Risk management and valuation of insurance liabilities

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Valuation of insurance liabilities

- Reserving: What is the least amount of capital we need to cover our liabilities at an acceptable level of risk?
- Pricing/underwriting: What is the least premium we can sell a financial product for without worsening our risk-return profile?
- Both questions come down to asset-liability management, where one looks for an investment strategy that covers the claims as well as possible.
- For replicable claims, we recover classical risk-neutral pricing formulas.
- In general, the valuations are subjective and difficult to compute.

Valuation of insurance liabilities

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Asset-liability management

- Consider an insurance portfolio with aggregate claims c_t payable at time $t = 1, \dots, T$.
- After paying c_t, the insurer invests the remaining wealth in financial markets over the next period $[t, t+1]$.
- Denote by $r_{t,j}$ the total return on investment class $j \in J$ over period $[t-1, t]$.
- Let $h_{t,j}$ be the amount of wealth invested in $j \in J$ at the beginning of period t.
- We model the claims and investment returns as random variables in a probability space $(\Omega, \mathcal{F}, P)$.
- Investment strategies are nonanticipative: the portfolio $h_t = (h_{t,j})_{j \in J}$ chosen at time t is measurable wrt the sigma field $\mathcal{F}_t$ generated by the variables observable by time t.

Asset-liability management

Given initial capital $-c_0$, claims $(c_t)_{t=1}^T$ and a risk measure $\mathcal{R} : L^0 \to \mathbb{R}$, consider the problem

$$
\begin{aligned}
\text{minimize} \quad & \mathcal{R}\left(-\sum_{j\in J} h_{T,j}\right) \quad \text{over} \quad h \in \mathcal{N}_D \\
\text{subject to} \quad & \sum_{j\in J} h_{0,j} + c_0 \le 0 \\
& \sum_{j\in J} h_{t,j} + c_t \le \sum_{j\in J} r_{t,j} h_{t-1,j},
\end{aligned}
\tag{ALM}
$$

where $\mathcal{N}_D$ is the set of investment strategies h with $h_t \in D_t$.

- Given $c = (c_t)_{t=0}^T$, we denote the optimal value of (ALM) by $\varphi(c)$.

Reserving

The least initial capital $\pi^0(c)$ whose investment returns cover the claims c at an acceptable level of risk is given by

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{j\in J} h_{0,j} + c_0 \quad \text{over} \quad h \in \mathcal{N}_D \\ \text{subject to} & \displaystyle\sum_{j\in J} h_{t,j} + c_t \leq \sum_{j\in J} r_{t,j} h_{t-1,j} \quad t = 1,\dots,T, \\ & \displaystyle\mathcal{R}\left(-\sum_{j\in J} h_{T,j}\right) \leq 0. \end{array}$$

- $\pi^0(c) = \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \leq 0\}$, where $p^0 = (1, 0, \dots, 0)$.
- If $\mathcal{R}$ is such that $\mathcal{R}(W) \leq 0 \iff W \leq 0$ a.s., then $\pi^0(c)$ becomes the superhedging cost $\pi^0_{\sup}(c)$.

Reserving

- The superhedging cost $\pi^0_{\sup}(c)$ gives the least initial capital needed to cover the claims c without any risk.
- $\pi^0_{\inf}(c) := -\pi^0_{\sup}(-c)$ gives the greatest initial capital one can risklessly give a way in exchange of receiving c.
- A claim c is replicable if $\pi^0_{\sup}(c) = \pi^0_{\inf}(c)$.
- In complete market models (like the Black–Scholes model), all claims are replicable.
- In practice, few claims are replicable.

Reserving

Theorem 1 *The function* π^0 *is convex, nondecreasing and* $\pi^0(0) \leq 0$*. We always have* $\pi^0(c) \leq \pi^0_{\sup}(c)$*. If* $\pi^0(0) \geq 0$*, then*

$$\pi^0_{\inf}(c) \leq \pi^0(c) \leq \pi^0_{\sup}(c)$$

with equalities throughout for replicable c*.*

- π^0 may be interpreted much like a risk measure in [Artzner, Delbaen, Eber and Heath, 1999]. However, we do not assume the existence of a numeraire so π^0 operates on sequences of cash-flows and it is not "cash invariant".
- For replicable claims, $\pi^0(c)$ is independent of the views P and the risk preferences $\mathcal{R}$.

"Best estimate"

- When the claims $c = (c_t)_{t=1}^T$ are deterministic and all wealth is invested in riskless bonds, we get the explicit solution

$$\pi^0(c) = \sum_{t=1}^{T} e^{-tY_t} c_t,$$

  where Y_t is the yield at maturity t.
- If c_t are the expectations of the future claims, this becomes the "best estimate" in Article 77.2 of Solvency II.
- Riskless yield curves are meant for valuation of deterministic cash-flows, not uncertain ones.
- For example, the "best estimate" of a European call-option is much higher than its market (or Black–Scholes) value.
- The "best estimate" is inherently procyclical.

Pricing

- If we already have liabilities $\bar{c}$, then the least premium rate/swap rate that allows us to deliver an additional claim c without worsening our risk-return profile is given by

$$\pi(\bar{c}; c) := \inf\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}.$$

- Typical examples of the premium process $p = (p_t)_{t=0}^T$
  - $p = (1, 0, \dots, 0)$ (initial payment),
  - $p = (1, \dots, 1)$ ("fixed leg"),
  - random sequence (pension insurance, CDS, CDO, ...).
- The buyer's swap rate is given by

$$\pi^b(\bar{c}; c) : = \sup\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} - c + \alpha p) \leq \varphi(\bar{c})\}.$$

Pricing

Theorem 2 *If* $\pi(\bar{c}; 0) \geq 0$*, then*

$$\pi_{\inf}(c) \leq \pi^b(\bar{c}; c) \leq \pi(\bar{c}; c) \leq \pi_{\sup}(c)$$

with equalities throughout for replicable c*.*

Here

$$\pi_{\sup}(c) = \inf\{\alpha \,|\, \pi^0_{\sup}(c - \alpha p) \leq 0\}.$$

and

$$\pi_{\inf}(c) = \sup\{\alpha \,|\, \pi^0_{\sup}(\alpha p - c) \leq 0\}.$$

Pricing

Indifference swap rates depend on an agents

1. future views P: description of the future development of the claims and the market.
2. risk preferences $\mathcal{R}$: level of risk at which the assets should cover the liabilities,
3. current financial position $\bar{c}$.

- Agents with identical characteristics have no incentive to trade with each other.
- When c is replicable and $p = (1, 0, \ldots, 0)$, the indifference swap rates coincide with capital requirements.

Case study

- We will calculate the reserves for the pension insurance portfolio of the Finnish private sector occupational pension system.
- The yearly claims c_t consist of aggregate old age, disability and unemployment pension benefits earned by the end of 2008 and become payable during year t.
- The claims depend on mortality and the price- and wage-inflation, etc.

Case study

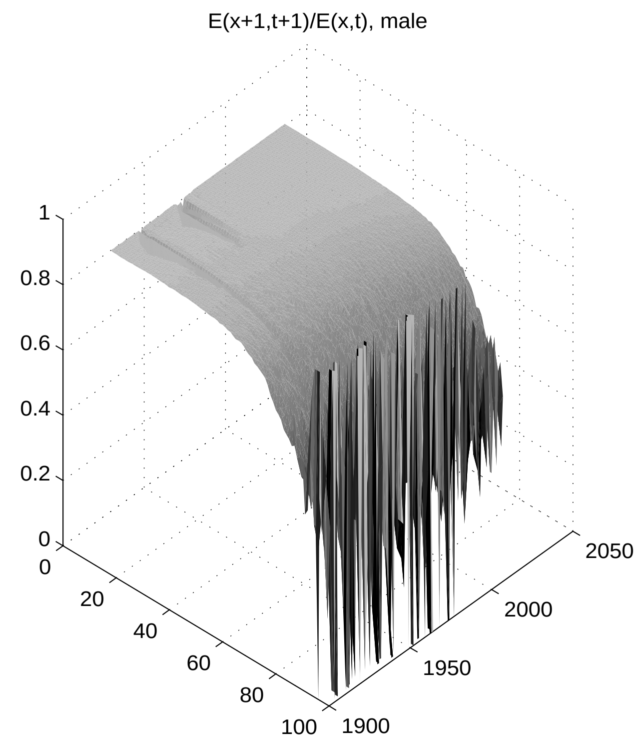
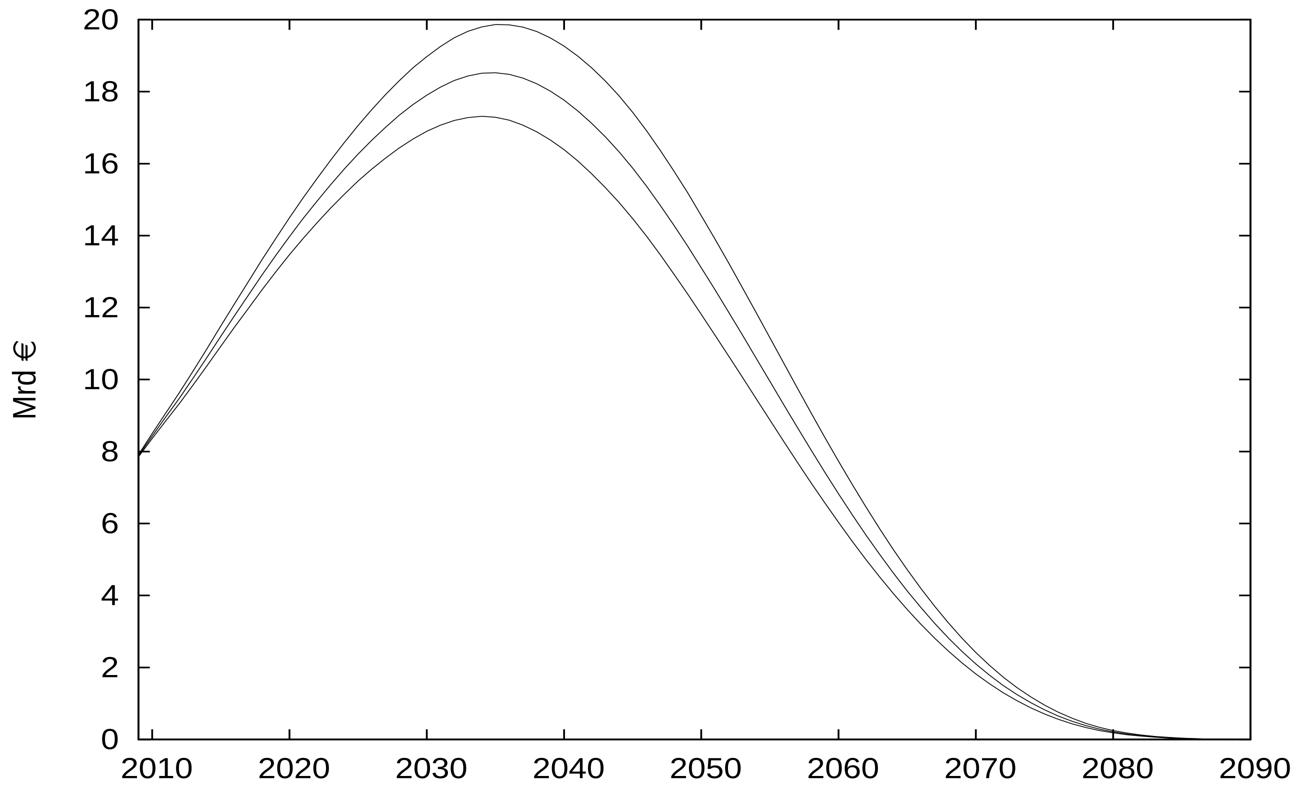


Figure 1: Survival rates of Finnish males

Case study

Figure 2: Yearly claims



Case study

- The traded assets consist of five equity indices and two bond indices.
- Bond returns in terms of the yield to maturity Y:

$$r_t = \exp(Y_t \Delta t - D \Delta Y_t + \frac{1}{2} C (\Delta Y_t)^2),$$

  where D is the duration and C the convexity.
- Log-indices and yields are modeled together with the liability risk factors (mortality, price- and wage-inflation) by a stochastic difference equation of the form

$$\Delta x_t = A x_{t-1} + b + \varepsilon_t$$

  so that indices have a given drift and yields are stationary.

Case study

- We find the minimum reserve

$$\pi^0(c) = \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \leq 0\}$$

by line search and numerical optimization.
- For given α, we approximate the optimum value $\varphi(c)$ of the ALM-problem with the Galerkin method which seeks optimal strategies among convex combinations of a finite collection of feasible trading strategies.

Case study

If $(h^i)_{i\in I}$ is a finite collection of feasible solutions to

$$\begin{aligned} &\text{minimize} \quad && \mathcal{R}\left(-\sum_{j\in J} h_{T,j}\right) \quad \text{over} \quad h \in \mathcal{N}_D \\ &\text{subject to} \quad && \sum_{j\in J} h_{0,j} + c_0 \leq 0 \\ & && \sum_{j\in J} h_{t,j} + c_t \leq \sum_{j\in J} r_{t,j} h_{t-1,j} \end{aligned} \tag{ALM}$$

then any convex combination of $(h^i)_{i\in I}$ is also feasible as long as the portfolio constraints D_t are convex.

Case study

The problem of finding the best convex combination can be written as the finite-dimensional convex optimization problem

$$\begin{aligned} &\text{minimize} \quad \mathcal{R}\left(-\sum_{i\in I}\alpha^i \sum_{j\in J} h^i_{T,j}\right) \quad \text{over} \quad \alpha \in \mathbb{R}^I_+ \\ &\text{subject to} \quad \sum_{i\in I}\alpha^i = 1. \end{aligned}$$

- When $\mathcal{R}(W) = Ev(W)$, the objective can be approximated by integration quadratures.
- Compare with the finite element method for elliptic PDEs with nonconstant coefficients.

Case study

Results with 529 basis strategies (buy and hold, constant proportions, portfolio insurance, target date fund).

Weight	Type	$CV@R_{97.5\%}$ (billion €)
0.665	BH	1569
0.029	BH	6567
0.104	BH	5041
0.022	CP	3324
0.039	PI	1420
0.099	PI	1907
0.042	PI	2417
	Best basis	1020
	Galerkin	251

Case study

	Confidence level				
	95%	90%	85%	80%	66%
Best basis	296	284	273	261	239
Optimized	288	271	254	236	202

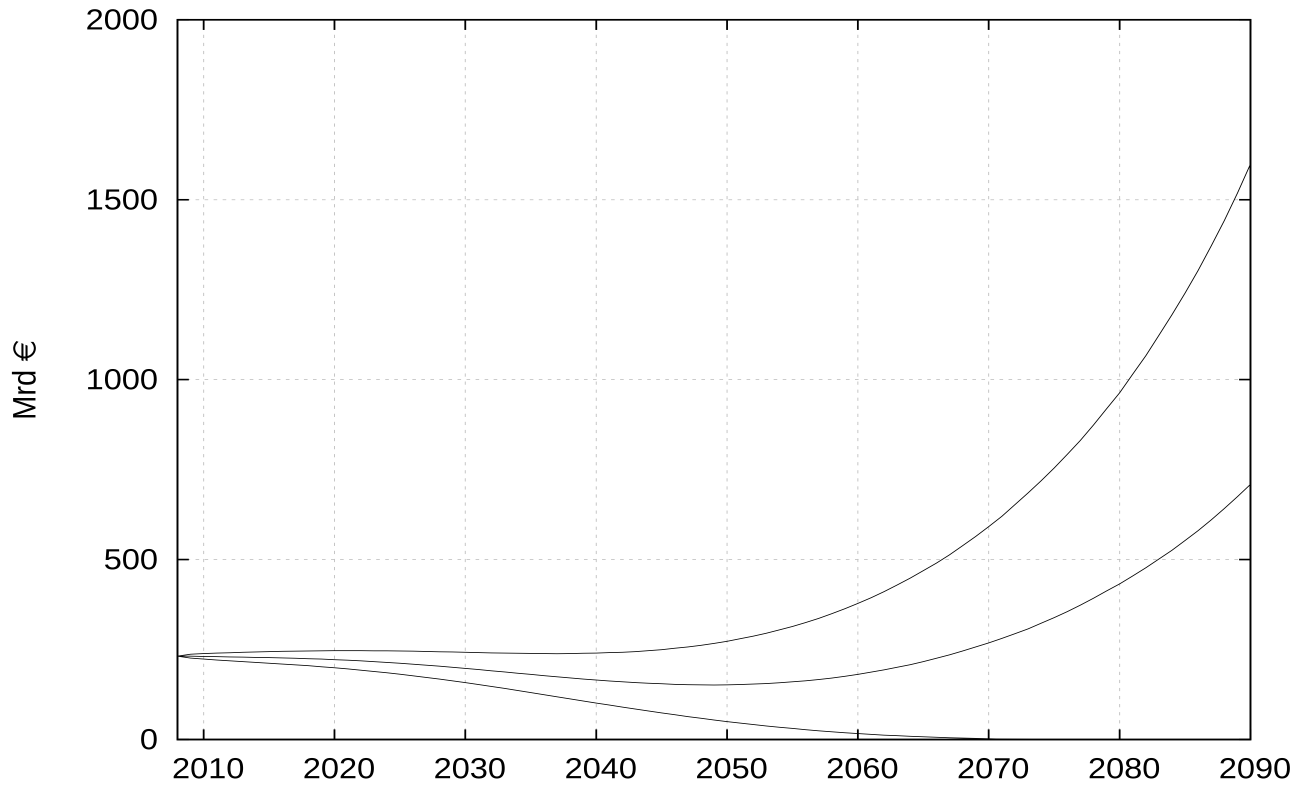
Table 1: Optimal reserves with varying risk tolerances

	Confidence level				
	95%	90%	85%	80%	66%
Best basis	24.3	25.4	26.4	27.6	30.1
Optimized	25.0	26.6	28.3	30.5	35.6

Table 2: Corresponding funding ratios

Case study

Figure 3: The development of 34%, 50%- and 66%-quantiles of net wealth when $\pi^0(c)$ is defined with $\mathcal{R} = V@R_{66\%}$.



Summary

- Call options and pension liabilities can be valued with the same principles.
- Both reserving and indifference pricing/swap rates come down to asset-liability management.
- In incomplete markets, the adequacy of reserves, prices and investment strategies is subjective (risk preferences, views, financial position, trading expertise).
- Quantitative risk management requires techniques from stochastics, statistics, optimization, numerical analysis, ...