Two-factor models for index linked bond portfolios

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April 11, 2012

Abstract

This paper studies portfolios of index linked bonds in terms yield to maturity and the underlying index. We find that, with appropriate parameterization, the two factors capture consistently over 97% of monthly return variations on fixed rate government bonds, inflation linked government bonds and investment grade corporate bonds. We derive simple return formulas that are easy to calibrate and to use in simulation based portfolio analysis.

1 Introduction

Bond portfolios are subject to a large number of risk factors. Even in the case of default-free fixed rate bonds, the portfolio return depends in general on the whole yield curve. Bond returns can also be described in terms of yield to maturity, which reduces return uncertainty into a single portfolio-specific factor. Such simple descriptions are easy to understand and to model, which makes them useful in portfolio analysis and risk management; see Fabozzi [2010] for a comprehensive treatment of the subject.

This paper shows that index-linked bonds allow for an equally simple description of returns over holding periods of fixed length. We derive simple return formulas in terms of the yield to maturity and the underlying index. The two risk factors are found to explain consistently over 97% of monthly return variations on fixed-rate government, inflation linked and corporate bonds over the past decade including the recent financial crisis. For government bonds, the underlying index reduces to a constant whereas for corporate bonds, it is the

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survivor index which, in turn, can be approximated by credit spreads. The models provide simple alternatives for modeling returns on bond indices for which yield and index data is available.

Any stochastic model for the yield and the underlying index translates through our return formulas into a stochastic model for portfolio returns. Such models are well suited for practical portfolio optimization and asset-liability management where large sets of scenarios over several time periods are often required. The models described in this paper have been applied to strategic portfolio optimization and the evaluation of insurance liabilities in Hilli et al. [2011a,b].

2 Returns on bond portfolios

Consider a bond or a bond portfolio with outstanding payments at times $t_1 < t_2 < \cdots < t_N$. We assume that the payments are tied to an underlying index $I$ so that the payment at time $t_n$ will be $c_n I_{t_n}$ where $c_n$ is a given nominal payment. The portfolio’s yield to maturity (or redemption yield) $Y_t$ at time $t < t_1$ is defined as the solution of the equation

$$P_t = \sum_{n=1}^{N} e^{-Y_t(t_n-t)} c_n I_t,$$

(1)

where $P_t$ is the portfolio’s market price and $c_n I_t$ are the nominal payments outstanding at time $t$. In the case of inflation linked bonds where $I$ is the consumer price index, $Y$ is known as the real yield. Fixed rate governments bonds correspond to a constant index. In survivor bonds, the underlying index $I$ is defined as the fraction of survivors in a given population. Such bonds have attracted a lot of attention in the recent literature on longevity risk management; see e.g. Biffis and Blake [2010]. The index $I$ can also be used to model default losses in the presence of counterparty risk. This will be studied in Section 5 below.

The anticipated development of the index $I$ affects the market price $P_t$ of the portfolio. In the case of inflation linked bonds, the amount $c_n I_{t_n}$ that will be received at time $t_n$ is usually expected to be higher than the nominal $c_n I_t$ at time $t$. This is reflected as a higher price and lower yield than for a fixed rate bond with a similar nominal payment structure. In the case of defaultable bonds, on the other hand, the eventual payments $c_n I_{t_n}$ may be strictly less than the outstanding payments $c_n I_t$ so the price is typically lower and the yield higher than for a corresponding “default-free” bond.

Our aim is to describe portfolio returns without explicit account of the outstanding payments. To this end, we will regard the price $P_t$ as a function of time $t$, yield $Y_t$ and the index $I_t$. We have $P_t = P(t,Y_t,I_t)$ where the function $P$ is defined for any $t$, $Y$ and $I$ by

$$P(t,Y,I) = \sum_{n=1}^{N} e^{-Y(t_n-t)} c_n I.$$
The log-return on the portfolio over a holding period \([t, s]\) can be approximated by

\[\Delta \ln P \approx \ln P(s, Y_s, I_s) - \ln P(t, Y_t, I_t).\]  

(2)

The approximation is exact if there are no payments and no portfolio updates during the holding period, i.e. if \(t_1 \geq s\). If \(s > t_n\), the \(n\)th payment has been collected and reinvested in the portfolio. An error may occur if the payment schedule of the updated portfolio differs essentially from the original one. If the payments during \([t, s]\) amount to a small fraction of all the outstanding payments (if length of the holding period is small compared to the maturity of the bonds in the portfolio), the resulting error will be small.

Formula (2) gives the log-return in terms of the time, yield and the index, but the definition of the function \(P\) involves the whole payment structure \((c_n)_{n=0}^N\). This dependence is greatly simplified in Taylor-approximations of the logarithmic price index. The first order approximation can be written as

\[\Delta \ln P \approx \frac{1}{P_t} \frac{\partial P}{\partial t}(t, Y_t, I_t) \Delta t + \frac{1}{P_t} \frac{\partial P}{\partial Y}(t, Y_t, I_t) \Delta Y + \frac{1}{P_t} \frac{\partial P}{\partial I}(t, Y_t, I_t) \Delta I\]  

(3)

where \(\Delta t = s - t\), \(\Delta Y = Y_s - Y_t\) and \(\Delta I = I_s - I_t\). The first two terms are quite familiar. Indeed, we have

\[\frac{1}{P_t} \frac{\partial P}{\partial t}(t, Y_t, I_t) = Y_t\]  

and

\[\frac{1}{P_t} \frac{\partial P}{\partial Y}(t, Y_t, I_t) = -D_t,\]

where

\[D_t = \frac{1}{P_t} \sum_{n=1}^{N} (t_n - t)e^{-Y_t(t_n - t)c_nI_t}\]

is the duration of the portfolio at time \(t\). The last term, where

\[\frac{1}{P_t} \frac{\partial P}{\partial I}(t, Y_t, I_t) = \frac{1}{P_t} \sum_{n=1}^{N} e^{-Y_t(t_n - t)c_n} = \frac{1}{P_t} \sum_{n=1}^{N} e^{-Y_t(t_n - t)}c_nI_t = \frac{1}{I_t}\]

arises from changes in the underlying index during the holding period \([s, t]\). The linear approximation (3) can be thus be written as

\[\Delta \ln P \approx Y_t \Delta t - D_t \Delta Y + \Delta I\]

(4)

The portfolio return over the holding period \([t, s]\) is then approximated by

\[\frac{P_s}{P_t} \approx \exp \left( Y_t \Delta t - D_t \Delta Y_t + \frac{\Delta I}{I_t} \right).\]

This gives the return in terms of only two risk factors, the yield \(Y\) and the index \(I\), while the payment structure is described by a single number, the duration \(D_t\). In a portfolio that is actively managed to maintain (approximately) a fixed duration, the main return uncertainty comes from uncertainties in the
development of the yield and the index. Such portfolios are typical ingredients in strategic asset management of many long-term investors. Moreover, common bond-index data is collected according to such portfolio updating rules.

In some cases, accuracy can be improved with a second order approximation. Denoting the convexity by

\[ C_t = \frac{1}{P} \frac{\partial^2 P}{\partial Y^2} = \frac{1}{P} \sum_{n=1}^{N} (t_n - t)^2 e^{-Y(t_n - t)} c_n I_t, \]

we get

\[ \frac{\partial^2 \ln P}{\partial Y^2} = 1, \quad \frac{\partial^2 \ln P}{\partial I^2} = C_t - D_t^2, \quad \frac{\partial^2 \ln P}{\partial I^2} = -\frac{1}{I_t^2}, \]

while all other second-order derivatives of the logarithmic price are zero. The second-order approximation of the log-return becomes

\[ \Delta \ln P \approx Y_s \Delta t - D_t \Delta Y + \Delta \ln I + \frac{1}{2} (C_t - D_t^2)(\Delta Y)^2 - \frac{1}{2} \frac{(\Delta I)^2}{I_t^2}. \]

This differs from the first order formula (4) only by the addition of two quadratic terms and by the replacement of \( Y_t \) by \( Y_s \) in the first term. Using the second order Taylor approximation of the logarithm,

\[ \Delta \ln I \approx \frac{\Delta I}{I_t} - \frac{1}{2} \frac{(\Delta I)^2}{I_t^2}, \]

we can simplify the formula to

\[ \Delta \ln P \approx Y_s \Delta t - D_t \Delta Y + \Delta \ln I + \frac{1}{2} (C_t - D_t^2)(\Delta Y)^2. \] (5)

Reduction of most of the second-order terms in the quadratic approximation suggests that the logarithm of the price is not far from being linear in time, the yield and the log-index.

Our analysis is closely related to Christensen and Sørensen [1994] and Chance and Jordan [1996]. While Christensen and Sørensen [1994] analyzed price sensitivities with respect to a general level of interest rates, we have focused explicitly on a given portfolio and its yield to maturity. Chance and Jordan [1996] developed second order approximations of bond returns in absolute terms rather than relative terms described above with the logarithm. In Chance and Jordan [1996] this resulted in more complicated formulas with five different terms for the effects of time and yield changes only. Neither Christensen and Sørensen [1994] nor Chance and Jordan [1996] considered index linked bonds.

3 Fixed rate government bonds

In the case of fixed rate default-free bonds, the underlying index is constant so the second order approximation becomes

\[ \Delta \ln P \approx Y_s \Delta t - D_t \Delta Y + \frac{1}{2} (C_t - D_t^2)(\Delta Y)^2. \] (6)
This is similar to the model studied e.g. in Ilmanen [1992] but there the time component was ignored. Chance and Jordan [1996] incorporated the time component but their model, like that of Ilmanen [1992], was based on a Taylor-approximation of the price instead of its logarithm.

3.1 Empirical results

We study the accuracy of the above approximations with monthly returns on fixed rate government bonds. Our dataset covers end of month observations of the Barclays’ market capitalization weighted total return indices, durations and yields for France, Germany, Italy, United Kingdom, United States and the Euro area\(^1\). The length of the time series for each country is given at the bottom of Table 1.

We fit the following two models to the data

\[
\text{Model 1: } \Delta \ln P_s \approx c + Y_s \Delta t - D_t \Delta Y_s, \\
\text{Model 2: } \Delta \ln P_s \approx c + Y_s \Delta t - D_t \Delta Y_s + \gamma (\Delta Y_s)^2
\]

by ordinary least squares regression. The parameters \(c\) and \(\gamma\) are estimated by ordinary least squares. The regression statistics provide us with diagnostic tools to evaluate the performance of the proposed models. By comparing the fits of the two models we can assess the significance of the quadratic term in explaining the returns.

Table 1 displays the estimation results for the two model specifications. Model 1 already provides an almost perfect fit to the total return data with \(R^2\) values ranging from 99.6\% to 99.8\%. The model fit has been consistent over time and the residuals have remained marginal even during the recent financial crisis. The \(R^2\) values being close to 100\% there is not much room for improvement when adding the second-order term. The estimated coefficients \(\gamma\) in Model 2 are generally significant at a 5\% level but, consistently with the findings of [Chance and Jordan, 1996], the improvements in the \(R^2\) statistics are marginal, less than 0.02\% points in all the studied markets. The Partial-\(R^2\) statistic is defined as the \(R^2\)-statistic obtained by regressing the residual of Model 1 with the quadratic term. The quadratic term of Model 2 explains less than 5.11\% of the residual variance of Model 1. The estimated constant \(c\) in Model 1 deviates from zero at 5\% significance level, but with varying signs, this may indicate that the model has not entirely captured all the systematic return components. The addition of the quadratic term in Model 2 mitigates this effect to some extent.

Our results are consistent with Ilmanen [1992] who finds that the explanatory power of duration has increased over time. In his empirical study, Ilmanen [1992] found that the duration explained 80\% to 90\% of return variance of fixed rate government bonds during the 1980s. The nearly 100\% \(R^2\)-values for Model 1 during the past decade suggest that the trend has continued. It should be noted, however, that our results are not strictly comparable since our model explains the log-returns and it includes the time component.

\(^1\)Further information is available online at https://ecommerce.barcap.com/indices/index.dxml
Table 1: Regression statistics for fixed-rate government bonds

<table>
<thead>
<tr>
<th></th>
<th>FRA</th>
<th>GER</th>
<th>IT</th>
<th>UK</th>
<th>US</th>
<th>EURO</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Model 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 * c</td>
<td>0.0147</td>
<td>0.0083</td>
<td>0.0084</td>
<td>-0.0195</td>
<td>-0.0202</td>
<td>0.015</td>
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<tr>
<td>(3.5842)</td>
<td>(1.7215)</td>
<td>(1.9339)</td>
<td>(-4.8403)</td>
<td>(-3.0658)</td>
<td>(4.1467)</td>
<td></td>
</tr>
<tr>
<td>R²</td>
<td>99.76%</td>
<td>99.64%</td>
<td>99.68%</td>
<td>99.84%</td>
<td>99.71%</td>
<td>99.79%</td>
</tr>
<tr>
<td><strong>Model 2</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 * c</td>
<td>0.007</td>
<td>0.0007</td>
<td>-0.0001</td>
<td>-0.0256</td>
<td>-0.0277</td>
<td>0.01</td>
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<td>(1.4255)</td>
<td>(0.1186)</td>
<td>(-0.0178)</td>
<td>(-5.7509)</td>
<td>(-3.6689)</td>
<td>(2.2687)</td>
<td></td>
</tr>
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<td>γ</td>
<td>24.4889</td>
<td>24.1545</td>
<td>32.5155</td>
<td>4.8396</td>
<td>11.1262</td>
<td>17.031</td>
</tr>
<tr>
<td>(2.7949)</td>
<td>(2.2625)</td>
<td>(2.5933)</td>
<td>(3.0699)</td>
<td>(1.9703)</td>
<td>(1.9131)</td>
<td></td>
</tr>
<tr>
<td>R²</td>
<td>99.77%</td>
<td>99.65%</td>
<td>99.70%</td>
<td>99.85%</td>
<td>99.72%</td>
<td>99.79%</td>
</tr>
<tr>
<td><strong>Partial-R²</strong></td>
<td>5.11%</td>
<td>3.41%</td>
<td>4.43%</td>
<td>2.63%</td>
<td>2.84%</td>
<td>2.46%</td>
</tr>
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</table>

**Data start**
- 1997-12
- 1997-12
- 1997-12
- 1980-12
- 1998-12
- 1997-12

**Data end**
- 2010-03
- 2010-03
- 2010-03
- 2010-03
- 2010-03
- 2010-03

**Notes.** The table contains estimation results for the models:

Model 1: $\Delta \ln P_s \approx c + Y_s \Delta t - D_t \Delta Y_s$,

Model 2: $\Delta \ln P_s \approx c + (Y_s + \pi_s) \Delta t - D_t \Delta Y_s + \gamma (\Delta Y_s)^2$,

Numbers in parentheses are the $t$-statistics for the estimated coefficients.

**Partial-R²** is defined as the $R²$-statistic obtained by regressing the residual of Model 1 with the quadratic term.

4 Inflation linked bonds

In inflation linked bonds, principal and coupon payments are usually tied to a time-lagged consumer price index. In other words, the underlying index $I_t$ at time $t$ is defined as the consumer price index reported for some time before $t$. The time-lag is known as the **indexation lag**. We refer the reader to [Deacon et al., 2004, Chapter 2] for a general account on inflation linked bonds.

For inflation linked bonds, the first-order approximation (4) can be written as

\[ \Delta \ln P_t \approx (Y_s + \pi_s) \Delta t - D_t \Delta Y_s, \]

where $\pi_s$ is the **annualized rate of inflation** over the period $[t - \delta, s - \delta]$ where $\delta$ denotes the indexation lag.

4.1 Empirical results

In our empirical study, we consider the two model specifications

Model 1: $\Delta \ln P_t \approx c + Y_s \Delta t - D_t \Delta Y_s$,

Model 2: $\Delta \ln P_t \approx c + (Y_s + \pi_s) \Delta t - D_t \Delta Y_s$.

The first one ignores the third term in the Taylor-approximation (3). This allows us to evaluate the significance of the changes in the outstanding payments in explaining portfolio returns.
We use monthly observations of Barclays inflation linked government bond index data. The data consists of total return indices, yields and durations on portfolios of inflation linked government bonds for Canada, France, South Africa, Sweden, United Kingdom and United States\(^2\). The bonds’ cash flows are linked to the evolution of country specific inflation indices (usually the general consumer price index). The monthly time series of appropriate inflation indices were obtained from Eurostat and the national statistical authorities’ websites. Since the indexation lag is not specified in the data, we select the lag \(\delta\) that gives the best fit to historical data. In all the studied countries, the data strongly supports one specific choice of the lag. The lags are given in Table 2 along with the regression statistics for the two models in the six markets.

The inclusion of the inflation-term in Model 2 results in a substantial improvement over Model 1. Model 2 gives a good fit to the total return data with \(R^2\) values ranging from 97.6% to 99.8%. The lowest \(R^2\) value for South Africa is mainly caused by one outlier observation caused by pricing distortions in the local conventional bond market in 2002 and the subsequent inversion of the break-even inflation curve.

The high values of Partial-\(R^2\) statistics mean that the inflation-term is able to explain most of the residual variance of Model 1. The reduction in the residuals between Model 2 and Model 1 is caused solely by the incorporation of the inflation term. The fact that the estimated constant terms in Model 2 are generally insignificant is another indication that the model is well specified.

Table 2: Regression statistics for inflation-indexed government bonds

<table>
<thead>
<tr>
<th></th>
<th>CAN</th>
<th>FRA</th>
<th>SA</th>
<th>SWE</th>
<th>UK</th>
<th>US</th>
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</thead>
<tbody>
<tr>
<td>Model 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100*c</td>
<td>0.1642</td>
<td>0.1328</td>
<td>0.4935</td>
<td>0.1114</td>
<td>0.2073</td>
<td>0.1966</td>
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<tr>
<td>t-stat</td>
<td>(6.201)</td>
<td>(5.544)</td>
<td>(9.367)</td>
<td>(3.678)</td>
<td>(7.611)</td>
<td>(5.888)</td>
</tr>
<tr>
<td>(R^2)</td>
<td>97.48%</td>
<td>95.46%</td>
<td>78.6%</td>
<td>91.29%</td>
<td>96.06%</td>
<td>94.41%</td>
</tr>
<tr>
<td>Model 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100*c</td>
<td>0.0046</td>
<td>-0.0098</td>
<td>-0.0069</td>
<td>0.0087</td>
<td>-0.0151</td>
<td>-0.0004</td>
</tr>
<tr>
<td>t-stat</td>
<td>(0.628)</td>
<td>(-0.839)</td>
<td>(-0.389)</td>
<td>(1.465)</td>
<td>(-1.918)</td>
<td>(-0.066)</td>
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<tr>
<td>(R^2)</td>
<td>99.80%</td>
<td>98.91%</td>
<td>97.58%</td>
<td>99.66%</td>
<td>99.67%</td>
<td>99.82%</td>
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<tr>
<td>Partial-(R^2)</td>
<td>92.22%</td>
<td>76.92%</td>
<td>88.69%</td>
<td>96.14%</td>
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<td>96.78%</td>
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<td>Indexation-lag</td>
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<td>3</td>
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<td>2010-03</td>
<td>2010-03</td>
<td>2010-03</td>
</tr>
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Notes. The table contains estimation results for the models:

Model 1: \(\Delta \ln P_s \approx c + Y_s \Delta t - D_t \Delta Y_s\),

Model 2: \(\Delta \ln P_s \approx c + (\pi_s + \pi_s) \Delta t - D_t \Delta Y_s\),

Numbers in parentheses are the \(t\)-statistics for the estimated coefficients.

\(^2\)Further information is available online at https://ecommerce.barcap.com/indices/index.dxml
5 Corporate bonds

Because of possible defaults, the outstanding payments of a portfolio of corporate bonds may decrease during a holding period. We will assume that the portfolio consists of bonds from \( M \) different issuers so that the notional payments are of the form

\[
c_n = \sum_{i=1}^{M} c_{in},
\]

where \( c_{in} \) is the payment an issuer \( i \) should make at time \( t_n \). The outstanding payments at any given time \( t \) consist of the notionals of those issuers that have not defaulted by time \( t \). The portfolio can thus be viewed as an index linked bond with the underlying index

\[
I_t^M = \frac{1}{c_n} \sum_{i=1}^{M} c_{in} \mathbb{1}_{\{\tau_i > t\}},
\]

where \( \tau_i \) is the default-time of issuer \( i \) and \( \mathbb{1}_{\{\tau_i > t\}} \) equals zero or one depending on whether the issuer has defaulted by \( t \) or not. In order to apply the models of Section 2 one could model the index \( I^M_t \) directly. We will, however, derive a further simplification where \( I^M_t \) is approximated by credit spreads for which extensive data is readily available.

As e.g. in Lando [1998] and Schönbucher [1998], we will assume that the defaults occur according to a Cox process with stochastic intensity \( \lambda \) so that, conditionally on \( \lambda \), the probability that issuer \( i \) survives until \( t \) is given by

\[
E[\mathbb{1}_{\{\tau_i > t\}} | \lambda] = I_t,
\]

where

\[
I_t = \exp \left( - \int_0^t \lambda_u du \right).
\]

Arguing like Jarrow et al. [2005] (who studied defaultable zero coupon bonds), one can show that the default event risk can be diversified away when the number of issuers in the portfolio is increased. It is assumed that all issuers have the same default intensity \( \lambda \) and that, conditionally on \( \lambda \), the defaults are independent.

**Proposition 1.** We have \( E[I_t^M | \lambda] = I_t \) and

\[
E(I_t^M - I_t)^2 \leq \frac{\sum_{i=1}^{M} (c_{in})^2}{[\sum_{i=1}^{M} (c_{in})]^2}.
\]

**Proof.** When all issuers have default intensity \( \lambda \), we get

\[
E[I_t^M | \lambda] = \frac{\sum_{i=1}^{M} c_{in} E[\mathbb{1}_{\{\tau_i > t\}} | \lambda]}{\sum_{i=1}^{M} c_{in}} = \frac{\sum_{i=1}^{M} c_{in} I_t}{\sum_{i=1}^{M} c_{in}} = I_t.
\]
Let \( w_i = c_n^i / \sum_{i=1}^M c_n^i \). By independence,

\[
E(I_t^M - I_t)^2 = E \left( \sum_{i=1}^M w_i (\mathbb{1}_{\tau_i > t} - I_t) \right)^2 \\
= \sum_{i=1}^M w_i^2 E(\mathbb{1}_{\tau_i > t} - I_t)^2,
\]

where both \( \mathbb{1}_{\tau_i > t} \) and \( I_t \) are between zero and one.

By Proposition 1, the outstanding payments of a well diversified portfolio of defaultable bonds can be approximated in the \( L^2 \)-norm by \( c_n I_t \), where

\[
I_t = \exp \left( - \int_0^t \lambda_u du \right)
\]

For example, if all issuers have equal weight in the portfolio, the upper bound on the variance in Proposition 1 converges to zero as the number \( M \) of issuers is increased. For such a portfolio, formula (5) gives the approximation

\[
\Delta \ln P \approx Y_s \Delta t - D_t \Delta Y - \int_t^s \lambda_u du + \frac{1}{2} (C_t - D_t^2)(\Delta Y)^2.
\]

In order to estimate the integral term, we will assume that there exists a risk neutral measure \( Q \) under which market prices of traded securities are equal to the expectations of their discounted cash-flows; see e.g. [Duffie and Singleton, 2003, Chapter 5] or [Brigo and Mercurio, 2006, Chapter 2]. Denoting the default-free short rate by \( r \) and applying Proposition 1, we get

\[
P_t \approx E^Q \sum_{n=1}^N \exp \left( - \int_t^{t_n} r_u du \right) c_n I_{t_n}
\]

where \( r_u^d = r_t + \lambda_t \) is the default adjusted short rate. This is analogous to the reduced form pricing formulas obtained in Lando [1998], Schönbucher [1998], Duffie and Singleton [1999]. The default intensity can thus be expressed as the short spread \( \lambda_t = r_u^d - r_t \).

Since neither \( r_u^d \) nor \( r_t \) is observable in practice, we will approximate them by yields on short maturity corporate and government bonds, respectively. This gives the approximation

\[
\Delta \ln P \approx (Y_s - S_s) \Delta t - D_t \Delta Y + \frac{1}{2} (C_t - D_t^2)(\Delta Y)^2.
\]

where \( S_s \) is the short-maturity credit-spread.
5.1 Empirical results

We will study the two models

Model 1: \[ \Delta \ln P_s \approx c + Y_s \Delta t - D_t \Delta Y_s, \]

Model 2: \[ \Delta \ln P_s \approx c + (Y_s - \alpha S_s) \Delta t - D_t \Delta Y_s, \]

which differ by the term \( \alpha S_s \), where \( S_s \) is the yield spread on short maturity bond indices. We allow the constant \( \alpha \) to deviate from one since even with the shortest maturity available (1-3 years in our numerical examples below), the spread \( S_s \) is likely to overestimate the mean loss rate \( l_s \). Indeed, the yield on corporate bonds contains a premium for unexpected variations of the loss rate that might occur before maturity. In addition, there may be a premium for default event risk which remains when the diversification is less than perfect.

We fit the models to monthly observations of Merrill-Lynch investment grade corporate bond index data. Our dataset covers the total return indices, yields and durations of market capitalization weighted investment grade corporate bond portfolios of maturities 1–3, 3–5, 5–7 and 7–10 years from the US and European markets. As a proxy for the instantaneous loss rate we will use the yield spread of 1–3 year maturity bonds.

Figures 1 and 2 display historical monthly values of \( S_s \Delta t \) and residual of Model 1 for European and US corporate bond markets during 1996/2–2010/1. The spread \( S_s \) is obtained as the difference between the yields of corporate and government bond portfolios of 1-3 years to maturity.

Table 3: Return regressions for the EU corporate bond portfolios.

<table>
<thead>
<tr>
<th></th>
<th>Model 1</th>
<th></th>
<th></th>
<th></th>
<th>Model 2</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1-3 Y</td>
<td>3-5 Y</td>
<td>5-7 Y</td>
<td>7-10 Y</td>
<td>1-3 Y</td>
<td>3-5 Y</td>
<td>5-7 Y</td>
<td>7-10 Y</td>
</tr>
<tr>
<td>100 * c</td>
<td>-0.022</td>
<td>-0.038</td>
<td>-0.064</td>
<td>-0.094</td>
<td>0.013</td>
<td>0.007</td>
<td>0.028</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td>(-3.431)</td>
<td>(-5.014)</td>
<td>(-4.357)</td>
<td>(-5.754)</td>
<td>(1.811)</td>
<td>(0.818)</td>
<td>(1.753)</td>
<td>(0.675)</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.509</td>
<td>0.633</td>
<td>1.312</td>
<td>1.502</td>
<td>(7.424)</td>
<td>(8.246)</td>
<td>(8.921)</td>
<td>(9.426)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>96.38%</td>
<td>98.63%</td>
<td>97.5%</td>
<td>98.21%</td>
<td>97.33%</td>
<td>99.05%</td>
<td>98.35%</td>
<td>98.87%</td>
</tr>
<tr>
<td></td>
<td>Partial-( R^2 )</td>
<td>26.35%</td>
<td>30.63%</td>
<td>34.07%</td>
<td>36.58%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Data start</td>
<td>1997-1</td>
<td>1997-1</td>
<td>1997-1</td>
<td>1997-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Data end</td>
<td>2010-1</td>
<td>2010-1</td>
<td>2010-1</td>
<td>2010-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes. The table contains estimation results for two different model specifications:

Model 1: \[ \Delta \ln P_s = c + Y_s \Delta t - D_t \Delta Y_s, \]

Model 2: \[ \Delta \ln P_s = c + (Y_s - \alpha S_s) \Delta t - D_t \Delta Y_s, \]

Numbers in parentheses are the \( t \)-statistics for the estimated coefficients.

Table 3 displays the estimation results for the European and Table 4 for the US data under the two model specifications. Model 1, which ignores the effects of default losses on the outstanding payments, already explains the majority of the return variations with \( R^2 \) values ranging between 96.4% and 99.2%. The
addition of the spread term in Model 2 improves the model fit in all maturities and markets. The spread term is able to explain roughly 20–40% of the residual return variation of Model 1. In addition to improving the models’ fit, the addition of the spread term also dilutes the significance of the estimated constant terms, which signals improved model specification. These quantitative results are well supported by Figures 1 and 2 that display the evolution of unexplained residual returns of Model 1 together with the short maturity yield spreads in the US and European markets, respectively.

The estimates of $\alpha$ as well as the Partial-$R^2$ values tend to increase with maturity. This is consistent with the rating compositions and seniorities of the bonds at different maturities. Figure 3 displays the evolution of the rating distributions of the US corporate bond indices by maturity bucket during the worst phases of the financial crises, when the default losses reached exceptionally high levels. The rating quality deteriorates with maturity which explains the increased default losses at the longer maturities. The composition of the US
Figure 2: Historical evolution of $S_t \Delta t$ (blue) and residual returns of Model 1 representing default losses (green) during 1996/2–2010/1 for the EU indices.

indices by instrument type are displayed in Figure 4. In line with the deterioration of the rating quality, the seniority of the bonds contained in the indices also declines as the maturity of the indices increases. The largest default losses were observed in the 5–7 year index which had the largest share of capital related debt instruments (under the Basel II regulation) and securitized exposures, both of which exhibited large losses during the financial crisis. On the other hand, the shorter maturities contained the highest share of senior debt and also exhibited the lowest default losses. Similar conclusions regarding the deterioration of the rating quality of the indices as a function of the maturity hold for the European indices as well.

The three terms in Model 2 explain majority of the return variation in the considered eight portfolios but, especially at the longer maturities, there remains residual spikes in the aftermath of the Lehman Brothers’ collapse in September 2008. Some idiosyncratic default event risk thus seems to remain, even though the indices contain bonds from hundreds of issuers. Another source of error may
be the approximation of the instantaneous default loss rate by the yield spread on 1–3 year bonds. On the average, however, the systematic default risk seems to be captured to a large extent by the spread term.

References


Figure 3: The composition of US corporate bond indices by rating classes AAA (dark blue), AA (light blue), A (yellow) and BBB (red) during 6/2009-5/2010.


Figure 4: The distribution of US corporate bond indices among senior debt (dark blue), subordinated debt (light blue), capital instruments (yellow) and securitized debt (red) during 6/2009-5/2010.