x_0^{\sigma^n}, c_2^{\sigma^n}, \dots, c_T^{\sigma^n}), \quad n \in \mathbb{N}$$

belongs to $A((C_t)_{t=1}^T)$ with the hedging sequence $\tilde{x}^n := (x_1^{\sigma^n}, \dots, x_{T-1}^{\sigma^n}, 0)$. Since $\tilde{c}^n \rightarrow (c_1 - x_0, c_2, \dots, c_T)$ almost surely, we apply the induction hypothesis to the T -step model $(C_t)_{t=1}^T$ and obtain an \mathcal{F} -measurable random subsequence $(\tau^n)_{n=0}^\infty$ of $(\sigma^n)_{n=0}^\infty$ and an $(\mathcal{F}_t)_{t=1}^T$ -adapted process $\tilde{x} = (x_1, \dots, x_T)$ such that $\tilde{x}_t^{\tau^n} \rightarrow \tilde{x}_t$ for all t almost surely, \tilde{x} hedges \tilde{c} and $x_t \in N_t^\perp$ for $t = 1, \dots, T$. This proves the claim, since $x_t^{\tau^n} \rightarrow x_t$ almost surely for all $t = 0, \dots, T$ and the process $x := (x_0, \dots, x_T)$ has the desired properties. Indeed, x is adapted, $x_t \in N_t^\perp$ for all t , $x_T = 0$ and we have

$$\Delta x_t + c_t \in C_t \quad P\text{-a.s. for all } t = 0, \dots, T$$

since $C_t(\omega)$ is a closed subset of \mathbb{R}^d for each ω .

Case 2: the sequence $(x_0^n)_{n=1}^\infty$ is not almost surely bounded. We will show that this leads to a contradiction with the robust no scalable arbitrage property. Indeed, assume that the set

$$A := \{\omega \in \Omega \mid \liminf |x_0^n(\omega)| = \infty\}$$

has positive probability and consider the sequences

$$\hat{x}^n := \alpha^n x^n \quad \text{and} \quad \hat{c}^n := \alpha^n c^n, \quad n \in \mathbb{N},$$

where $\alpha^n = \frac{\chi_A}{\max\{|x_0^n|, 1\}}$. The processes $(\hat{x}^n)_{n=1}^\infty$ and $(\hat{c}^n)_{n=1}^\infty$ are adapted, $\hat{x}_t^n \in L^0(N_t^\perp, \mathcal{F}_t)$ for all t , $\hat{x}_T^n = 0$, $\hat{c}_t^n \rightarrow 0$ almost surely for each t and the sequence $(\hat{x}_0^n)_{n=0}^\infty$ is almost surely bounded. Moreover, we have

$$\hat{x}_t^n - \hat{x}_{t-1}^n + \hat{c}_t^n \in \alpha^n C_t \quad P\text{-a.s.}, \quad (6.1)$$

where $\alpha^n C_t \subset C_t$ since C_t is convex, $0 \in C_t$ and $|\alpha^n| \leq 1$. Thus we have the same situation as in case 1, and using the same reasoning we obtain an \mathcal{F} -measurable random sequence $(\tau^n)_{n=0}^\infty$ and an adapted process x such that $\hat{x}^{\tau^n} \rightarrow x$ almost surely, $x_t \in N_t^\perp$ for all t and $x_T = 0$. Moreover, since $\alpha^n \rightarrow 0$ a.s., (6.1) and (3.2) imply

$$\Delta x_t \in C_t^\infty \quad P\text{-a.s. for all } t = 0, \dots, T.$$

Thus $x \in \mathcal{N}(C^0)$, by Lemma 6.1, and then $x_0 \in N_0$ almost surely, by Lemma 6.2. Since also $x_0 \in N_0^\perp$, we have $x_0 = 0$ almost surely. But we also have $|\hat{x}_0^n| \rightarrow 1$ and hence $|x_0| = 1$ on the nontrivial set A , which is a contradiction. Thus, case 2 cannot occur under the robust no scalable arbitrage condition. \square

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