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A stochastic programming model for asset liability management of a Finnish pension company

Petri Hilli · Matti Koivu · Teemu Pennanen · Antero Ranne

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Abstract This paper describes a stochastic programming model that was developed for asset liability management of a Finnish pension insurance company. In many respects the model resembles those presented in the literature, but it has some unique features stemming from the statutory restrictions for Finnish pension insurance companies. Particular attention is paid to modeling the stochastic factors, numerical solution of the resulting optimization problem and evaluation of the solution. Out-of-sample tests clearly favor the strategies suggested by our model over static fixed-mix and dynamic portfolio insurance strategies.

Keywords Stochastic optimization · Asset-liability management · Econometric modeling · Discretization

1 Introduction

Stochastic programming has proven to be an efficient approach in designing effective strategies in wealth- and asset liability management in practice. This is due to its ability to cope with the dynamics and complex constraint structures usually inherent in such problems. In principle, stochastic programming is not tied to any particular form of objective function or model

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P. Hilli · T. Pennanen (✉)
Department of Business Technology, Helsinki School of Economics, PL 1210, 00101 Helsinki, Finland
e-mail: pennanen@hse.fi

M. Koivu
Risk Management Division, European Central Bank, D-60311 Frankfurt am Main, Germany

A. Ranne
Actuarial Department, Ilmarinen Mutual Pension Insurance Company
e-mail: antero.ranne@ilmarinen.fi

of the stochastic factors as long as the distribution of the stochastic factors is independent of the decision variables in the model. Successful applications of stochastic programming to asset liability management have been reported e.g. in Nielsen and Zenios (1996), Cariño et al. (1998), Cariño and Ziemba (1998), Høyland (1998), Consigli and Dempster (1998), Kouwenberg (2001), and Geyer et al. (2003). See also the collections Ziemba and Mulvey (1998) and Zenios and Ziemba (2004) and the references therein. For a general introduction to stochastic programming we refer the reader to the official (COSP) stochastic programming site: *www.stoprog.org*.

This paper describes a stochastic programming model and its computer implementation for asset liability management of a Finnish pension insurance company. Finnish pension companies manage large investment funds and, like most pension companies in Europe, they are facing a large number of retiring policyholders at around 2010–2020. Our model addresses a long term dynamic investment problem where the aim is to cover the uncertain future liabilities with dynamic investment strategies. The assets are considered as the aggregate investment classes of cash, bonds, stocks, property and loans to policyholders. In addition to investment decisions, our model looks for optimal bonus payments and it takes explicitly into account various portfolio and transaction restrictions as well as some legal restrictions coming from the intricate pension system in Finland which is based on the defined benefit rule. The legal restrictions form a unique part of the model not present in earlier applications of stochastic programming.

We pay particular attention to describing the uncertain factors in the model which include investment returns, cash-flows, and the technical reserves used in the definition of the statutory restrictions. This is important since the solution of a stochastic programming model depends usually heavily on the underlying model for the stochastic factors. We use the stochastic model for assets and liabilities developed in Koivu et al. (). This model is based on a *vector equilibrium correction* model which, in addition to short term dynamics, takes into account long term equilibrium relations between certain economic factors (Engle and Granger, 1987).

Stochastic models such as the one in Koivu et al. () are based on an infinite sample spaces, which result in infinite dimensional optimization problems. We solve these problems numerically through discretization as described in Pennanen and Koivu (2002) and analyzed in (Pennanen). This is convenient for the user who only needs to come up with an appropriate econometric description of the stochastic factors. The discretization and numerical solution of the discretized models are fully automated and hidden from the user.

The model was implemented and tested against static fixed-mix and dynamic portfolio insurance strategies. Fixed-mix strategies are simple decision rules that always rebalance the investment portfolio to maintain fixed asset proportions. Portfolio insurance strategies are based on the constant proportion portfolio insurance framework of Perold and Sharpe (1988) and Black and Jones (1988), where the proportion of risky assets is kept as a constant multiple of the difference between the portfolio value and a protective floor. If the portfolio value hits or falls below the floor, all the funds are invested in less risky assets.

These decision strategies are by no means realistic models for the behavior of a real pension insurance company. However, they are often used for various simulation purposes in practice, which motivates their use as benchmarks. Other, more sophisticated but computation-intensive, choices of benchmarks have been used in Høyland (1998), Kouwenberg (2001); see also Fleten et al. (2002). We used the out-of-sample testing procedure recommended e.g. by Dardis and Mueller (2001) of Tillinghast-Towers Perrin. In the tests, the strategies based on our stochastic programming model clearly outperform both the fixed-mix and portfolio insurance strategies. Similar results have been obtained with the more sophisticated benchmarks in Høyland (1998); Kouwenberg (2001); Fleten et al. (2002).

The rest of the paper is organized as follows. A mathematical model of the ALM problem is presented in Section 2. Section 3 outlines the model for the underlying stochastic factors. Section 4 describes the procedure used for discretization of the optimization model. Section 5 outlines a computer implementation of our model and reports the results of numerical tests.

2 The optimization model

Our model is a multistage stochastic program where a sequence of decisions (asset allocations etc.) is interlaced with a sequence of observations of random variables (asset returns etc.). At each stage, decisions are made based on the information revealed up to that point, so the decision variables at a stage are functions of the random variables observed up to that stage. This kind of interdependent dynamics of information and decisions is typical in sequential decision making under uncertainty, which is what ALM and many other wealth management problems are; see for example Ziemba and Mulvey (1998) or Föllmer and Schied (2002).

The decision stages are indexed by $t = 0, 1, \dots, T$, where $t = 0$ denotes the present time, and the set of assets is indexed by $j \in J$, with

$$J = \{\text{cash, bonds, stocks, property, loans to policyholders}\}.$$

The decision variables characterize the asset management strategy as well as the company's solvency situation and the bonus strategy. Uncertainties result from random future investment returns as well as from random cash flows and technical reserves described below. There are several constraints stemming from the regulations of the Finnish pension system. The objective is to optimize the development of the company's solvency situation as described by the Ministry of Social Affairs and Health as well as the amount of bonuses paid to policyholders.

We will first describe the asset management model, followed by the model of statutory restrictions and finally the objective. Decision variables are random variables for all t except for $t = 0$. For parameters, randomness will be indicated explicitly.

2.1 Asset management

Asset management constitutes a central part of the model. The following formulation is fairly standard in asset management applications of stochastic programming.

Inventory constraints describe the dynamics of holdings in each asset class:

$$\begin{aligned} h_{0,j} &= h_j^0 + p_{0,j} - s_{0,j} \\ h_{t,j} &= R_{t,j} h_{t-1,j} + p_{t,j} - s_{t,j} \quad t = 1, \dots, T-1, \quad j \in J, \end{aligned}$$

where

$$\begin{aligned} h_j^0 &= \text{initial holdings in asset } j, \\ R_{t,j} &= \text{return on asset } j \text{ over period } [t-1, t] \ (\textit{random}) \end{aligned}$$

are parameters, and

$$\begin{aligned} p_{t,j} &= \text{(nonnegative) purchases of asset } j \text{ at time } t, \\ s_{t,j} &= \text{(nonnegative) sales of asset } j \text{ at time } t, \end{aligned}$$

$$h_{t,j} = \text{holdings in asset } j \text{ in period } [t, t+1]$$

are decision variables. As usual, we do not allow portfolio rebalancing at the horizon, which is why the index t goes only up to $T-1$ in the inventory constraints. Also, the company does not have control over the loans since, according to the Finnish law the policyholders have the right to borrow money from the company against their paid pension premiums. The amount invested in loans is thus determined by the policyholders. Holdings in loans are stochastic and we will assume them to be proportional to the technical reserves; see Section 2.2.1 below.

Budget constraints guarantee that the total expenses do not exceed revenues:

$$\sum_{j\in J}\left(1+c_j^p\right)p_{0,j} + H_{-1} \le \sum_{j\in J}\left(1-c_j^s\right)s_{0,j} + F_0,$$

$$\sum_{j\in J}\left(1+c_j^p\right)p_{t,j} + \tau_t H_{t-1} \le \sum_{j\in J}\left(1-c_j^s\right)s_{t,j} + \sum_{j\in J} D_{t,j}h_{t-1,j} + F_t \quad t=1,\dots,T-1,$$

where

$$\begin{aligned}
c_j^p &= \text{transaction cost for buying asset } j,\\
c_j^s &= \text{transaction cost for selling asset } j,\\
\tau_t &= \text{length of period } [t-1,t] \text{ in years},\\
H_{-1} &= \text{transfers to the bonus reserve a year before stage } t=0,\\
D_{t,j} &= \text{dividend paid on asset } j \text{ over period } [t-1,t]\ (\textit{random}),\\
F_t &= \text{cash flows in period } [t-1,t]\ (\textit{random})
\end{aligned}$$

are parameters and

$$H_t, \quad t=0,\dots,T-1 = \text{transfers to the bonus reserve per year during period } [t,t+1]$$

are decision variables. The net cash flow F_t is the difference between pension contributions and expenditures during period $[t-1,t]$. The company can pay a proportion of its accumulated wealth as bonuses to its policyholders. These bonuses are paid as reductions of the pension contributions. The amount of the total bonuses is determined at the end of each year, and the sum is transferred to the bonus reserve. The whole bonus reserve is then paid out during the following year. For periods longer than one year, we assume that H_t is kept constant throughout the period, hence $\tau_t H_{t-1}$ gives the value of bonuses paid to policyholders during period $[t-1,t]$.

Portfolio constraints give bounds for the allowed range of portfolio weights:

$$l_j w_t \le h_{t,j} \le u_j w_t \quad t=0,\dots,T-1, \quad j\in J,$$

where

$$w_t = \sum_{j\in J} h_{t,j} = \text{total wealth at time } t=0,\dots,T-1,$$

Table 1 Lower and upper bounds for investment proportions

j	l_j	u_j
Cash	0.01	1
Bonds	0	1
Stocks	0	0.5
Property	0	0.4

Table 2 Upper bounds for transactions

j	b_j^p	b_j^s
Cash	0.2	0.2
Bonds	0.2	0.2
Stocks	0.2	0.2
Property	0.01	0.01

and

$$l_j = \text{lower bound for the proportion of } w_t \text{ in asset } j,$$
$$u_j = \text{upper bound for the proportion of } w_t \text{ in asset } j$$

are parameters whose values are given in Table 1. The upper bounds for stocks and property are statutory restrictions. The lower bound for cash investments is set to guarantee sufficient liquidity.

Note that the total wealth w_t at stage $t = 0, \ldots, T-1$ is computed after portfolio rebalancing. At the horizon, there is no rebalancing so we define it as

$$w_T = \sum_{j \in J} (R_{T,j} + D_{T,j}) h_{T-1,j} + F_T - \tau_T H_{T-1}.$$

Transaction constraints bound the sales and purchases to a given fraction of w_t:

$$p_{t,j} \leq \tau_t b_j^p w_t \quad t = 0, \ldots, T-1, \quad j \in J,$$
$$s_{t,j} \leq \tau_t b_j^s w_t \quad t = 0, \ldots, T-1, \quad j \in J,$$

where

$$b_j^p = \text{upper bound for purchases of asset } j \text{ per year as a fraction of total wealth},$$
$$b_j^s = \text{upper bound for sales of asset } j \text{ per year as a fraction of total wealth}$$

are parameters. The values of b_j^p and b_j^s are displayed in Table 2. The tight rebalancing restrictions for property are set because of illiquidity of the Finnish property markets. For other asset classes the yearly rebalancing is restricted to be at most 20% of the total wealth. These restrictions model the policies of the company as well as the requirement that the size of transactions should be kept at levels that do not affect market prices.

2.2 Statutory restrictions

The statutory restrictions for Finnish pension insurance companies are quite strict, and they form a unique part of our stochastic programming model. Besides imposing constraints on the decision variables, these rules form the basis for defining the objective function in our model.

2.2.1 Solvency capital

The Finnish pension insurance companies are obliged to comply with several restrictions described in the legislation, government decrees or regulations given by the Ministry of Social Affairs and Health. A fundamental restriction is that the assets of a company must always cover its *technical reserves* L_t, which corresponds to the present value of future pension expenditure discounted with the so called "technical interest rate". A detailed description for determining the value of L_t is given in Koivu et al. () The assets include, besides the total amount of investments w_t, a transitory item of the net amount of other debts and credits in the balance sheet. This relatively small amount is calculated approximately as a fixed proportion c^G of the technical reserves. The difference

$$C_t = w_t + c^G L_t - L_t = w_t - (1 - c^G)L_t$$

of assets and the technical reserves is called the *solvency capital.* If at any time, C_t becomes negative, the company is declared bankrupt.

2.2.2 Solvency limits

Besides bankruptcy ($C_t \leq 0$), there are several target levels that have been set to characterize the pension insurance companies' solvency situation. These levels form an early warning system, so that the company and the supervising authorities can take action before a bankruptcy actually happens. A fundamental concept in the system is the *solvency border* $\tilde{B}_t$, defined in (1) below. If the solvency capital C_t falls below this limit the financial position is considered to be at risk, and the company is required to present to the authorities a plan for recovering a safe position. In addition, the company is not allowed to give any bonuses to its policyholders.

The target zone for the ratio $C_t/\tilde{B}_t$ is [2, 4]. In this zone, the financial position of a company is considered to be quite good. There is still discussion about how strictly the upper limit should be observed (in practice, no company has yet exceeded the upper limit). Therefore, we will ignore the upper limit in the model.

The concept of the solvency border corresponds to the solvency requirements in the European Union (EU) insurance directives. There is, however, an essential difference in the calculation method. The Finnish solvency border is based on the investment portfolio of a company. The fluctuation of the solvency capital is mainly caused by the investment market, and therefore the risk of going bankrupt is strongly dependent of the company's investment risk. The starting point of the Finnish system is that the probability of ruin in one year at the solvency border should be approximately 2.5%, and therefore the value of the border is required to be dependent on the investment portfolio. In contrast, the EU directives take no account of the company's investments. It is widely regarded that the EU regulations are insufficient, and a project is now established to renew the EU solvency requirements. The solvency border $\tilde{B}_t$ is given by

$$\tilde{B}_t = \left(a \sum_{j \in J} m_j h_{t,j} + b \sqrt{\sum_{j,k \in J} \sigma_{j,k} h_{t,j} h_{t,k}} \right) \frac{(L_t + H_t)}{w_t}, \tag{1}$$

where $a = -0.972/100$, $b = 1.782/100$, and the parameters

$$m = \begin{bmatrix} 0.18 \\ 0.66 \\ 6.20 \\ 3.70 \\ 0.72 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0.93 & 0.01 & 3.08 & 1.05 & -0.02 \\ 0.01 & 11.47 & 12.80 & -3.62 & 11.19 \\ 3.08 & 12.80 & 460.51 & 91.50 & 9.67 \\ 1.05 & -3.62 & 91.50 & 176.55 & -1.31 \\ -0.02 & 11.19 & 9.67 & -1.31 & 11.18 \end{bmatrix}$$

give the means and covariances for the asset classes (in the order: cash, bonds, stocks, property, loans to policyholders), according to the government decree, of one-year rate of returns over the technical interest rate. For asset classes like stocks, the parameter $\sigma_{j,j}$ is substantially larger that for safer classes like bonds. In reality, the values of m and σ are not fixed for eternity, but are updated by the Ministry of Social Affairs and Health on an irregular basis. The current values were set in 1999. We have decided to keep the values m and σ fixed in our optimization model partly because of the infrequent updating and also because any uncertainty in these parameters would be hard to model. Note that $\tilde{B}_t$ is a nonconvex function of the variables in the model.

2.2.3 Upper bound for bonuses

Finnish pension insurance companies compete with each other by paying out bonuses to their policyholders. To attract new customers companies would like to keep the amount of bonuses very high, but because the pension system is statutory, the government has aimed to restrict the amount of bonuses so that a sufficient proportion of the assets is preserved in the system to guarantee future pensions. Therefore, the Ministry of Social Affairs and Health imposes a formula for the maximum amount of each year's bonus transfers. The maximum depends on the solvency capital C_t and the solvency border $\tilde{B}_t$ of the company according to the formula

$$\tilde{H}_t^{\max} = \phi(C_t/\tilde{B}_t)(C_t - \tilde{B}_t)$$

where $\phi(z)$ is a piecewise linear function which has the minimum value of 0 when $z \leq 1$ and the maximum value of 0.04 when $z \geq 4$. It follows that $H_t^{\max}$ is also a nonconvex function of the variables in the model.

2.2.4 Convex approximations

In the optimization model, the nonconvex solvency border is replaced by

$$B_t = a \sum_{j \in J} m_j h_{t,j} + b \sqrt{\sum_{j,k \in J} \sigma_{j,k} h_{t,j} h_{t,k}},$$

which is convex in the variables. We have $B_t \geq \tilde{B}_t$ since $(L_t + H_t)/w_t \leq 1$ unless the company is bankrupt. Replacing $\tilde{B}_t$ by B_t in the model, makes the constraints in the model more restrictive, so we will stay on the safe side, except when the company is bankrupt. In the case of bankruptcy, the solvency border is underestimated by a factor of $(L_t + H_t)/w_t$.

We will also replace the nonconvex function $\tilde{H}_t^{\max}$ by a convex approximation, namely,

$$H_t^{\max} = 0.03 \max\{C_t - B_t, 0\}.$$

This is based on the fact that the historical average of $\phi(z)$ has been close to 0.03.

2.3 Objective function

There are many possibilities for measuring the performance of a company. Natural candidates would be expected utility of wealth or solvency capital under various utility functions. Here, we will describe a utility function that takes explicitly into account the unique features of the Finnish pension system.

As described in Section 2.2.2, the Ministry of Social Affairs and Health measures pension insurance companies' solvency situation by the ratio $C_t/\tilde{B}_t$ of the solvency capital and the solvency border. The Ministry defines four zones according to which companies' solvency situation is classified:

$$\begin{aligned}
C_t/\tilde{B}_t &\in [2, \infty) : \text{target} \\
C_t/\tilde{B}_t &\in [1, 2) : \text{below target} \\
C_t/\tilde{B}_t &\in [0, 1) : \text{crisis} \\
C_t/\tilde{B}_t &\in (-\infty, 0) : \text{bankrupt.}
\end{aligned}$$

We replace $\tilde{B}_t$ throughout by its convex approximation B_t given above, and we define three shortfall variables:

$$\begin{array}{ll}
SF_{t,1} \geq 2B_t - C_t & t = 1, \ldots, T-1, \\
SF_{t,2} \geq B_t - C_t + H_t/0.03 & t = 1, \ldots, T-1, \\
SF_{t,3} \geq -C_t & t = 1, \ldots, T,
\end{array}$$

each of which gives the amount by which a zone is missed. These will be penalized in the objective function. The inequality for $SF_{t,2}$ incorporates the constraint

$$H_t \leq H_t^{\max}$$

for bonus transfers. The penalty for $SF_{t,2}$ will be chosen large enough to guarantee that, at the optimum, the upper bound is satisfied.

For $t = 0, \ldots, T-1$, the state of the company will be evaluated by the following utility function

$$u(C_t, B_t, H_t, L_t) = C_t/L_t - \sum_{z=1}^{3} \gamma_z SF_{t,z}/L_t + u^b(H_t/L_t),$$

where γ_z are positive parameters and u^b is a nondecreasing concave function that will be specified according to the preferences of the company. However, the choice of u^b has to be made in accordance with the penalty parameter γ_2 in order to guarantee that the upper bound for H_t is not violated at the optimum. At stage T, the utility is measured by

$$u_T(C_T, L_T) = C_T/L_T - \gamma_3 SF_{T,3}/L_T.$$

The overall objective function in our model is the discounted expected utility

$$E^P\left\{\sum_{t=1}^{T-1} d_t u(C_t, B_t, H_t, L_t) + d_T u_T(C_T, L_T)\right\},$$

where d_t is the discount factor for stage t. The problem is to maximize this expression over all the decision variables and subject to all the constraints described above.

2.4 Problem summary

Deterministic parameters:

$$\begin{aligned}
h_j^0 &= \text{initial holdings in asset } j,\\
c_j^p &= \text{transaction cost for buying asset } j,\\
c_j^s &= \text{transaction cost for selling asset } j,\\
l_j &= \text{lower bound for wealth in asset } j \text{ as a fraction of total wealth},\\
u_j &= \text{upper bound for wealth in asset } j \text{ as a fraction of total wealth},\\
b_j^p &= \text{upper bound for purchases of asset } j \text{ per year as a fraction of total wealth},\\
b_j^s &= \text{upper bound for sales of asset } j \text{ per year as a fraction of total wealth},\\
c^G &= \text{the amount of transitory items as a fraction of the technical reserves},\\
a &= \text{the (negative) weight for the return component in the solvency border},\\
b &= \text{the weight for the standard deviation component in the solvency border},\\
m_j &= \text{mean yearly return of asset } j \text{ according to the government decree},\\
\sigma_{j,k} &= \text{covariance of one-year returns according to the government decree},\\
\tau_t &= \text{length of period } [t-1, t] \text{ in years},\\
\gamma_z &= \text{penalty parameters in the objective function},
\end{aligned}$$

Stochastic parameters:

$$\begin{aligned}
R_{t,j} &= \text{return on asset } j \text{ over period } [t-1, t],\\
D_{t,j} &= \text{dividend paid on asset } j \text{ over period } [t-1, t],\\
F_t &= \text{cash flows from period } [t-1, t],\\
L_t &= \text{technical reserves at time } t,
\end{aligned}$$

Decision variables:

$$
\begin{aligned}
h_{t,j} &= \text{holdings in asset } j \text{ from period } t \text{ to } t+1,\\
p_{t,j} &= \text{purchases of asset } j \text{ at time } t,\\
s_{t,j} &= \text{sales of asset } j \text{ at time } t,\\
w_t &= \text{total wealth at time } t,\\
H_t &= \text{transfers to bonus reserve at time } t,\\
C_t &= \text{solvency capital at time } t,\\
B_t &= \text{solvency border at time } t,\\
SF_{t,z} &= \text{shortfall from zone } z \text{ at time } t.
\end{aligned}
$$

Our stochastic programming model is

$$
\text{maximize} \quad E^P\left\{\sum_{t=1}^{T-1} d_t u(C_t, B_t, H_t, L_t) + d_T u_T(C_T, L_T)\right\}
$$

$$
\begin{aligned}
h_{0,j} &= h_j^0 + p_{0,j} - s_{0,j},\\
h_{t,j} &= R_{t,j} h_{t-1,j} + p_{t,j} - s_{t,j},\\
p_{t,j}, s_{t,j} &\geq 0,\\
\sum_{j\in J}\left(1 + c_j^p\right) p_{0,j} + H_{-1} &\leq \sum_{j\in J}\left(1 - c_j^s\right) s_{0,j} + F_0,\\
\sum_{j\in J}\left(1 + c_j^p\right) p_{t,j} + \tau_t H_{t-1} &\leq \sum_{j\in J}\left(1 - c_j^s\right) s_{t,j} + \sum_{j\in J} D_{t,j} h_{t-1,j} + F_t,\\
w_t &= \sum_{j\in J} h_{t,j},\\
l_j w_t &\leq h_{t,j} \leq u_j w_t,\\
p_{t,j} &\leq \tau_t b_j^p w_t,\\
s_{t,j} &\leq \tau_t b_j^s w_t,\\
C_t &= w_t - (1 - c^G) L_t,\\
B_t &\geq a \sum_{j\in J} m_j h_{t,j} + b \sqrt{\sum_{j,k\in J} \sigma_{j,k} h_{t,j} h_{t,k}},\\
SF_{t,1} &\geq 2B_t - C_t,\\
SF_{t,2} &\geq B_t - C_t + 100 H_t/3,\\
SF_{t,3} &\geq -C_t,\\
\text{for all} \quad t &= 1, \ldots, T-1, \quad j \in J,
\end{aligned}
$$

$$
\begin{aligned}
w_T &= \sum_{j\in J}(R_{T,j}+D_{T,j})h_{T-1,j}+F_T-\tau_T H_{T-1}.\\
C_T &= w_T-(1-c^G)L_T,\\
SF_{T,3} &\geq -C_T,\\
(h,p,s,w,H,C,B,SF) &\in \mathcal{N}
\end{aligned}
$$

where P is the probability distribution of the random parameters, E^P denotes the expectation operator, and the constraints are required to hold almost surely with respect to P. The symbol $\mathcal{N}$ stands for the subspace of *nonanticipative* decision rules, i.e. the set of strategies where the decision at each stage depends only on the random variables whose values have been observed by that stage.

Our model is a convex optimization problem that is nonlinear both in the objective and the constraints. There are 19 decision variables in each stage $t=0,\ldots,T-1$ (recall that for loans to policyholders, $h_{t,j}$, $p_{t,j}$ and $s_{t,j}$ are determined by L_t) and 3 in the last stage.

The probability distribution P of the random parameters is an important part of the model, and the solution will depend on it in an essential way. We assume that the random parameters follow the stochastic model developed in Koivu et al. () This model is briefly outlined in the following section. Since this model has continuous distributions, the resulting optimization problem is infinite-dimensional. Solutions of the problem are then sought numerically through discretization. As in Pennanen and Koivu (2002), the discretization is obtained by approximating the continuous probability measure by a discrete one. This is described in Section 4.

3 Modeling the stochastic factors

The stochastic factors in the optimization model are first expressed in terms of seven economic variables, namely short term interest rate sr, long term bond yield br, stock price index S, dividend index Div, property price index P, rental index $Rent$ and wage index W. These variables are then modeled by a time series model.

The formulas for calculating $R_{t,j}$ and $D_{t,j}$ for each asset class are displayed in Table 3, where τ_t denotes the length of the time period in years and the parameter D_M denotes the average duration of the company's bond portfolio.

The return for cash investments is approximated by the geometric average of the short term interest rate during the holding period. The formula for bond returns is based on a duration

Table 3 Return and dividend formulas

Asset class	$R_{t,j}$	$D_{t,j}$
Cash	$((1+sr_t)(1+sr_{t-1}))^{\frac{\tau_t}{2}}$	1
Bonds	$\left(\frac{1+br_{t-1}}{1+br_t}\right)^{D_M}$	$\frac{1}{2}(br_{t-1}+br_t)\tau_t$
Stocks	$\frac{S_t}{S_{t-1}}$	$\frac{1}{2}\left(\frac{Div_{t-1}}{S_{t-1}}+\frac{Div_t}{S_t}\right)\tau_t$
Property	$\frac{P_t}{P_{t-1}}$	$\left(\frac{1}{2}\left(\frac{Rent_{t-1}}{P_{t-1}}+\frac{Rent_t}{P_t}\right)-0.03\right)\tau_t$
Loans	1	$\frac{1}{2}(br_{t-1}+br_t)\tau_t$

approximation as in Campbell et al. (1997), Chapter 10. The parameter D_M is set equal to five years. The dividends for stock and property investments present the average dividend and rental yield, respectively, during the holding period. For property investments the maintenance costs, which are assumed to be a constant 3% of the property value, are deducted from the rental yield. Similarly to bonds, the cash income for loans is approximated by an average of bond yield. This is based on the fact that the interest on newly given loans is usually set equal to the current bond yield. The return for loans is equal to one because these instruments are not traded in the market.

The Finnish earnings-related pension scheme follows the defined benefit principle, where the pension insurance company guarantees the pension payments which are tied to the development of the policyholder's salaries. It follows that, the technical reserves L and cash flows F depend on policyholder's wages and population dynamics. These are assumed independent, so that their development can be modeled separately. The values of L and F depend also on the technical interest rate, which determines the total growth rate for the reserves. In the model, the technical interest rate is calculated based on recent asset returns and it is an important part of the model because, to a great extent, it determines the correlations between the investment variables and the reserves. The development of wages is described by the general Finnish wage index. For a more detailed description of the development of L and F, see Koivu et al. ()

The quarterly development of

$$x_t = \begin{bmatrix} \ln sr_t \\ \ln br_t \\ \ln S_t \\ \ln Div_t \\ \ln P_t \\ \ln Rent_t \\ \ln W_t \end{bmatrix}$$

will be modeled with a Vector Equilibrium Correction (VEqC) model, popularized by Engle and Granger (1987) and Johansen (1995). During the last decade VEqC models have been widely used in modeling and forecasting economic and financial time series, see e.g. Campbell and Shiller (1987); Clements and Hendry (1998, 1999) and Anderson et al. (2000). We consider a VEqC model

$$\Delta_\delta x_t = \sum_{i=1}^{k} A_i \Delta_\delta x_{t-i} + \alpha(\beta' x_{t-1} - \mu) + \epsilon_t, \tag{2}$$

where $A_i \in \mathbb{R}^{7\times 7}$, $\beta \in \mathbb{R}^{7\times l}$, $\mu \in \mathbb{R}^l$, $\alpha \in \mathbb{R}^{7\times l}$, Δ_δ denotes the shifted difference operator

$$\Delta_\delta x_t := \Delta x_t - \delta$$

with $\delta \in \mathbb{R}^7$, and ϵ_t are independent normally distributed random variables with zero mean and variance matrix $\Sigma \in \mathbb{R}^{7\times 7}$. When the model is stationary the parameter vector δ determines the average drift for the time series. The term $\alpha(\beta' x_{t-1} - \mu)$ takes into account the long-term behavior of x_t around statistical equilibria described by the linear equations $\beta' x = \mu$. It is assumed that, in the long run,

$$E[\beta' x_t] = \mu,$$

and that if x_t deviates from the equilibria it will tend to move back to them. The matrix α determines the speed of adjustment toward the equilibria. In a sense, VEqC-models incorporate long-run equilibrium relationships (often derived from economic theory) with short-run dynamic characteristics deduced from historical data.

We take δ and μ as user specified parameters. This enables incorporation of expert information in specifying the expected growth rates for x_t as well as long term equilibrium values for such quantities as mean reversion levels, interest rate spread and dividend yield. In particular, this gives control over mean returns which have been shown (in the context of the Markowitz model) to have a big impact on the optimal portfolio choice, see Chopra and Ziemba (1993). The appropriate lag-length k and the remaining parameters are estimated from quarterly data from Finland and the EU-area. The estimated parameter values used in the numerical tests of Section 5 are given in the Appendix. For a more detailed description of the model; see Koivu et al. ()

4 Discretization

In our optimization model, we are interested in the conditional distributions of x_{t+h}, given x_t, typically for $h \geq 4$. This can be calculated conveniently as follows. After specifying the model (2), we write it as a Vector Auto-Regressive (VAR) model in levels

$$x_t = (I + A_1 + \Gamma)x_{t-1} + \sum_{i=2}^{k}(A_i - A_{i-1})x_{t-i} - A_k x_{t-k-1} + c + \epsilon_t,$$

where $\Gamma = \alpha\beta'$ and $c = -\alpha\mu + (I - \sum_{i=1}^{k} A_i)\delta$. This, in turn, can be written in the companion form

$$\bar{x}_t = \bar{A}\bar{x}_{t-1} + \bar{c} + \bar{\epsilon}_t,$$

where

$$\bar{x}_t = \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-k} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} I + A_1 + \Gamma & A_2 - A_1 & \cdots & A_k - A_{k-1} & -A_k \\ I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

$$\bar{c} = \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \bar{\epsilon}_t = \begin{bmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It follows that

$$\bar{x}_{t+h} = \bar{A}^h \bar{x}_t + \sum_{i=1}^{h} \bar{A}^{h-i} \bar{c} + e_h, \tag{3}$$

where $e_h = \sum_{i=1}^{h} \bar{A}^{h-i} \bar{\epsilon}_i$. The random term e_h is normally distributed with zero mean, and from the independence of $\bar{\epsilon}_i$ it follows that e_h has the variance matrix

$$\bar{\sum_h} = \sum_{i=1}^{h} \bar{A}^{h-i} \begin{bmatrix} \sum & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} (\bar{A}^T)^{h-i}.$$

A convenient feature of (3) is that the dimension of the random term never exceeds $7(k+1)$ even if h is increased. In the model of Koivu et al. (), $k = 1$, so the dimension will be at most 14.

We discretize the model (3) using integration quadratures as described in Pennanen and Koivu (2002). This results in scenario trees that converge weakly to the original process as the number of branches is increased. This technique is just as easy to implement as the better known method of conditional sampling, where a scenario tree with a given period structure $(\tau_1, \ldots, \tau_T)$ and branching structure $(\nu_1, \ldots, \nu_T)$ can be generated as follows; see e.g. Chiralaksanakul (2003). For each $t = 0, \ldots, T$, denote by $\mathcal{N}_t$ the set of nodes in the scenario tree at stage t. The set $\mathcal{N}_0$ consists only of the root node which is labeled by 0. The rest of the nodes will be labeled by positive integers in the order they are generated. The number $h_t = 4\tau_t$ gives the length of period $[t-1, t]$ in quarters.

Set $m := 0$, $\bar{x}_m :=$ the current state of the world, and $\mathcal{N}_0 := \{m\}$.
for $t := 1$ **to** T
  $\mathcal{N}_t := \emptyset$
  for $n \in \mathcal{N}_{t-1}$
    Draw a random sample of ν_t points $\{e^i_{h_t}\}_{i=1}^{\nu_t}$ from $N(0, \bar{\Sigma}_{h_t})$
    for $i := 1$ **to** ν_t
      $m := m + 1$
      $\bar{x}_m = \sum_{i=1}^{h} \bar{A}^{h-i} \bar{c} + \bar{A}^h \bar{x}_n + e^i_{h_t}$
      $\mathcal{N}_t := \mathcal{N}_t \cup \{m\}$
    end
  end
end

The random samples required above are easily generated by computing the spectral decomposition

$$\bar{\Sigma}_h = \sum_{i=1}^{7(k+1)} \lambda^i_h u^i_h \left(u^i_h\right)^T,$$

where λ_h^i are the eigenvalues of $\bar{\Sigma}_{h_t}$ in decreasing order and u_h^i are the corresponding eigenvectors. If $\bar{\Sigma}_{h_t}$ has rank d_t, we have

$$\bar{\Sigma}_h = C_h C_h^T,$$

where $C_h = [\sqrt{\lambda_h^1} u_h^1, \ldots, \sqrt{\lambda_h^{d_t}} u_h^{d_t}]$, and then the desired sample is obtained as

$$e_{h_t}^i := C_h F_{d_t}^{-1}(u_{h_t}^i),$$

where $\{u_{h_t}^i\}_{i=1}^{\nu_t}$ is a random sample from U_{d_t}, the d_t-dimensional uniform distribution on $[0, 1]^{d_t}$ and F_{d_t} is the d_t-fold Cartesian product of univariate standard normal distribution functions. An advantage of computing the spectral decomposition (instead of the Cholesky decomposition as e.g. in Høyland et al. (2003)) is that when $\bar{\Sigma}_{h_t}$ is singular, d_t gives the true dimension of the random term. For example, when $h = 1$, $d_t = 7$.

The random samples $\{u_{h_t}^i\}_{i=1}^{\nu_t}$ above can be viewed as discrete approximations of U_{d_t}. As in Pennanen and Koivu (2002), we will replace these random samples by *low discrepancy point-sets* that have been designed to give good approximations of U_{d_t}. In the numerical tests in the next section we will use point-sets from the Sobol sequence; see for example Niederreiter (1992), Jäckel (2002). This produces a scenario tree with the same branching structure as the above conditional sampling procedure but a potentially better approximation of the original stochastic process, because the low discrepancy points are constructed to be more evenly distributed over U_{d_t} than typical random points. The computation times with Sobol sequences is roughly equal to that with Monte Carlo. Another advantage of using the spectral decomposition (instead of Cholesky decomposition) in forming the square root of the covariance matrix is that it allows for significant variance reduction in connection with low discrepancy point-sets; see e.g. Acworth et al. (1998).

5 Numerical results

5.1 Implementation

Figure 1 sketches the structure of the overall optimization system. The scenario generator (written in C programming language) takes as input the period and branching structures of the scenario tree and the time series model for the stochastic factors and generates the scenario tree for the assets and liabilities. The tree can be visually and otherwise inspected e.g. in spreadsheet programs until the outcomes are satisfactory. The scenario tree is then written into a text file in AMPL format described in Fourer et al. (2002). The optimization model written in AMPL modeling language and the data from the scenario generator are processed in AMPL and fed to MOSEK[1], which is an interior-point solver for convex (nonlinear) programs. The solution details and statistics produced by AMPL/MOSEK can again be visualized e.g. in spreadsheet programs. The system can be used under most Unix and Windows platforms.

[1] www.mosek.com

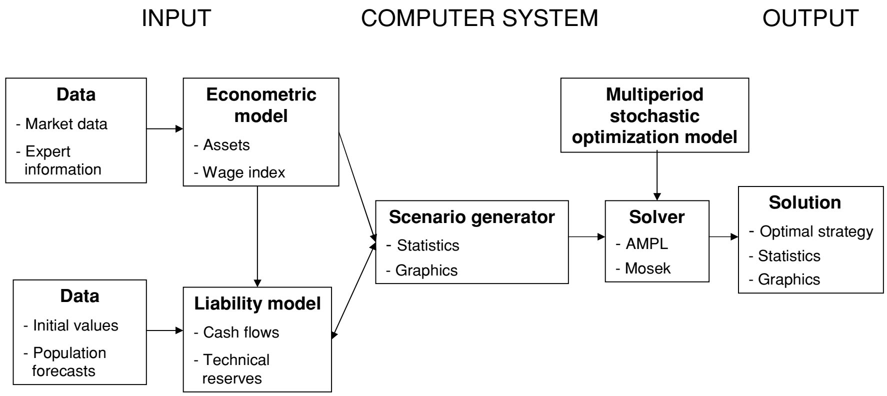


Fig. 1 Stochastic optimization system

Table 4 Shortfall penalty coefficients in the example

	γ_1	γ_2	γ_3
SP 1	1	10	10
SP 2	0.5	10	10
SP 3	1	1	1
SP 4	0.1	10	10
SP 5	0	0	0

5.2 Computational experiments

We chose the beginning of year 2002 as the first stage $t = 0$ in our experiments. The initial values for the time series model and the model parameters

$$h_0 = (1563, 622, 5573, 3914, 2158)$$

and $H_{-1} = 151.341$ (million euros) were chosen accordingly. As an example, we generated a scenario tree with period structure (1, 3, 6) years and branching structure (25, 10, 10) (2500 scenarios). This takes less than a second on Intel Pentium 4, 2.33GHz, with 1Gb of SDRam. Figure 2 plots the values of some important parameters on the scenario tree.

We solved the corresponding stochastic programming model for five sets of shortfall penalty coefficients given in Table 4. These (somewhat arbitrarily chosen) values correspond to different attitudes towards the attainment of the various target zones described in Subsection 2.3. In all cases we used the piecewise linear utility function

$$u^b(\cdot) = 1.5\gamma_2 \min\{\cdot, 0.01\}$$

for bonuses. The solution of the corresponding optimization models takes less than 10 seconds each. Figure 3 displays the optimal portfolio weights in stage $t = 0$ for the five sets of parameter choices. The first column gives the actual portfolio of the company in the beginning of year 2002. One can also examine the development of the optimized decision variables along the scenario tree. Figures 4 (a) and (b) plot the optimized C_t/L_t and H_t/L_t ratios, respectively, for SP1 of Table 4. The solvency capital C_t is always nonnegative (no bankruptcy) in every scenario while the bonus transfer/liability ratio H_t/L_t is equal to 0.01 in almost every scenario.

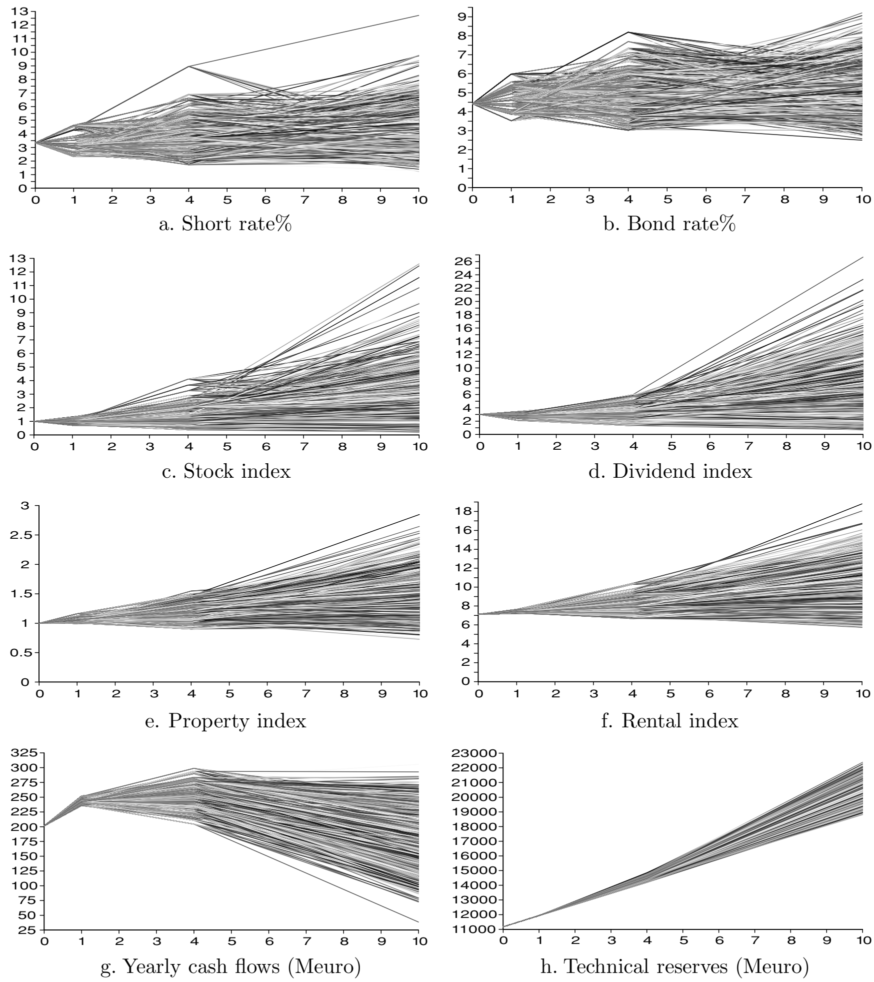


Fig. 2 Scenario tree of the example.

To gain some insight on the effect of the the shortfall penalties associated with the target zones, we solved the optimization model SP1 for varying levels of initial wealth w_0. This was done by rescaling the initial portfolio so that the relative portfolio weights remained unchanged. The model was resolved and the optimized first stage portfolio recorded. The resulting portfolios are graphed in Figure 5 (a) as a function of the ratio w_0/L_0. For comparison, we did the same for SP5, where there is no penalty for the shortfalls; see Figure 5 (b). The original wealth-liability ratio at the beginning of year 2002 was $w_0/L_0 = 1.238$.

Compared to SP5, the optimal portfolios for SP1 have considerably more wealth allocated to the short interest rate and bonds when $w_0/L_0 \leq 1.5$. This is natural since putting more wealth to the "safer" instruments reduces the solvency border and also the shortfalls. When w_0/L_0 approaches 2, the portfolios begin to look alike. This is caused by the fact that for high levels of initial wealth the probability of a shortfall is reduced and the effect of penalties becomes negligible. The most interesting phenomenon is that when the company approaches bankruptcy ($w_0/L_0 < 1$), it moves wealth from short interest rate to bonds and stocks, even

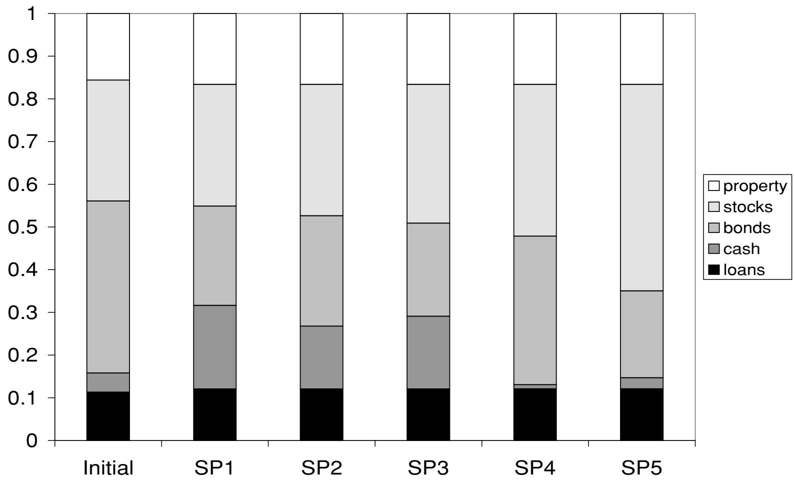


Fig. 3 Initial portfolio h^0 and the optimal portfolios corresponding to the parameter values in Table 4.

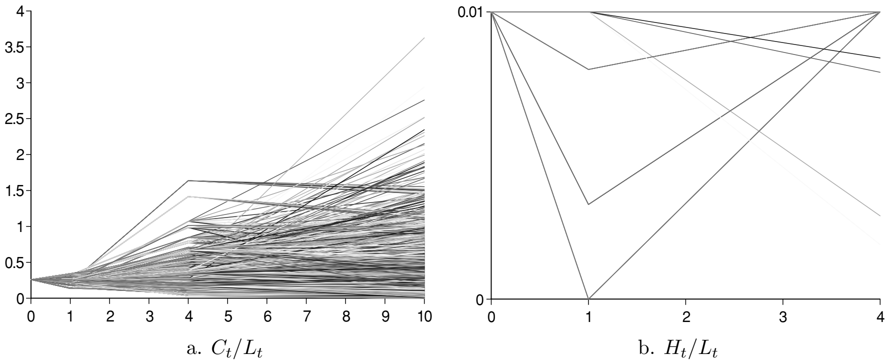


Fig. 4 Optimized solvency capital and bonus ratios along the scenario tree for SP1.

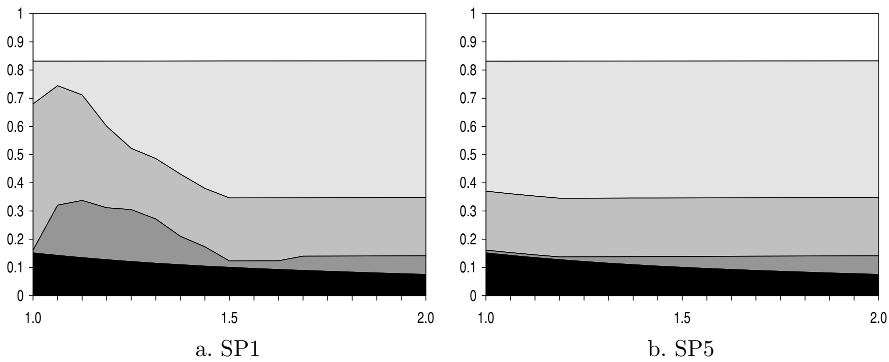


Fig. 5 The optimal first stage portfolio as a function of w_0/L_0 (The legend of Figure 3 applies).

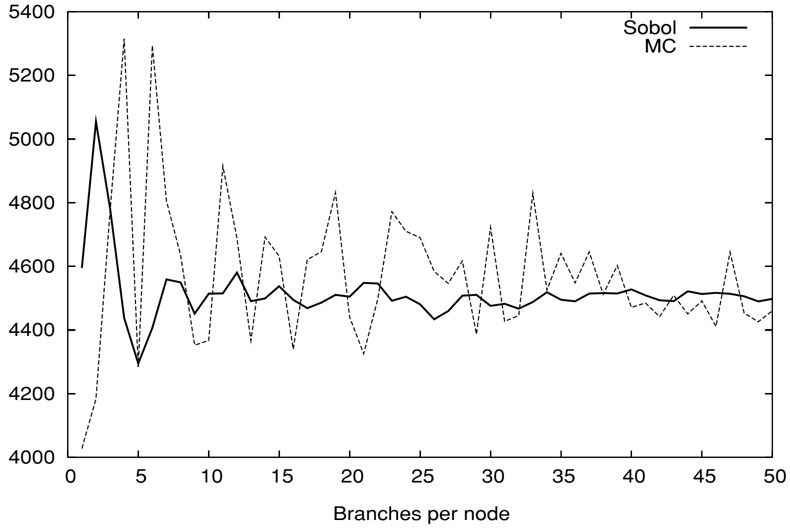


Fig. 6 Convergence of the optimal value

though this results in higher solvency border and higher shortfall penalties for the first two zones. This is probably due to the fact that the company is anticipating the solvency situation in later periods and trying to make safe portfolio allocations by ignoring to some extent the recommendations embodied in the definition of the solvency border.

5.3 Convergence of discretizations

Being forced to approximate the continuous distribution of the uncertain parameters by finite distributions, it is natural to ask how the corresponding optimization problems depend on the number of scenarios. A simple test is to study the behavior of the optimal values as the number of scenarios is increased. We will do the test for SP1 of Table 4 using the Sobol sequence as described in Section 4. For simplicity, we only considered fully symmetric scenario trees where each node has an equal number of branches, i.e. branching structure is (k, k, k) for $k = 1, 2, 3 \ldots$. The solid line in Figure 6 plots the objective value as a function of the size of the scenario tree. For low values of k, the optimal value goes through large variations, but as k is increased the optimal value seems to stabilize. In fact, it stabilizes close to 4504 which is what we obtained with the branching structure (25, 10, 10) in the above example. Convergence of discretizations of multistage stochastic programs has been studied for example by Olsen (1976), Casey and Sen (2003) and (Pennanen).

For comparison, we did the same test using Monte Carlo sampling in generating the scenario trees. This resulted in the dotted line in Figure 6. The optimal values obtained with Monte Carlo seem to converge too but not nearly as fast as the optimal values obtained with the Sobol sequence.

5.4 Out-of-sample test

We implemented an out-of-sample testing procedure to evaluate the performance of our stochastic programming model. Optimized strategies corresponding to the five sets of shortfall penalty coefficients in Table 4 were compared with a variety of static fixed-mix and dynamic portfolio insurance (PI) strategies meeting the statutory restrictions of Table 1. The fixed-mix portfolio weights were chosen according to a grid in order to evenly cover the region of feasible portfolios.

In the PI strategies the portfolio weights for cash π_c and property π_p are varied according to the same rules as in the fixed-mix case. The rest of the wealth is divided between bonds and stocks and the proportion of stocks in the portfolio at time t is given by,

$$\pi_{s,t} = \begin{cases} \min\left\{(1-\pi_c-\pi_p)\min\left\{\rho\frac{C_t}{w_t}, 1\right\}, 0.5\right\} & \text{if } C_t \geq 0, \\ 0 & \text{if } C_t < 0, \end{cases}$$

where ρ is a risk tolerance parameter indicating how the proportion invested in stocks increases with the company's solvency ratio, C_t/w_t. The percentage invested in stocks is a constant multiple of the company's solvency ratio, which was close to 22% initially, with higher values of ρ resulting in higher stock market allocations. When the company's solvency capital is negative the stock market allocation is set to zero and the remaining wealth is invested in bonds. PI strategy seems appropriate for a pension insurance company because it allocates more wealth to risky assets, stocks when the company's solvency ratio improves and reduces the stock market exposure as the company approaches insolvency.

As pointed out in the introduction, fixed-mix and PI strategies should not be considered as fully realistic decision rules. Rather, we view them as the first benchmarks that any practical decision support system should be able to outperform. Note however, that with these decision strategies, there is no guarantee that the transaction constraints will be satisfied. To simplify the comparison of different strategies, bonus transfers H_t were set to zero in each model. In addition, transaction costs were ignored in the case of fixed-mix and PI strategies to simplify computations. Note that this causes a bias in favor of the fixed-mix and PI strategies. The scenario trees used in optimization had the same structure as in the example of Section 5.1, that is, period structure (1, 3, 6) years and branching structure (25, 10, 10).

In the test, we evaluated the performance of each strategy over 325 randomly simulated scenarios of the stochastic parameters over 20 years. Portfolio rebalancing was made every year, i.e. fixed-mix portfolios are rebalanced to fixed proportions, PI portfolios are rebalanced and stochastic programming problems were solved with a new scenario tree, based on the current values of the stochastic parameters along the simulated scenario. We considered PI strategies with $\rho \in \{0.5, 1, \ldots, 20\}$. The following describes the testing procedure. As outlined in Section 3, the stochastic factors in each year can be expressed in terms of a 14-dimensional vector. Below, $\bar{x}_{s,y}$ denotes the value of this vector in year y along a randomly generated scenario s.

for $s := 1$ **to** 325
    Set $\bar{x}_{s,0} = \bar{x}_0$ (the current state of the world).
    for $y := 0$ **to** 19
        Generate a scenario tree rooted at $\bar{x}_{s,y}$.
        Solve the corresponding optimization problems and rebalance all the portfolios.
        Randomly sample $\bar{x}_{s,y+1}$ from the time series model and calculate the resulting portfolios and cash-flows at time $y+1$.
    end
end

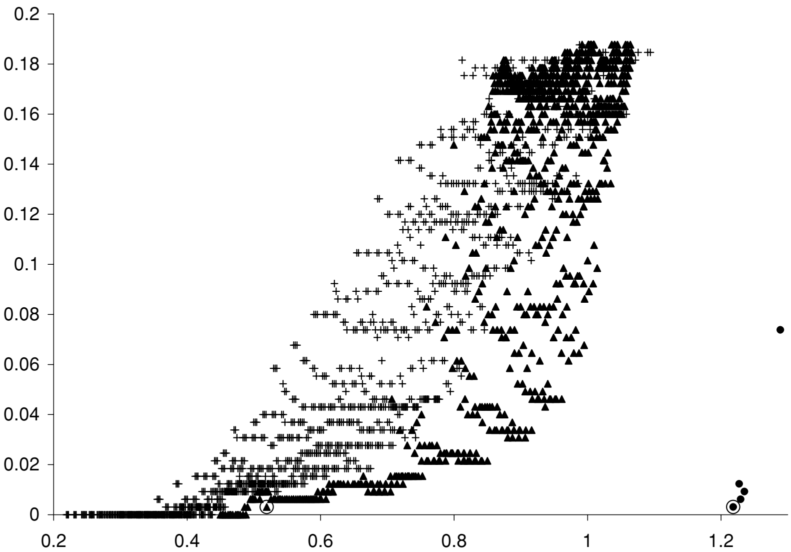


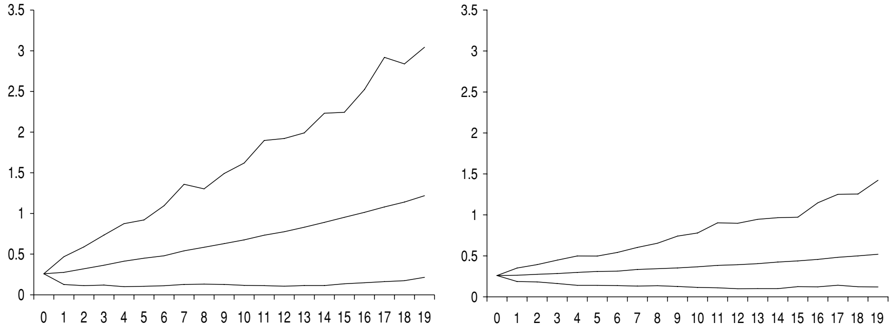
Fig. 7 C_T/L_T vs. bankruptcy probability for fixed-mix (+), PI (▲) and stochastic programming (•) strategies.

Figure 7 plots the performance of all the fixed-mix and PI strategies and the 5 stochastic programming strategies with respect to the average solvency capital at the end of the simulation period versus the bankruptcy probability during the period. Considering the main risk of the company, bankruptcy, and average solvency capital, the stochastic programming strategies clearly dominate both the fixed-mix and PI strategies, even though the probability of bankruptcy was not explicitly minimized. It is also worth noting that the best PI strategies outperform the best fixed-mix strategies at all reasonable risk levels. The riskiest stochastic programming strategy, SP5 of Table 4, went bankrupt in 25 simulations out of the 325 and the safest, SP1, in only one.

We will compare SP1 more closely with the best performing PI strategy circled in Figure 7, having the same bankruptcy probability as SP1. In the selected PI strategy $\pi_c = 0.04$, $\pi_p = 0.15$ and $\rho = 1$. The development of the solvency capital-reserves ratio for both strategies is described in Figure 8. The three lines represent the development of the sample average and the 95% confidence interval computed from the 325 scenarios. A higher mean and upwards skewed distribution indicates that the stochastic programming model can hedge against risks without losing profitability.

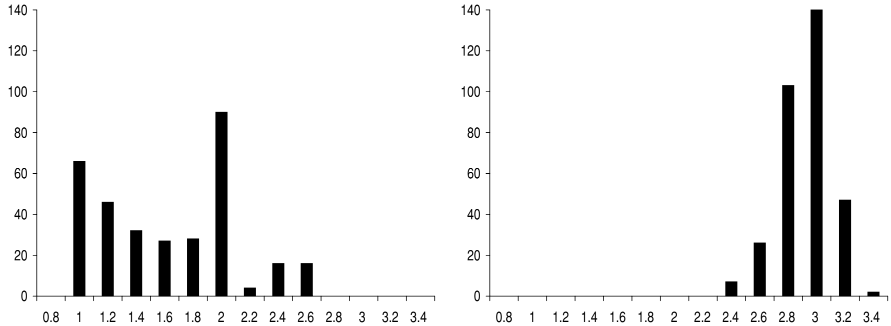
Figure 9 shows the distribution of the solvency capital-solvency border ratio C_t/B_t at the beginning of the second year. Due to the aim for high investment returns, the stochastic programming strategy avoids unnecessarily high levels of C_t/B_t, and consequently, it hits the lower border of the target zones frequently.

Figure 10 displays the development of the distribution of C_t/B_t in the 325 scenarios over the four zones defined in Subsection 2.3. If we compare the two strategies according to the target zones, the PI strategy seems to perform better than SP1. However, in the long run the stochastic programming strategy produces superior returns compared to the PI strategy, without increasing the company's bankruptcy risk.



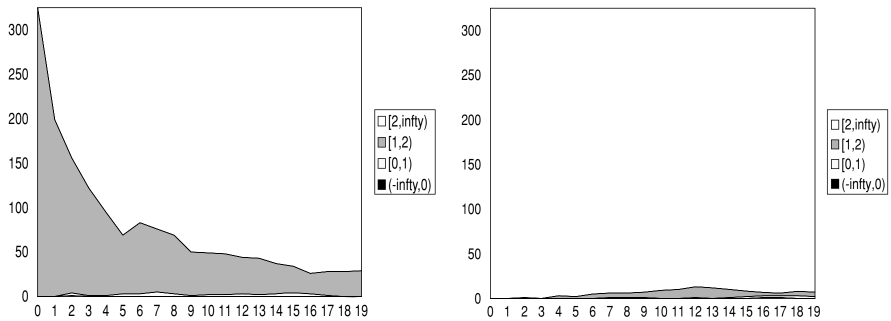
a. Stochastic programming. b. Portfolio insurance.

Fig. 8 C_t/L_t averages and 95% confidence intervals.



a. Stochastic programming. b. Portfolio insurance.

Fig. 9 Distribution of C_2/B_2 at the beginning of the second period.



a. Stochastic programming. b. Portfolio insurance.

Fig. 10 Development of the distribution of C_t/B_t over the different zones.

6 Conclusion

This paper presented a stochastic programming model that was developed for asset-liability management of a Finnish pension insurance company. The modeling was done in two phases:

1. modeling of the decision problem, which included the specification of the decision variables, stochastic factors, objective and constraints,
2. modeling of the stochastic factors. For this we used the model developed in Koivu et al. ()

This resulted in an infinite-dimensional stochastic optimization problem, which was solved in two steps:

1. discretization, which resulted in a finite dimensional optimization problem where the uncertainty was approximated by a scenario tree,
2. numerical solution of the discretized model. This was done using an algebraic modeling language and an interior point solver for nonlinear convex optimization.

Numerical results indicated that the model is robust and superior to more traditional ALM approaches.

Appendix

The parameters for the time series model described in Section 3 were estimated using full information maximum likelihood and are as follows. The number of lags $k = 1$,

$$\delta = \begin{bmatrix} 0 \\ 0 \\ 0.0114 \\ 0.0114 \\ 0.007 \\ 0.007 \\ 0.009 \end{bmatrix}, \qquad \mu = \ln \begin{bmatrix} 3.7 \\ 1.2 \\ 2.5 \\ 7.0 \end{bmatrix},$$

$$A_1 = 10^{-1} \begin{bmatrix} 3.672 & 3.467 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.855 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -59.11 \\ 0 & 0 & -2.425 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.629 & 0 & 3.617 & 0 & 0 \\ 0 & -0.209 & 0 & 0 & -0.663 & 8.533 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.638 & 8.712 \end{bmatrix},$$

$$\alpha = 10^{-1} \begin{bmatrix} 0 & 0.964 & 0 & 0 \\ -1.061 & -1.499 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1.449 & 0 \\ -0.238 & 0 & 0 & 0.637 \\ 0 & 0.080 & 0 & 0 \\ 0 & 0 & -0.024 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Sigma = 10^{-4}\begin{bmatrix} 53.711 & 7.155 & -4.795 & -15.978 & 0.112 & 0.273 & -0.233 \\ 7.155 & 55.719 & 10.741 & -11.647 & 0.228 & 0.465 & -0.263 \\ -4.795 & 10.741 & 116.726 & 45.187 & 6.345 & -0.587 & 0.126 \\ -15.978 & -11.647 & 45.187 & 62.235 & -1.042 & -0.218 & 0.517 \\ 0.112 & 0.228 & 6.345 & -1.042 & 2.752 & -0.007 & 0.003 \\ 0.273 & 0.465 & -0.587 & -0.218 & -0.007 & 0.167 & -0.018 \\ -0.233 & -0.263 & 0.126 & 0.517 & 0.003 & -0.018 & 0.033 \end{bmatrix}.$$

The initial values for the time series at the beginning of year 2002 were

$$x_0 = \ln\begin{bmatrix} 3.35 \\ 4.42 \\ 279.6 \\ 843.7 \\ 118.0 \\ 839.8 \\ 140.6 \end{bmatrix}, \qquad x_{-1} = \ln\begin{bmatrix} 4.16 \\ 4.33 \\ 242.9 \\ 776.0 \\ 117.7 \\ 831.3 \\ 139.1 \end{bmatrix}.$$

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