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Epi-convergent discretization of the generalized Bolza problem in dynamic optimization

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Abstract The paper is devoted to well-posed discrete approximations of the so-called generalized Bolza problem of minimizing variational functionals defined via extended-real-valued functions. This problem covers more conventional Bolza-type problems in the calculus of variations and optimal control of differential inclusions as well of parameterized differential equations. Our main goal is find efficient conditions ensuring an appropriate epi-convergence of discrete approximations, which plays a significant role in both the qualitative theory and numerical algorithms of optimization and optimal control. The paper seems to be the first attempt to study epi-convergent discretizations of the generalized Bolza problem; it establishes several rather general results in this direction.

Keywords Optimization and optimal control · Generalized problem of Bolza · Discrete approximations · Epi-convergence

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1 Introduction and problem formulation

The paper concerns a dynamic optimization problem called the "generalized problem of Bolza," which extends the classical Bolza problem in the calculus of variations as well Bolza-type problems in constrained optimal control. Let $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ and $f\colon \mathbb{R}^n \times \mathbb{R}^n \times [a,b] \to \overline{\mathbb{R}}$ be proper lower semicontinuous (l.s.c.) extended-real-valued functions that may take the infinite value $+\infty$. We always assume that f is a *normal integrand* in the sense of [9, Definition 14.27], which means that its epigraphical set-valued mapping $S_f\colon [a,b] \rightrightarrows \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined by

$$S_f(t) := \operatorname{epi} f(t,\cdot,\cdot) = \big\{(x,v,\alpha) \in \mathbb{R}^n \times]\mathbb{R}^n \times \mathbb{R} \big|\ \alpha \geq f(x,v,t)\big\}$$

is closed-valued and measurable; the latter signifies a very general and natural class of functions under integration. Consider the following *generalized Bolza problem*:

$$\text{minimize}\ \ J[x] := \varphi\big(x(a), x(b)\big) + \int_a^b f\big(x(t), \dot{x}(t), t\big)\,\mathrm{d}t \tag{1.1}$$

on the space X of absolutely continuous vector functions $x\colon [a,b] \to \mathbb{R}^n$. Observe that, due to the possible infinite values of φ and f, problem (1.1) implicitly incorporates both *endpoint* and *dynamic constraints*

$$(x(a), x(b)) \in \operatorname{dom}\varphi, \quad (x(t), \dot{x}(t)) \in \operatorname{dom} f(\cdot,\cdot,t)\ \text{ for a.e. }\ t \in [a,b],$$

where "dom" stands for the *effective domain* (points of finite values) of an extended-real-valued function. The case of $\operatorname{dom} f(\cdot,\cdot,t) = \mathbb{R}^n \times \mathbb{R}^n$ corresponds to the classical Bolza problem in the calculus of variations (or the problem with *finite Lagrangians*), where both functions φ and f are considered to be sufficiently smooth. Admitting extended values of f allows us to include in the framework of (1.1) *optimal control* problems governed by *differential inclusions* of the type

$$\dot{x}(t) \in F(x(t), t) \quad \text{for a.e. }\ t \in [a,b]. \tag{1.2}$$

The differential inclusion model (1.2) covers in turn *parameterized systems* of optimal control

$$\dot{x}(t) = g(x(t), u(t), t), \quad u(t) \in U(x(t), t)\ \text{ for a.e. }\ t \in [a,b], \tag{1.3}$$

which can be reduced to (1.2) with $F(x,t) = g(x, U(x,t), t)$. Note that, besides *standard* open-loop control systems with state-independent control regions U, the differential inclusion framework (1.2) makes it possible to study *state-dependent* control regions $U = U(x,t)$ in (1.3), which are much more complicated and important for applications reflecting a certain *feedback effect* in control; see the book [10] for more details and references.

Along with the generalized Bolza problem (1.1), we consider its *discrete approximation* built as follows. For any natural number $k \in \mathbb{N} := \{1, 2 \ldots\}$, let $t_0^k := a$ and

$$t_j^k := t_0^k + \frac{b-a}{k} j \quad \text{as } j = 1, \ldots, k;$$

this forms the *uniform grid* $\{t_j^k\}_{j=0}^k$ on $[a, b]$ with $t_k^k = b$. A natural discretization of the Bolza problem (1.1) reads as

$$\text{minimize } \varphi(x_0, x_k) + h_k \sum_{j=0}^{k-1} f\left(x_j, \frac{x_{j+1} - x_j}{h_k}, t_j^k\right), \quad x \in X^k, \tag{1.4}$$

where $X^k := \{(x_0, \ldots, x_k) \mid x_j \in \mathbb{R}^n\}$ and $h_k := (b-a)/k$, $k \in \mathbb{N}$. Furthermore, identifying X^k with the subspace of *piecewise linear* continuous functions in X, we can write (1.4) in the *continuous-time form*

$$\text{minimize } \varphi(x(a), x(b)) + \int_a^b f\left(x(s^k(t)), \dot{x}(t), s^k(t)\right) \mathrm{d}t, \quad x \in X^k,$$

where the piecewise constant functions $s^k \colon [a, b] \to \mathbb{R}$ are given by

$$s^k(t) := t_j^k \text{ for } t \in \left[t_j^k, t_{j+1}^k\right), \quad k \in \mathbb{N}. \tag{1.5}$$

For each $k \in \mathbb{N}$, define now the functional

$$J^k[x] := \begin{cases} \varphi(x(a), x(b)) + \int_a^b f\left(x(s^k(t)), \dot{x}(t), s^k(t)\right) \mathrm{d}t & \text{if } x \in X^k, \\ +\infty & \text{otherwise} \end{cases} \tag{1.6}$$

on the space X of absolutely continuous functions. The *primary goal* of this paper is to establish verifiable conditions ensuring an appropriate *epi-convergence* of the sequence $\{J^k[x]\}_{k=1}^\infty$ in (1.6) to the original Bolza functional $J[x]$ given by (1.1). Epi-convergence is understood in the conventional sense of variational analysis uniquely defined in finite-dimensional spaces [9], while in the infinite-dimensional case under consideration there are various possibilities to specify epi-convergence depending on the topology used. In what follows, we consider epi-convergence of the *Mosco type* that distinguishes between the *strong* and *weak* convergence in the corresponding lower limit and upper limit relationships; see [1] and the precise definition below.

Impose on X the standard Sobolev structure with the $W^{1,2}[a,b]$-norm

$$\|x\|_{W^{1,2}} := \max_{t\in[a,b]} |x(t)| + \left(\int_a^b |\dot{x}(t)|^2 \, dt \right)^{1/2}$$

($|\cdot|$ stands for the norm on $\mathbb{R}^n$) and say that the sequence $\{J^k[x]\}_{k=1}^{\infty}$ in (1.6) *Mosco epi-converges* to $J[x]$ in (1.1) if for any $x \in X$ we have

$$\liminf_{k\to\infty} J^k[x^k] \geq J[x] \text{ for every sequence } x^k \xrightarrow{w} x \quad \text{and} \tag{1.7}$$

$$\limsup_{k\to\infty} J^k[x^k] \leq J[x] \text{ for some sequence } x^k \xrightarrow{s} x, \tag{1.8}$$

where the symbols $\xrightarrow{w}$ and $\xrightarrow{s}$ signify the convergence of the sequence $x^k \to x$ as $k \to \infty$ in the weak and strong/norm topology of $W^{1,2}[a,b]$, respectively. One of the strongest motivations to study the Mosco epi-convergence is the following general result on *value convergence* given in [1]:

If $\{J^k\}_{k=1}^{\infty}$ Mosco epi-converges to J, then

$$\limsup_{k\to\infty} \left(\inf J^k \right) \leq \inf J,$$

*where the infima of J^k and J are taken over $x \in W^{1,2}[a,b]$. Furthermore, if there is a sequence $x^m \xrightarrow{w} x$ such that $x^m \in \varepsilon^{k^m}$-*argmin *$J^{k^m}$ for some sequences $k^m \to \infty$ as $m \to \infty$ and $\varepsilon^k \downarrow 0$ as $k \to \infty$, then $x \in$* argmin *J and one has* inf *$J^{k^m} \to$* inf *J as $m \to \infty$. In particular, if there is a weakly compact set $C \subset X$ such that ε^k-*argmin *$(J^k \cap C) \neq \emptyset$ for all $k \in \mathbb{N}$, then* inf *$J^k \to$* inf *J as $k \to \infty$.*

The reader can find more information on the history of epi-convergence, its relationships with other types of convergence, and a number of applications to various problems of optimization and control in [1,2,6–9] and the references therein; let us particularly mention [2] devoted to epi-convergent discretizations for standard parameterized systems of optimal control of type (1.3) with $U(x,t) = U$ and [6,7] containing applications of epi-convergence to numerical algorithms in deterministic and stochastic optimization and optimal control. We are not familiar with any research on epi-convergence of discrete approximations for the generalized Bolza problem (1.1) and its basic specifications considered in this paper, although traces of some similar constructions can be found in [4] for the case of differential inclusions of type (1.2); see, in particular [4, Theorem 3.3], where relationship (1.7) in Mosco convergence was proved for optimal solutions to (1.2)—but not for the generalized Bolza problem (1.1)—under stronger assumptions in comparison with this paper.

The paper is organized as follows. In Sect. 2 we establish broad sufficient conditions ensuring the fulfillment of the "lower part" (1.7) in Mosco epi-convergence for the generalized Bolza problem. Section 3 contains two major results on the “upper part” (1.8) related to special (but rather general) structures of the integrand in (1.1), which particularly cover the case of differential inclusions. Observe that, while both lower and upper parts in Mosco epi-convergence are of their independent interest with valuable applications, we have the “full” Mosco epi-convergence when all the assumptions in the corresponding “lower” and “upper” results of Sects. 3 and 4 are satisfied. Finally, we discuss some further possible extensions of the results obtained to the case of higher dimensions; see Remark 3.4.

2 Lower epi-convergence of discrete approximations

This section deals with the lower limit part (1.7) in Mosco epi-convergence of the discrete approximations. First we justify the following relationship between the weak convergence of sequences in $W^{1,2}[a,b]$ and the strong convergence in $L^2[a,b]$ of their compositions with piecewise constant functions $s^k(\cdot)$ from (1.5).

Lemma 2.1 (strong convergence of compositions) *Let $x^k \stackrel{w}{\to} x$ as $k \to \infty$, and let s^k, $k \in \mathbb{N}$, be the piecewise constant functions defined in* (1.5). *Then one has the strong L^2-convergence of the compositions $x^k \circ s^k \stackrel{L^2}{\to} x$ as $k \to \infty$.*

Proof It follows from the definitions and the Newton–Leibniz formula that

$$x^k(s^k(t)) - x^k(t) = x^k\big(t_j^k\big) - \left[x^k\big(t_j^k\big) + \int_{t_j^k}^{t} \dot{x}^k(\tau)\mathrm{d}\tau\right] = -\int_{t_j^k}^{t} \dot{x}^k(\tau)\mathrm{d}\tau,$$

for all $t \in \big[t_j^k, t_{j+1}^k\big)$, $j = 0,\ldots,k-1$, $k \in \mathbb{N}$. Involving Jensen's inequality, we have

$$\begin{aligned}
\|x^k \circ s^k - x\|_{L^2} &\le \|x^k \circ s^k - x^k\|_{L^2} + \|x^k - x\|_{L^2} \\
&= \left[\sum_{j=0}^{k} \int_{t_j^k}^{t_{j+1}^k} \left|\int_{t_j^k}^{t} \dot{x}^k(\tau)\,\mathrm{d}\tau\right|^2 \mathrm{d}t\right]^{\frac{1}{2}} + \|x^k - x\|_{L^2} \\
&\le \left[\sum_{j=0}^{k} \int_{t_j^k}^{t_{j+1}^k} \int_{t_j^k}^{t} |\dot{x}^k(\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}t\right]^{\frac{1}{2}} + \|x^k - x\|_{L^2} \\
&\le \left[\sum_{j=0}^{k} h^k \int_{t_j^k}^{t_{j+1}^k} |\dot{x}^k(t)|^2\,\mathrm{d}t\right]^{\frac{1}{2}} + \|x^k - x\|_{L^2} \\
&\le \sqrt{h^k}\|\dot{x}^k\|_{L^2} + \|x^k - x\|_{L^2}.
\end{aligned}$$

Since $x^k \overset{w}{\to} x$ as $k \to \infty$, the sequence of $\|\dot{x}^k\|_{L^2}$ remains bounded. Furthermore, the weak convergence of $x^k \to x$ in $W^{1,2}$ easily implies the strong convergence of this sequence in L^2. Thus, we arrive at the conclusion of the lemma by $h^k \to 0$ as $k \to \infty$. □

The next result provides general growth conditions on the integrand f in (1.1) ensuring the lower limit relationship (1.7) of Mosco epi-convergence.

Theorem 2.2 (lower epi-convergence for the generalized Bolza problem) *Let the integrand $f(x, v, t)$ be convex in v and satisfy the lower growth condition*

$$f(x, v, t) \geq -\gamma\left(|x|^2 + t^2\right) - h(|v|) \quad \textit{for all } \; x, v \in \mathbb{R}^n \; \textit{and a.e.} \; t \in [a, b], \tag{2.1}$$

where $\gamma \in \mathbb{R}$ and the function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and such that $h(\alpha)/\alpha^2 \to 0$ as $\alpha \to \infty$. Then the lower limit relationships (1.7) *hold.*

Proof Take arbitrary $x^k, x \in X$ with $x^k \overset{w}{\to} x$ as $k \to \infty$ in the weak topology of $W^{1,2}[a, b]$. It is easy to observe from the definitions that the weak convergence $x^k \overset{w}{\to} x$ in $W^{1,2}[a, b]$ implies that $x^k(a) \to x(a)$ and $x^k(b) \to x(b)$ as $k \to \infty$. Since the endpoint function φ in (1.1) is assumer to be l.s.c., we get

$$\liminf_{k\to\infty} \varphi(x^k(a), x^k(b)) \geq \varphi(x(a), x(b)). \tag{2.2}$$

Further, consider the extended-real-valued function

$$g(u, v) := f(x, v, t) \quad \text{with } \; u = (x, t), \tag{2.3}$$

which is l.s.c. in both variables and convex in v by the assumed properties of f, and then define the integral functional $G\colon L^2[a, b] \times L^2[a, b] \to \overline{\mathbb{R}}$ by

$$G(u, v) := \int_a^b g(u(t), v(t)) \mathrm{d}t. \tag{2.4}$$

From the growth condition (2.1) we get

$$g(u, v) \geq -\gamma|u|^2 - h((|v|) \quad \text{whenever } \; u \in \mathbb{R}^n \times [a, b] \; \text{ and } \; v \in \mathbb{R}^n$$

for the function g defined in (2.3) with $\gamma \in \mathbb{R}$ and $h(\cdot)$ having the aforementioned properties. Employing [3, Theorem 7], we conclude that the integral functional G in (2.4) is l.s.c. on $L^2[a, b] \times L^2[a, b]$ with respect to the strong convergence in u and the weak convergence in v. Observing that $s^k \overset{L^2}{\to} I$ (the identity function on $[a, b]$) by construction, that $x^k \circ s^k \overset{L^2}{\to} x$ by Lemma 2.1, and that $\dot{x}^k \to \dot{x}$ weakly in L^2 due to $x^k \xrightarrow{w} x$, we conclude from definitions (2.3), (2.4) and from the lower semicontinuity of G asserted above that

$$\liminf_{k\to\infty} \int_a^b f(x^k(s^k(t)), \dot{x}^k(t), s^k(t))\mathrm{d}t \geq \int_a^b f(x(t), \dot{x}(t), t)\mathrm{d}t. \tag{2.5}$$

Combing finally (2.2) and (2.5) with the constrictions of $J[x]$ and $J^k[x]$ in (1.1) and (1.6), we arrive at the relationships

$$\begin{aligned}\liminf_{k\to\infty} J^k[x^k] &\geq \liminf_{k\to\infty}\left[\varphi(x^k(a), x^k(b)) + \int_a^b f(x^k(s^k(t)), \dot{x}^k(t), s^k(t))\mathrm{d}t\right]\\ &\geq \liminf_{k\to\infty} \varphi(x^k(a), x^k(b)) + \liminf_{k\to\infty}\int_a^b f(x^k(s^k(t)), \dot{x}^k(t), s^k(t))\mathrm{d}t\\ &\geq \varphi(x(a), x(b)) + \int_a^b f(x(t), \dot{x}(t), t)\mathrm{d}t = J[x]\end{aligned}$$

and thus complete the proof of the theorem. □

3 Upper epi-convergence of discrete approximations

In this section we establish the second (upper limit) part (1.8) in Mosco epi-convergence of discrete approximations for two particular (but rather broad and important for both optimization theory and applications) classes of the generalized problem of Bolza.

First, we consider the generalized Bolza problem (1.1) with the following structures of the endpoint function φ and the integrand f:

$$\varphi(x_a, x_b) = \varphi_0(x_a, x_b) + \delta(x_a; \Omega) \quad \text{and} \quad f(x, v, t) = f_0(x, v, t) + \delta(v; F(x, t)), \tag{3.1}$$

where $\delta(\cdot; C)$ stands for the *indicator function* of the set C equal 0 on C and ∞ otherwise, and where the functions φ_0 and f_0 take only (*finite*) real values. The generalized Bolza problem (1.1) with data (3.1) is clearly equivalent to the following problem on minimizing the standard (not extended-real-valued) Bolza functional on absolutely continuous trajectories of the differential inclusion:

$$\begin{aligned}&\text{minimize } \ \varphi_0(x(a), x(b)) + \int_a^b f_0(x(t), \dot{x}(t), t)\mathrm{d}t\\ &\text{subject to } \ x(a) \in \Omega,\\ &\dot{x}(t) \in F(x(t), t) \quad \text{for a.e. } t \in [a, b],\end{aligned} \tag{3.2}$$

where the sets Ω and $F(x,t)$ are assumed to be closed while may *not be convex*. The following theorem presents sufficient conditions for the fulfillment of the upper limit relationships (1.8) in Mosco epi-convergence of discrete approximations to the Bolza problem (3.2) for nonconvex differential inclusions.

Theorem 3.1 (upper epi-convergence in the Bolza problem for differential inclusions) *Let for every feasible trajectory $x(\cdot)$ to* (3.2) *there exist $U \subset \mathbb{R}^n$ such that*:

- *$x(t) \in U$ for all $t \in [a,b]$*;
- *φ_0 and f_0 are continuous on $U \times U$ and $U \times U \times [a,b]$, respectively*;
- *the mapping $F = F(x,t)$ is bounded on $U \times [a,b]$, Lipschitz continuous in x on U uniformly in $t \in [a,b]$, and a.e. Hausdorff continuous in t on $[a,b]$ uniformly in $x \in U$.*

Impose furthermore the upper growth condition on f_0:

$$f_0(x,v,t) \le \gamma(|x|^2 + |v|^2 + t^2) + \beta \quad \textit{whenever} \;\; (x,v) \in \mathbb{R}^2,\; t \in [a,b] \tag{3.3}$$

with some $\gamma, \beta \in \mathbb{R}$. Then the upper limit relationships (1.8) *hold.*

Proof Without loss of generality take $x \in \operatorname{dom} J$; otherwise (1.8) holds trivially. To find a sequence of $x_k \in X$ satisfying both conditions in (1.8), we use [4, Theorem 3.1] that ensures the existence of $x^k \in X^k$ for which $x^k(a) = x(a)$, $\|\dot{x}^k - \dot{x}\|_{L^1} \to 0$ as $k \to \infty$, and

$$\dot{x}^k(t) \in F(x^k(s^k(t)), s^k(t)) \quad \text{for a.e. } t \in [a,b]. \tag{3.4}$$

It follows from the boundedness assumption of F that in fact $\|x - x^k\|_{W^{1,2}} \to 0$ as $k \to \infty$. Furthermore, we have the strong convergence $x^k \circ s^k \overset{L^2}{\to} x$ by Lemma 2.1.

Consider now the function

$$g(u) := -f_0(x,v,t) \quad \text{with } w = (x,v,t), \tag{3.5}$$

which is obviously a normal integrand, and the integral functional

$$G(u) := \int_a^b g(u(t))\mathrm{d}t \quad \text{on } L^2[a,b],$$

which is l.s.c. in the strong topology of L^2 by [3, Theorem 7] due to (3.5) and the upper growth condition (3.3) imposed on f_0. Therefore

$$\limsup_{k\to\infty} \int_a^b f_0(x^k(s^k(t)), \dot{x}^k(t), s^k(t))\mathrm{d}t \le \int_a^b f_0(x(t), \dot{x}(t), t)\mathrm{d}t.$$

Combining this with the dynamic constraints (3.4) and the definition of f in (3.1), we get

$$\limsup_{k\to\infty} \int_a^b f(x^k(s^k(t)), \dot{x}^k(t), s^k(t))\mathrm{d}t \le \int_a^b f(x(t), \dot{x}(t), t)\mathrm{d}t.$$

To finish the proof, it suffices to observe that

$$\lim_{k\to\infty} \varphi(x^k(a), x^k(b)) = \lim_{k\to\infty} \varphi_0(x(a), x^k(b)) = \varphi_0(x(a), x(b)) = \varphi(x(a), x(b)),$$

where the first and the last equalities follow from $x^k(a) = x(a) \in \Omega$ while the second one follows from the continuity of φ_0 and the fact that the strong convergence $\|x^k - x\|_{W^{1,2}} \to 0$ obviously yields $x^k(b) \to x(b)$ as $k \to \infty$. □

Remark 3.2 (differential inclusions with endpoint constraints) The above arguments allow us to extend the result of Theorem 3.1 to the Bolza problem for differential inclusions with the general *endpoint constraints* of the type

$$(x(a), x(b)) \in \Omega \subset \mathbb{R}^n.$$

To accomplish this in the way of proving Theorem 3.1, we have to *perturb* the endpoint constraints (in fact only those on the right-hand end) *consistently* with the step of discretization h_k as $k \to \infty$; cf. the proof of [4, Theorem 3.1] and also [5, Theorem 6.4] that holds in infinite dimensions. A challenging issue is to derive efficient *numerical estimates* for appropriate endpoint constraint perturbations ensuring the upper epi-convergence of discrete approximations.

The next theorem concerns the case of *general endpoint constraints* (via an arbitrary l.s.c. extended-real-valued function φ in the generalized Bolza problem) and ensures the upper epi-convergence (1.8) with *no perturbations* of endpoint constraints. However, it covers a special (but rather general) class of (1.1) with integrands f admitting the representation

$$f(x, v, t) = f_0(x, v, t) + \delta(x; C(t)) + g(v), \tag{3.6}$$

where $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is an extended-real-valued convex function. Note that the special form (3.6) implies the *separated dynamic* constraints on the state and velocity variables:

$$x(t) \in C(t) \quad \text{and} \quad \dot{x}(t) \in \operatorname{dom} g \quad \text{for a.e. } t \in [a, b].$$

Theorem 3.3 (upper epi-convergence for the Bolza problem with arbitrary endpoint and separated dynamic constraints) *Consider the generalized Bolza problem* (1.1)*, where* $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ *is proper and l.s.c., while the normal integrand* f *is given by* (3.6)*. Assume that the mapping* $C\colon [a,b] \rightrightarrows \mathbb{R}^n$ *is of closed graph, that the velocity function* $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *is convex and l.s.c., and that the real-valued function* $f\colon \mathbb{R}^n \times \mathbb{R}^n \times [a,b] \to \mathbb{R}$ *is continuous and satisfies the upper growth condition* (3.3)*. Then one has* (1.8)*.*

Proof Take $x \in \operatorname{dom} J$ and define $x^k \in X^k$ by

$$x^k\big(t_j^k\big) = x\big(t_j^k\big) \quad \text{for all } j = 0, \dots, k \text{ and } k \in \mathbb{N}. \tag{3.7}$$

We obviously have that $x^k(a) = x(a)$ and $x^k(b) = x(b)$ for all $k \in \mathbb{N}$. Let us justify that $x^k \stackrel{s}{\to} x$ as $k \to \infty$, where $\stackrel{s}{\to}$ stands as above for the strong convergence in $W^{1,2}[a,b]$. Indeed, $x^k \in X^k$ means that $\dot{x}^k$ is a piecewise constant function on $[a,b]$, and (3.7) gives

$$\begin{aligned}\dot{x}^k(t) &= \frac{x\left(t_{j+1}^k\right) - x\left(t_j^k\right)}{h^k} \\ &= \frac{1}{h^k}\int_{t_j^k}^{t_{j+1}^k} \dot{x}(t)\mathrm{d}t \quad \text{for all } t \in \left[t_j^k, t_{j+1}^k\right), \quad j = 0, \dots, k-1.\end{aligned} \tag{3.8}$$

We can represent $\dot{x}^k$ as $\dot{x}^k = P^k\dot{x}$, where P^k stands for the *orthogonal projection* on the subspace of piecewise constant functions. Now pick any $\varepsilon > 0$ and $y \in C^\infty$ satisfying $\|y - \dot{x}\|_{L^2} \le \varepsilon$. Since the operator P^k is *nonexpansive*, we get

$$\begin{aligned}\|\dot{x}^k - \dot{x}\|_{L^2} &\le \|P^k\dot{x} - P^k y\|_{L^2} + \|P^k y - y\|_{L^2} + \|y - \dot{x}\|_{L^2} \\ &\le \|\dot{x} - y\|_{L^2} + \|P^k y - y\|_{L^2} + \|y - \dot{x}\|_{L^2} \\ &\le \|P^k y - y\|_{L^2} + 2\varepsilon\end{aligned}$$

with $\|P^k y - y\|_{L^2} \to 0$ as $k \to \infty$ by the uniform continuity of y in $[a,b]$. This clearly implies that $x^k \stackrel{s}{\to} x$ as $k \to \infty$ for x^k defined in (3.7).

To justify the other relationship in (1.8), we get by (3.8), the v-convexity of f in (3.6), and Jensen's inequality that

$$\begin{aligned}
J^k[x^k] &= \varphi(x(a),x(b)) + \sum_{j=0}^{k} h_k f\left(x\left(t_j^k\right), \frac{x\left(t_{j+1}^k\right) - x\left(t_j^k\right)}{h^k}, t_j^k\right) \\
&= \varphi(x(a),x(b)) + \sum_{j=0}^{k} h_k f\left(x\left(t_j^k\right), \frac{1}{h^k}\int_{t_j^k}^{t_{j+1}^k} \dot{x}(t), t_j^k\right) \\
&\leq \varphi(x(a),x(b)) + \sum_{j=0}^{k} \int_{t_j^k}^{t_{j+1}^k} f\left(x\left(t_j^k\right), \dot{x}(t), t_j^k\right) \mathrm{d}t \\
&= \varphi(x(a),x(b)) + \int_a^b f(x(s^k(t)), \dot{x}(t), s^k(t))\mathrm{d}t \\
&= \varphi(x(a),x(b)) + \int_a^b [f_0(x(s^k(t)), \dot{x}(t), s^k(t)) + g(\dot{x}(t))]\mathrm{d}t,
\end{aligned}$$

where the last equality holds since $x \in \operatorname{dom} J$ implies that

$$x(s^k(t)) \in C(s^k(t)) \quad \text{for all } t \in [a,b].$$

Indeed, it follows from $x \in \operatorname{dom} J$ that $x(t) \in C(t)$ for a.e. $t \in [a,b]$, which yields $x(t) \in C(t)$ *for all* $t \in [a,b]$ by the continuity of x and the closed-graph property of C on $[a,b]$.

Define now the normal integrand

$$g(u,t) := -f_0(x, x(t), \tau) \quad \text{with } u = (x,\tau)$$

and consider the integral functional

$$G(u) := \int_a^b g\big(u(t),t\big)\mathrm{d}t \quad \text{on } L^2[a,b]. \tag{3.9}$$

By the upper growth condition (3.3) for f_0 we have

$$g(u,t) \geq -\gamma(|u|^2 + |x(t)|^2) - \beta = -\gamma|u|^2 - \alpha(t),$$

where $\alpha(t) := \gamma|x(t)|^2 - \beta$. Since $x \circ s^k \xrightarrow{L^2} x$ by Lemma 2.1 and since $s^k \xrightarrow{L^2} I$ (the identity function), we have by the lower semicontinuity result of [3, Theorem 7] applied to (3.9) that

$$\limsup_{k\to\infty}\int_a^b f_0(x(s^k(t)),\dot{x}(t),s^k(t))\mathrm{d}t \le \int_a^b f_0(x(t),\dot{x}(t),t)\mathrm{d}t,$$

which implies the first relationship in (1.8) and completes the proof of the theorem. □

Remark 3.4 (extensions to higher dimensions) The proof of Theorem 3.3 works also for the *finite element method* (FEM) in *higher dimensions*. In the latter case, the uniform grid $t_0 < \cdots < t_k$ should be replaced by the corner points of a triangulation of a bounded domain in $\mathbb{R}^d$, and the corresponding counterparts of the functions s^k in (1.5) are the piecewise constant functions giving the barycenter of the triangle that contains its argument. Such a higher-dimensional analog of Theorem 3.3 gives *consistency* results for FEM approximations applied, e.g., to *obstacle problems*.

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