*The Quarterly Journal of Mathematics* Advance Access published on January 3, 2013
**Quart. J. Math. 64 (2013), 1057–1089; doi:10.1093/qmath/has035**

# MEAN VALUES OF $\zeta'/\zeta(s)$, CORRELATIONS OF ZEROS AND THE DISTRIBUTION OF ALMOST PRIMES

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[Received 14 March 2012]

**Abstract**

We establish relationships between mean values of products of logarithmic derivatives of the Riemann zeta-function near the critical line, correlations of the zeros of the Riemann zeta-function and the distribution of integers representable as a product of a fixed number of prime powers.

## 1. Introduction

Goldston *et al.* [9], building on earlier work [5, 7, 8], gave an equivalence, assuming the Riemann hypothesis (RH), between the three quantities

$$I(\sigma; T) = \int_1^T \left| \frac{\zeta'}{\zeta}(\sigma + \mathrm{i}t) \right|^2 \mathrm{d}t,$$
$$F(\alpha; T) = N(T)^{-1} \sum_{0<\gamma,\gamma'<T} T^{\mathrm{i}\alpha(\gamma-\gamma')} w(\gamma - \gamma'), \tag{1.1}$$
$$P(\beta; T) = \int_1^\infty \left( \psi\left(x + \frac{x}{T}\right) - \psi(x) - \frac{x}{T} \right)^2 x^{-2-2\beta}\, \mathrm{d}x.$$

In $F(\alpha; T)$, the sum is over pairs of ordinates of zeros $\rho = \frac{1}{2} + \mathrm{i}\gamma$ of the Riemann zeta-function, $N(T) = (T/2\pi)\log(T/2\pi e) + O(\log T)$ is the number of zeros with $0 < \gamma < T$ and $w(u) = 4/(4+u^2)$ is a weight function. In $P(\beta; T)$, the function $\psi(x)$ is defined as $\psi(x) = \sum_{n\le x} \Lambda(n)$, where $\Lambda(n)$ is von Mangoldt's function; that is, $\Lambda(n) = \log p$ if $n = p^m$ with $p$ a prime, and $\Lambda(n) = 0$ otherwise.

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The main results of Goldston *et al.* [9] are captured by the following asymptotic formulas. Assume RH and suppose that $A > 0$ is fixed. If there exists a number $f(A)$ such that one of the following asymptotic formulas is true as $T \to \infty$, then all of them are true:

$$\begin{aligned} I\left(\frac{1}{2} + \frac{A}{\log T};\ T\right) &\sim f(A) T \log^2 T, \\ \int_{0^+}^{\infty} F(\alpha;\ T)\, \mathrm{e}^{-2\alpha A}\, \mathrm{d}\alpha &\sim f(A), \\ P\left(\frac{A}{\log T};\ T\right) &\sim f(A) \frac{\log^2 T}{T}. \end{aligned} \tag{1.2}$$

In the second formula, the lower limit is interpreted as $\varepsilon(T) > 0$ with $\varepsilon(T) \to 0$ slowly so that the spike of $F(\alpha;\ T)$ at $\alpha = 0$ gives no contribution. It turns out that taking $\varepsilon(T) = \log\log T/(2\log T)$ is permissible. We discuss this point further in Sections 2 and 3.

The above asymptotics give an equivalence between a mean value of $\zeta'/\zeta(s)$, the zeros of the zeta-function and prime powers. In particular, the equivalence shows that Montgomery's conjecture for the pair correlation of the zeros of the zeta-function is equivalent to a statement about a weighted variance for the number of primes in short intervals. The purpose of this paper is to extend these results to the case of mean values of the product of several $\zeta'/\zeta(s)$'s, higher correlation functions of the zeros and weighted variances for the number of integers in short intervals that are a product of a specific number of prime powers. These integers are the 'almost primes' we refer to in the title. The phrase 'almost primes' is not precisely accurate; we have used it for lack of a better established term and because it conveys the idea reasonably well. Usually a $J$-almost prime is a number that is a product of $J$ not necessarily distinct primes. However, we are using the term to mean a product of $J$ not necessarily distinct *prime powers*.

In the next section, we introduce the higher analogues of $I(\sigma;\ T)$, $F(\alpha;\ T)$ and $P(\beta;\ T)$ and state our main results. In Section 3, we prove Theorem 2.1, which gives an equivalence between mean values of the $N$-fold product of $\zeta'/\zeta(s)$'s and averages of the $N$-level form factor of the zeros of the zeta-function. This is followed by the proof of an upper bound for mean values of a product of $\zeta'/\zeta(s)$'s given in Corollary 2.1. In Sections 4 and 5, we prove Theorems 2.2 and 2.3, respectively, both of which give equivalences between mean values of products of several $\zeta'/\zeta(s)$'s and the distribution of numbers representable as a product of a fixed number of prime powers. In Section 6, we provide several explicit evaluations of mean values of products of $\zeta'/\zeta(s)$'s, assuming that the random matrix conjectures hold for the zeros of the zeta-function.

## 2. Higher analogues of *I*, *F* and *P*, and main results

We let $N = J + K$ with $J \geq 0, K \geq 1$ and always assume that $N \geq 2$. The vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ consists of $J$ ones followed by $K$ negative ones. We set $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ with $a_n > 0$ and $a_n \approx 1/\log T$ for $1 \leq n \leq N$. Here $a_n \approx 1/\log T$ means that there exist constants $0 < A_n \leq A_n'$ such that $A_n/\log T \leq |a_n| \leq A_n'/\log T$. We also write

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{N-1}) \quad \text{and} \quad d\boldsymbol{\alpha} = d\alpha_1 \cdots d\alpha_{N-1},$$

with $\alpha_n \in \mathbb{R}$, and always assume that

$$\alpha_N = -\sum_{n<N} \alpha_n.$$

For $1 \le n \le N-1$, we set $\mathbf{e}_n = (0, \dots, 1, \dots 0) \in \mathbb{R}^{N-1}$, where the $n$th component is 1 and the rest are 0. We also define $\mathbf{e}_N = (-1, -1, \dots, -1) \in \mathbb{R}^{N-1}$. Expressions like $\boldsymbol{\alpha} \cdot \mathbf{e}_n$ represent the standard dot product in $\mathbb{R}^{N-1}$. Observe that $\boldsymbol{\alpha} \cdot \mathbf{e}_n = \alpha_n$, even when $n = N$.

## 2.1. *Higher analogues of I and F*

Our generalization of the mean value $I(\sigma; T)$ in (1.1) is

$$I(\sigma, \mathbf{a}, \boldsymbol{\varepsilon}; T) = \int_0^T \prod_{n=1}^{N} \frac{\zeta'}{\zeta}(\sigma + a_n + \mathrm{i}\varepsilon_n t)\,\mathrm{d}t.$$

Note that we might just as well write $I(\sigma, \mathbf{a}, J; T)$.

Our generalization of $F(\alpha; T)$ in (1.1) is

$$\begin{aligned} F(\boldsymbol{\alpha}; T) &= F(\alpha_1, \dots, \alpha_{N-1}; T) \\ &= \frac{1}{N(T)} \sum_{0<\gamma_1,\dots,\gamma_N<T} T^{i\sum_{n<N}\alpha_n(\gamma_n-\gamma_N)} w(\gamma_1-\gamma_N, \dots, \gamma_{N-1}-\gamma_N), \end{aligned} \tag{2.1}$$

where

$$w(x_1, x_2, \dots, x_{N-1}) = \prod_{n=1}^{N-1} \frac{4}{4+x_n^2} \tag{2.2}$$

is a weight function. In the random matrix theory literature, $F(\boldsymbol{\alpha}; T)$ is referred to as the *N-level form factor*. It is the Fourier transform of the $N$-level correlation function. When $N = 2$, $F(\boldsymbol{\alpha}; T) = F(\alpha_1; T)$ agrees with our previous definition.

In Section 3.1, we describe various properties of $F(\boldsymbol{\alpha}; T)$. However, it is necessary to mention two of these here in order to state our first main result. Along the hyperplanes $\boldsymbol{\alpha} \cdot \mathbf{e}_n = \alpha_n = 0$, $1 \le n \le N$, the $N$-level form factor $F(\boldsymbol{\alpha};\ T)$ essentially degenerates into an $(N-1)$-level form factor. Moreover, it degenerates into an even lower order form factor on the intersections of these hyperplanes. These lower order factors have a 'spike' of width approximately $\log\log T/(2\log T)$. The model for this is the term $T^{-2|\alpha|}\log T$ in Montgomery's function $F(\alpha;\ T)$. We write $F^*(\boldsymbol{\alpha}; T)$ for the part of $F(\boldsymbol{\alpha}; T)$ that is supported outside the spikes from the lower correlation terms. Thus, (1.2) can be written as

$$\int_0^\infty F^*(\alpha;\ T)\,\mathrm{e}^{-2\alpha A}\,\mathrm{d}\alpha \sim f(A).$$

See Section 3.1 for a precise definition of $F^*$ and the statement of our Hypothesis LC describing the behaviour of the lower correlation terms.

We also require a hypothesis that asserts that averages of $F$ are bounded. This is less technical than Hypothesis LC, so we state it here.

HYPOTHESIS AC *We have*

$$\int_{x_1}^{x_1+1} \cdots \int_{x_{N-1}}^{x_{N-1}+1} |F^*(\boldsymbol{\alpha};\ T)|\,\mathrm{d}\boldsymbol{\alpha} \ll 1$$

*uniformly for* $(x_1, x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$.

The case $N = 2$ of Hypothesis AC is known to hold under the assumption of RH (see Goldston [6]). Our first main result is a generalization of the equivalence between $I$ and $F$ implied by the first two formulas in (1.2); it relates $I(\sigma, \mathbf{a}, \boldsymbol{\varepsilon}; T)$ to the Laplace transform of $F^*(\boldsymbol{\alpha};\ T)$ over a certain sector in $\mathbb{R}^{N-1}$. To describe the sector, we let $\beta \in \mathbb{R}$ and define

$$U_{N,\boldsymbol{\varepsilon}}(\beta) = \{(\alpha_1, \ldots, \alpha_{N-1}) \in \mathbb{R}^{N-1} \mid \varepsilon_1\alpha_1, \ldots, \varepsilon_N\alpha_N > \beta\},$$

where, as always, $\alpha_N = -\sum_{n<N} \alpha_n$.

THEOREM 2.1 *Assume RH, Hypotheses AC and LC. Let* $\mathbf{a} = (a_1, \ldots, a_N)$, *where the* $a_n \asymp 1/\log T$ *and are positive, and let* $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N)$ *consist of* $J \geq 0$ *ones followed by* $K \geq 1$ *negative ones. Then, with* $U_{N,\boldsymbol{\varepsilon}}(\beta)$ *as above, we have*

$$I\left(\frac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T\right) = T \log^N T \int_{U_{N,\boldsymbol{\varepsilon}}(0)} F^*(\boldsymbol{\alpha};\ T) T^{-\sum_{n\leq N} a_n\varepsilon_n\alpha_n}\,\mathrm{d}\boldsymbol{\alpha} + o(T \log^N T). \tag{2.3}$$

Note that Hypothesis LC has enabled us to state Theorem 2.1 in a way that suppresses the degenerate parts of the form factor.

Goldston *et al.* [9] proved that

$$I\left(\frac{1}{2} + \frac{A}{\log T}; T\right) \sim T \log^2 T \int_0^\infty F^*(\alpha;\ T)\,\mathrm{e}^{-2A\alpha}\,\mathrm{d}\alpha, \tag{2.4}$$

from which the equivalence between the first two formulas in (1.2) is immediate. Observe that (2.4) follows from Theorem 2.1 on taking $\mathbf{a} = (A/\log T, A/\log T)$ and $\boldsymbol{\varepsilon} = (1, -1)$.

If $J = 0$, $U_{N,\boldsymbol{\varepsilon}}$ is empty, so the integral in Theorem 2.1 equals zero. This makes sense because, in the integral defining $I(\frac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T)$, one can move the path of integration to the right, expand the integrand as a Dirichlet series and then integrate term-by-term to show that the integral is small. (For the same reason, if $K = 0$, the integral is also small, but our statement of the theorem requires that $K \geq 1$.)

From Theorem 2.1, we deduce the following corollary.

COROLLARY 2.1 *With the same hypotheses as in Theorem* 2.1, *we have*

$$I(\tfrac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T) \ll T \log^N T. \tag{2.5}$$

To compare this with other estimates for $I(\frac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T)$, write $a_n = A_n/\log T$ with $A_n > 0$, $1 \leq n \leq N$. Using a well-known approximation of $\zeta'/\zeta(s)$ by Dirichlet polynomials due to Selberg [11] (or see Titchmarsh [12, Chapter 14]), one can show that, on RH, if the $A_n$ are sufficiently large (as functions of $N$), then

$$I(\tfrac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T) \ll T \log^N T. \tag{2.6}$$

The extra hypotheses of the conjecture allow us to assert that the upper bound here holds for *all* $A_n > 0$.

In the special case where $N$ is even, $\mathbf{a} = (a_1, \ldots, a_{N/2}, a_1, \ldots, a_{N/2})$, $a_n = A_n/\log T > 0$ and $\boldsymbol{\varepsilon}$ consists of $N/2$ ones followed by $N/2$ minus ones, one can use a Bessel's inequality type of argument to show that if RH is true, then

$$I(\tfrac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T) \gg T \log^N T. \tag{2.7}$$

Here any positive $A_n$'s work, and there is no need to assume Hypothesis AC or LC. The idea is to square out the integrand in

$$0 \le \int_0^T \left| \prod_{n=1}^{N/2} \frac{\zeta'}{\zeta}\left(\frac{1}{2} + a_n + \mathrm{i}t\right) - \prod_{n=1}^{N/2}\left( -\sum_{n \le X} \frac{\Lambda(n)}{n^{1/2+a_n+\mathrm{i}t}}\right)\right|^2 \mathrm{d}t,$$

and estimate three of the resulting four terms by standard methods. Thus, for these special cases of $I$, the first term on the right-hand side of (2.3) is the main term.

Assuming that the zeros of the zeta-function follow the statistics predicted by random matrix theory, one can obtain an explicit formula for $F^*(\boldsymbol{\alpha};\ T)$ and, in principle, evaluate the right-hand side of the equation in Theorem 2.1. Although this would lead to an explicit formula for $I(\sigma, \mathbf{a}, \boldsymbol{\varepsilon}; T)$, in general it is a cumbersome task. We discuss the first few cases in Section 6. In any case, given $F(\boldsymbol{\alpha};\ T)$, one can in theory obtain an asymptotic formula for $I(\sigma, \mathbf{a}, \boldsymbol{\varepsilon}; T)$.

### 2.2. *Higher analogues of P*

Our second and third main results generalize the equivalence between $I$ and $P$ implied by the first and last formulas in (1.2).

To define our analogue of $P(\beta; T)$, let $\mathbf{b} = (b_1, b_2, \ldots, b_L)$ with $b_l > 0$ for $1 \le l \le L$. We define $\Lambda_{\mathbf{b}}(n)$ by

$$\prod_{l=1}^{L} \frac{\zeta'}{\zeta}(s + b_l) = (-1)^L \sum_n \frac{\Lambda_{\mathbf{b}}(n)}{n^s}, \tag{2.8}$$

where $\sigma > 1$. Then

$$\Lambda_{\mathbf{b}}(n) = \sum_{p_1^{\nu_1} p_2^{\nu_2} \cdots p_L^{\nu_L} = n} \frac{\log p_1 \cdots \log p_L}{p_1^{b_1\nu_1} p_2^{b_2\nu_2} \cdots p_L^{b_L\nu_L}}.$$

Thus, $\Lambda_{\mathbf{b}}(n)$ is supported on those positive integers $n$ that are representable as a product of $L$, not necessarily distinct, prime powers.

We define $R_{\mathbf{b}}(x)$ to be the sum of the residues of

$$\prod_{l=1}^{L} \frac{\zeta'}{\zeta}(s + b_l)\, \frac{x^s}{s},$$

at the points $s = 1 - b_l$. For example, if the $b_l$ are distinct, we have

$$R_{\mathbf{b}}(x) = -\sum_{l=1}^{L} \frac{x^{1-b_l}}{1-b_l} \prod_{\substack{j=1 \\ j\neq l}}^{L} \frac{\zeta'}{\zeta}(1 - b_l + b_j).$$

Next we set

$$\Psi_{\mathbf{b}}(x) = (-1)^L {\sum_{n\leq x}}' \Lambda_{\mathbf{b}}(n),$$

where the prime on the sum indicates that the term $\Lambda_{\mathbf{b}}(x)$ is counted with weight $\frac{1}{2}$. We also write

$$\Delta_{\mathbf{b}}(x) = \Psi_{\mathbf{b}}(x) - R_{\mathbf{b}}(x).$$

Thus, $\Delta_{\mathbf{b}}$ measures the difference between $\Psi_{\mathbf{b}}(x)$ and its expected value.

Now let $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ with $a_n > 0$ and $a_n \approx 1/\log T$ as before. Also let $\beta > 0$ and $1 \leq J < N$. Writing $\mathbf{a}_J = (a_1, a_2, \ldots, a_J)$ and $\mathbf{a}'_J = (a_{J+1}, a_{J+2}, \ldots, a_N)$, we set

$$P(\beta, \mathbf{a}, J;\ T) = \int_1^\infty \left(\Delta_{\mathbf{a}_J}\left(x + \frac{x}{T}\right) - \Delta_{\mathbf{a}_J}(x)\right)\left(\Delta_{\mathbf{a}'_J}\left(x + \frac{x}{T}\right) - \Delta_{\mathbf{a}'_J}(x)\right)\frac{\mathrm{d}x}{x^{2+2\beta}}.$$

This is our analogue of $P(\beta; T)$ in (1.2).

Our next two results relate $P$ to weighted and unweighted versions of $I$, respectively.

THEOREM 2.2 *Assume RH and let* $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ *with* $a_n = A_n/\log T$ *and* $A_n > 0$ *for* $1 \leq n \leq N$. *Also let* $1 \leq J < N$ *and* $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$, *where* $\varepsilon_1, \ldots, \varepsilon_J$ *are all one*, *and* $\varepsilon_{J+1}, \ldots, \varepsilon_N$ *are all negative one. Then*, *for* $\frac{1}{2} \leq \sigma \leq \frac{9}{10}$, *we have*

$$\int_{-\infty}^{\infty} \left(\prod_{n=1}^{N} \frac{\zeta'}{\zeta}(\sigma + a_n + \mathrm{i}\varepsilon_n t)\right)\left(\frac{\sin t/2T}{t}\right)^2 \mathrm{d}t = \frac{\pi}{2} P\left(\sigma - \frac{1}{2}, \mathbf{a}, J;\ T\right) + O\left(\frac{\log^{2N+1} T}{T^2}\right). \tag{2.9}$$

*The constant implied by the* $O$*-term depends on* $A_1, \ldots, A_N$ *but not on* $\sigma$, $J$ *or* $T$.

THEOREM 2.3 *Assume RH*, *Hypotheses AC and LC. Suppose that* $C$ *is fixed and positive*, *and that* $\mathbf{a} = (a_1, \ldots, a_N)$ *with* $a_n = A_n/\log T$ *and each* $A_n$ *fixed and positive. Define*

$$I_{\pm}(\sigma, \mathbf{a}, \boldsymbol{\varepsilon}; T) = \int_{-T}^{T} \prod_{n=1}^{N} \frac{\zeta'}{\zeta}(\sigma + a_n + \mathrm{i}\varepsilon_n t)\,\mathrm{d}t.$$

*If there exists a number* $f(C, \mathbf{A}, J)$ *such that one of the following asymptotic formulas holds*, *then the other also holds*:

$$\begin{aligned} I_{\pm}\left(\frac{1}{2} + \frac{C}{\log T}, \frac{\mathbf{A}}{\log T}, \boldsymbol{\varepsilon};\ T\right) &\sim f(C, \mathbf{A}, J)\, T\, \log^N T, \\ P\left(\frac{C}{\log T}, \frac{\mathbf{A}}{\log T}, J;\ T\right) &\sim f(C, \mathbf{A}, J) \frac{\log^N T}{2T}. \end{aligned} \tag{2.10}$$

The case $J = K = 1$ of (2.9) was proved by Goldston *et al.* [9]. Moreover, in this case we recover the equivalence between $I$ and $P$ implied by (1.2) from (2.10). Note that, in the formulas for $P(A/\log T; T)$ and $P(C/\log T, \mathbf{A}/\log T, J; T)$ in (1.2) and (2.10), the extra factor of $\frac{1}{2}$ in the latter appears because $I_{\pm}(1/2 + C/\log T, \mathbf{A}/\log T, \boldsymbol{\varepsilon}; T)$ is an integral over $[-T, T]$, whereas $I(1/2 + A/\log T; T)$ is over $[0, T]$.

To remove the sine weight from the left-hand side of (2.9) requires a Tauberian argument (see Lemma 5.2). It does not seem possible to prove an unweighted asymptotic such as $I_{\pm}(\sigma, \mathbf{a}, \boldsymbol{\varepsilon};\ T) \sim 2T^2 P(\sigma - 1/2, \mathbf{a}, J;\ T)$ without some additional assumption.

One might expect the $N$-correlation of the zeros to be related to a statement about prime $N$-tuples or some other generalization of the twin prime conjecture. However, the integral $P(\beta, \mathbf{a}, J;\ T)$ measures correlations in the discrepancy of the distributions of $\Lambda_{\mathbf{a}_J}$ and $\Lambda_{\mathbf{a}'_J}$. Thus, $N$-correlation is related to the representation of numbers as products of $J$ (not necessarily distinct) prime powers for all $J < N$.

## 3. Proof of Theorem 2.1 and Corollary 2.1

As the proof of Theorem 2.1 is involved, we carry it out in several stages. Our first goal is to precisely state the properties of $F(\boldsymbol{\alpha}; T)$ we shall need.

### 3.1. *Main properties of F*

As we have already mentioned in Section 2.1, we require several properties of $F(\boldsymbol{\alpha}; T)$. One of these, Hypothesis AC, was stated there and requires no further discussion.

Another follows directly from the definition of $F(\boldsymbol{\alpha}; T)$. Write $L = (1/2\pi)\log T$ and $\tilde{\gamma} = \gamma L$, so that $\mathrm{e}^{2\pi \mathrm{i}\alpha\tilde{\gamma}} = T^{\mathrm{i}\alpha\gamma}$. If $r$ is a real-valued function of bounded variation in $L^1(\mathbb{R}^{N-1})$, then

$$N(T)\int_{\mathbb{R}^{N-1}} F(\boldsymbol{\alpha}; T)\hat{r}(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} = \sum_{0<\gamma_1,\ldots,\gamma_N<T} r(\tilde{\gamma}_1 - \tilde{\gamma}_N, \ldots, \tilde{\gamma}_{N-1} - \tilde{\gamma}_N)w(\gamma_1 - \gamma_N, \ldots, \gamma_{N-1} - \gamma_N), \tag{3.1}$$

where

$$\hat{r}(\boldsymbol{\alpha}) = \int_{\mathbb{R}^{N-1}} r(u_1, u_2, \ldots, u_{n-1})\,\mathrm{e}^{-2\pi\mathrm{i}(\sum_{n<N}\alpha_n u_n)}\,\mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_{N-1}$$

is the Fourier transform of $r$.

The final property of $F(\boldsymbol{\alpha}; T)$ we need was mentioned briefly in Section 2.1, namely, that $F(\boldsymbol{\alpha}; T)$ contains essentially all the lower level form factors. We now elaborate on this with an example.

Suppose that $N = 3$, so that $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, and suppose that $\alpha_2 = 0$. Then, from (2.1), we see that

$$F(\alpha_1, 0; T) = \frac{1}{N(T)} \sum_{0<\gamma_1,\gamma_2,\gamma_3<T} T^{\mathrm{i}\alpha_1(\gamma_1-\gamma_3)} w(\gamma_1 - \gamma_3, \gamma_2 - \gamma_3).$$

Summing over $\gamma_2$, we find that

$$\sum_{0<\gamma_2<T} w(\gamma_1 - \gamma_3, \gamma_2 - \gamma_3) = w(\gamma_1 - \gamma_3) \sum_{0<\gamma_2<T} \frac{4}{4 + (\gamma_2 - \gamma_3)^2}.$$

The sum on the right is $\sim \log \gamma_3$ when $\log T < \gamma_3 < T - \log T$, and is $\ll \log T$ for the remaining $\gamma_3$ in $(0, T)$. Hence,

$$\sum_{0<\gamma_2<T} w(\gamma_1 - \gamma_3, \gamma_2 - \gamma_3) \sim w(\gamma_1 - \gamma_3) \log \gamma_3,$$

except for $O(\log^2 T)$ values of $\gamma_3$ in $(0, T)$. We therefore expect that

$$\begin{aligned} F(\alpha_1, 0; T) &\sim \frac{\log T}{N(T)} \sum_{0<\gamma_1,\gamma_3<T} T^{\mathrm{i}\alpha_1(\gamma_1-\gamma_3)} w(\gamma_1 - \gamma_3) \\ &= \log T\, F(\alpha_1; T). \end{aligned}$$

This calculation suggests that $F(\alpha_1, \alpha_2; T)$ should also be $\approx \log T\, F(\alpha_1; T)$ when $\alpha_2$ is so small that the term $T^{\mathrm{i}\alpha_2(\gamma_2-\gamma_3)}$ does not oscillate enough to cause significant cancellation. Since the 'spike' term in $F(\alpha_2; T)$ is $(1 + o(1))T^{-2|\alpha_2|} \log T$, we expect that $F(\alpha_1, \alpha_2; T)$ is approximately $T^{-2|\alpha_2|} \log T\ F(\alpha_1; T)$ when $|\alpha_2| \le \log\log T/(2\log T)$. Obviously, the same argument applies when $\alpha_1$ is near 0.

A similar phenomenon occurs when $\alpha_1 + \alpha_2$ is near 0. For suppose that $\alpha_1 + \alpha_2 = 0$. Then

$$F(\alpha_1, \alpha_2; T) = F(\alpha_1, -\alpha_1; T) = \frac{1}{N(T)} \sum_{0<\gamma_1,\gamma_2<T} T^{\mathrm{i}\alpha_1(\gamma_1-\gamma_2)} \sum_{0<\gamma_3<T} w(\gamma_1 - \gamma_3, \gamma_2 - \gamma_3).$$

The sum over $\gamma_3$ is more complicated than before, but one can show that, for most $\gamma_1, \gamma_2 \in (0, T)$, it is $\sim (1/2)w((\gamma_1 - \gamma_2)/2) \log T$. Thus, we expect that

$$F(\alpha_1, -\alpha_1; T) \approx \log T\, F(\alpha_1; T),$$

and that

$$F(\alpha_1, \alpha_2; T) \approx T^{-2|\alpha_1+\alpha_2|} \log T\, F(\alpha_1; T),$$

when $|\alpha_1 + \alpha_2| \le \log\log T/(2\log T)$. Clearly, $F(\alpha_1; T)$ may be replaced by $F(\alpha_2; T)$ in these two approximations.

More generally, $F(\boldsymbol{\alpha}; T)$ degenerates into a lower level sum on the set $S = \bigcup_{n=1}^{N} S_n$, where

$$S_n = \{\boldsymbol{\alpha} \in \mathbb{R}^{N-1} \mid \boldsymbol{\alpha} \cdot \mathbf{e}_n = 0\} \quad (1 \le n \le N). \tag{3.2}$$

The example above suggests that there are also lower correlation contributions for $\boldsymbol{\alpha}$ close to $S$. These lower correlations contribute a power of $\log T$ to $F(\boldsymbol{\alpha}; T)$ near $S$, but away from $S$ we expect $F(\boldsymbol{\alpha}; T)$ to be bounded in bounded regions.

To be more precise, we formulate a 'lower correlation' hypothesis. Let $Y$ be a subset of $\mathbb{R}^{N-1}$ and define a neighbourhood of $Y$ by

$$\eta(Y, \Delta) = \{\mathbf{t} \in \mathbb{R}^{N-1} \mid |\mathbf{t} - \mathbf{y}| < \Delta \text{ for some } \mathbf{y} \in Y\},$$

where $\Delta > 0$ and $|\mathbf{x}| = \max |x_j|$. If $n \le N - 1$, let $\tilde{\boldsymbol{\alpha}}_n$ be the $(N - 2)$-tuple obtained from $\boldsymbol{\alpha}$ by deleting $\alpha_n$. We then write $F(\tilde{\boldsymbol{\alpha}}_n;\ T)$ for the corresponding $(N - 1)$-level form factor summed over all ordinates except $\gamma_n$. If $n = N$, we let $\tilde{\boldsymbol{\alpha}}_N$ be the $(N-2)$-tuple obtained from $\boldsymbol{\alpha}$ by deleting any one of $\alpha_1, \dots, \alpha_{N-1}$ from $\boldsymbol{\alpha}$, say $\alpha_k$. We then let $F(\tilde{\boldsymbol{\alpha}}_N;\ T)$ be the $(N-1)$-level form factor

$$F(\tilde{\boldsymbol{\alpha}}_N; T) = \frac{1}{N(T)} \sum_{0<\gamma_1,\dots,\gamma_{N-1}<T} T^{\mathrm{i}\sum_{n<N}\alpha_n(\gamma_n-\gamma_k)} w(\gamma_1-\gamma_k, \dots, \gamma_{N-1}-\gamma_k).$$

We can now state our lower correlation hypothesis.

HYPOTHESIS LC *Let* $\epsilon = \epsilon(T) = \log\log T/(2\log T)$ *and let* $S_1, \dots, S_N$ *be the hyperplanes in* $\mathbb{R}^{N-1}$ *defined in* (3.2). *Then*

$$F(\boldsymbol{\alpha}; T) = F_*(\boldsymbol{\alpha}; T) + F^*(\boldsymbol{\alpha}; T),$$

*where* $F_*(\boldsymbol{\alpha}; T)$ *is supported on* $\eta(S, \epsilon)$ *and, for any fixed* $K > 0$, $F^*(\boldsymbol{\alpha}; T)$ *is bounded on the* $(N-1)$*-dimensional cube* $[-K, K]^{N-1}$, *as* $T \to \infty$. *Furthermore, if* $\boldsymbol{\alpha} \in \eta(S_n, \epsilon)$, $1 \le n \le N$, *then*

$$F_*(\boldsymbol{\alpha}; T) \ll |F(\tilde{\boldsymbol{\alpha}}_n; T)|T^{-2|\alpha_n|}\log T.$$

The final property of $F(\boldsymbol{\alpha}; T)$ we will need is that

$$\int_{x_1}^{x_1+1} \cdots \int_{x_{N-1}}^{x_{N-1}+1} |F(\boldsymbol{\alpha};\ T)|\,\mathrm{d}\boldsymbol{\alpha} \ll 1 \tag{3.3}$$

uniformly for $(x_1, x_2, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$. This follows easily by repeated applications of Hypotheses AC and LC.

### 3.2. *An expression for I in terms of F*

Our next goal is to express

$$I(\sigma, \mathbf{a}, \boldsymbol{\varepsilon};\ T) = \int_0^T \prod_{n=1}^N \frac{\zeta'}{\zeta}(\sigma + a_n + \mathrm{i}\varepsilon_n t)\,\mathrm{d}t$$

in a form on which we may use the Fourier inversion formula (3.1). Our starting point is Titchmarsh [**12**, Formula (14.4.1)], which says that, on RH,

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s}\,\mathrm{e}^{-\delta n} + \sum_\rho \delta^{s-\rho}\Gamma(\rho - s) + O(\delta^{\sigma-1/4}\log t)$$

uniformly for $\mathrm{e}^{-\sqrt{t}} \le \delta \le 1$ and $\frac{1}{2} \le \sigma \le \frac{9}{8}$. By the prime number theorem, the first sum on the right is bounded by $\int_1^\infty t^{-1/2}\,\mathrm{e}^{-\delta t}\,\mathrm{d}t \ll \delta^{-1/2}$. Thus, setting $X = (\log T)^{4/3} = \delta^{-1}$, we obtain the following lemma.

LEMMA 3.1 *Assume RH. Let* $X = (\log T)^{4/3}$, $a \approx 1/\log T$, *with* $a > 0$ *and* $\varepsilon = \pm 1$. *Then, for* $|t| < T$, *we have*

$$\frac{\zeta'}{\zeta}\left(\frac{1}{2} + a + \mathrm{i}\varepsilon t\right) = -\sum_{\gamma} R(-a + \mathrm{i}\varepsilon(\gamma - t)) + O(X^{1/2}),$$

*where* $R(z) = X^z \Gamma(z)$.

Using the lemma, we find that

$$I\left(\frac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon};\ T\right) = (-1)^N M(\mathbf{a}, \boldsymbol{\varepsilon};\ T) + E(\mathbf{a}, \boldsymbol{\varepsilon};\ T),$$

where

$$M(\mathbf{a}, \boldsymbol{\varepsilon};\ T) = \int_0^T \prod_{n=1}^{N} \left(\sum_{\gamma_n} R(-a_n + \mathrm{i}\varepsilon_n(\gamma_n - t))\right) \mathrm{d}t \tag{3.4}$$

and

$$E(\mathbf{a}, \boldsymbol{\varepsilon};\ T) \ll \sum_{S \subset \{1,2,\ldots,N\}} X^{(N-|S|)/2} \int_0^T \left|\prod_{n\in S}\left(\sum_{\gamma_n} R(-a_n + \mathrm{i}\varepsilon_n(\gamma_n - t))\right)\right| \mathrm{d}t.$$

Here the sum is over all proper subsets $S$ of $\{1, 2, \ldots, N\}$. By the Cauchy–Schwarz inequality

$$E(\mathbf{a}, \boldsymbol{\varepsilon};\ T) \ll T^{1/2} \sum_{S \subset \{1,2,\ldots,N\}} X^{(N-|S|)/2} \left(\int_0^T \left|\prod_{n\in S}\left(\sum_{\gamma_n} R(-a_n + \mathrm{i}\varepsilon_n(\gamma_n - t))\right)\right|^2 \mathrm{d}t\right)^{1/2}.$$

Note that the integral here has the form $M(\mathbf{a}', \boldsymbol{\varepsilon}';\ T)$, with $\boldsymbol{\varepsilon}'$ a vector of $|S|$ ones followed by $|S|$ negative ones and $\mathbf{a}'$ a vector of type $(a_1, a_2, \ldots, a_{|S|}, a_1, a_2, \ldots, a_{|S|})$. Later we will see that, under the hypotheses of Theorem 2.1, $M(\mathbf{a}, \boldsymbol{\varepsilon}; T) \ll TL^N$. Hence,

$$\begin{aligned} E(\mathbf{a}, \boldsymbol{\varepsilon}; T) &\ll T^{1/2} \sum_{S \subset \{1,2,\ldots,N\}} X^{(N-|S|)/2}\, T^{1/2} L^{|S|} \\ &\ll T X^{1/2} L^{N-1} = T L^{N-1/3}. \end{aligned}$$

Thus,

$$I(\tfrac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T) = (-1)^N M(\mathbf{a}, \boldsymbol{\varepsilon}; T) + O(T\, L^{N-1/3}), \tag{3.5}$$

with $M(\mathbf{a}, \boldsymbol{\varepsilon}; T)$ given by (3.4).

Our next goal is to estimate $M(\mathbf{a}, \boldsymbol{\varepsilon}; T)$. As in previous treatments of such expressions (see [**9**, **10**]), we may truncate the sums over the zeros, removing the ordinates with $\gamma < 0$ and $\gamma > T$. They make a negligible contribution, namely $O(L^B)$ for some positive integer $B$, because the gamma function decays so quickly. Similarly, we may extend the integral from $-\infty$ to $\infty$ with a change of at most $O(L^B)$. After doing so, making the change of variable $t \to t + \gamma_N$, and recalling that $\tilde{\gamma}_j = \gamma_j L$, we find that

$$M(\mathbf{a},\boldsymbol{\varepsilon};T)=\sum_{0<\gamma_1,\ldots,\gamma_N<T}\int_{-\infty}^{\infty}\prod_{n=1}^{N}R(-a_n+\mathrm{i}\varepsilon_n(\gamma_n-\gamma_N-t))\,\mathrm{d}t+O(L^B),$$
$$=\sum_{0<\gamma_1,\ldots,\gamma_N<T}\mathcal{R}(\tilde{\gamma}_1-\tilde{\gamma}_N,\ldots,\tilde{\gamma}_{N-1}-\tilde{\gamma}_N)+O(L^B), \tag{3.6}$$

where

$$\mathcal{R}(\mathbf{u})=\int_{-\infty}^{\infty}\prod_{n=1}^{N}R(-a_n+\mathrm{i}\varepsilon_n(u_n/L-t))\,\mathrm{d}t. \tag{3.7}$$

Here and for the remainder of the argument, we set $\mathbf{u}=(u_1,\ldots,u_{N-1})$ and $u_N=0$. Note that, in the definition of $\mathcal{R}(\mathbf{u})$, $u_N$ only occurs on the right. We also write $\mathrm{d}\mathbf{u}=\mathrm{d}u_1\cdots\mathrm{d}u_{N-1}$.

We would like to use (3.1) to express the sum on the second line of (3.6) in terms of $F(\boldsymbol{\alpha};T)$. However, the weight function $w$ defined in (2.2) is missing, so we need to insert it. To this end, we define

$$r(\mathbf{u})=\frac{\mathcal{R}(\mathbf{u})}{w(\mathbf{u}/L)}.$$

From (3.5) and (3.6), we now obtain

$$I\left(\frac{1}{2},\mathbf{a},\boldsymbol{\varepsilon};\ T\right)=(-1)^N\sum_{0<\gamma_1,\ldots,\gamma_N<T}r(\tilde{\gamma}_1-\tilde{\gamma}_N,\ldots,\tilde{\gamma}_{N-1}-\tilde{\gamma}_N)w(\gamma_1-\gamma_N,\ldots,\gamma_{N-1}-\gamma_N)$$
$$+O(TL^{N-1/3}). \tag{3.8}$$

By an application of equation (3.1), we then find that

$$I\left(\frac{1}{2},\mathbf{a},\boldsymbol{\varepsilon};T\right)=(-1)^N N(T)\int_{\mathbb{R}^{N-1}}F(\boldsymbol{\alpha};T)\hat{r}(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha}+O(T\,L^{N-1/3}). \tag{3.9}$$

Our next task is to find a useful expression for the Fourier transform of $r(\mathbf{u})$.

### 3.3. *The Fourier transforms of* $r(\mathbf{u})$ *and* $\mathcal{R}(\mathbf{u})$

To determine $\hat{r}$, we must also determine $\hat{\mathcal{R}}$, and to do this, we use the following lemma.

LEMMA 3.2 *Let* $0<a<1$, $A\in\mathbb{R}$ *and* $\varepsilon=\pm1$. *Then*

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}A\xi}\Gamma(-a+\mathrm{i}\varepsilon\xi)\,\mathrm{d}\xi=2\pi\,\mathrm{e}^{\varepsilon aA}(\mathrm{e}^{-\mathrm{e}^{-\varepsilon A}}-1).$$

*Proof*. We treat the integral as a contour integral along the imaginary axis. When $\varepsilon=1$, the integral is $-\mathrm{i}\,\mathrm{e}^{aA}\int_{-a-\mathrm{i}\infty}^{-a+\mathrm{i}\infty}\mathrm{e}^{Az}\Gamma(z)\,\mathrm{d}z$. Moving the contour left to $-\infty$, we pick up a sum of residues from poles at the negative integers that is easily recognized to be the Taylor series for the expression on the right. The case when $\varepsilon = -1$ follows from this by conjugation, which has the effect of replacing $A$ by $-A$. □

We are now ready to calculate $\hat{\mathcal{R}}$. Recalling that $u_N = 0$, we see that

$$\begin{aligned}\hat{\mathcal{R}}(\boldsymbol{\alpha}) &= \int_{\mathbb{R}^{N-1}} \mathcal{R}(\mathbf{u})\, \mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{\alpha}\cdot\mathbf{u}}\, \mathrm{d}\mathbf{u} \\ &= \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{\infty} \left( \prod_{n=1}^{N} R(-a_n + \mathrm{i}\varepsilon_n(u_n/L - t))\, \mathrm{e}^{-2\pi \mathrm{i}\alpha_n u_n} \right) \mathrm{d}t\, \mathrm{d}\mathbf{u}.\end{aligned}$$

We substitute $\xi_n$ for $u_n/L - t$ (note that $\xi_N = -t$) and obtain

$$\begin{aligned}\hat{\mathcal{R}}(\boldsymbol{\alpha}) &= L^{N-1} \int_{\mathbb{R}^N} \prod_{n=1}^{N} R(-a_n + \mathrm{i}\varepsilon_n \xi_n)\, \mathrm{e}^{-2\pi \mathrm{i} L \sum_{n<N} \alpha_n(\xi_n - \xi_N)}\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_N \\ &= L^{N-1} \prod_{n=1}^{N} \left( \int_{-\infty}^{\infty} R(-a_n + \mathrm{i}\varepsilon_n \xi_n)\, \mathrm{e}^{-2\pi \mathrm{i}\alpha_n \xi_n L}\, \mathrm{d}\xi_n \right).\end{aligned}$$

The second line follows from the first on using $\alpha_N = -\sum_{n<N} \alpha_n$. By Lemma 3.2, and since $\varepsilon_n^2 = 1$, we find that

$$\begin{aligned}&\int_{-\infty}^{\infty} R(-a_n + \mathrm{i}\varepsilon_n \xi_n)\, \mathrm{e}^{-2\pi \mathrm{i}\alpha_n \xi_n L}\, \mathrm{d}\xi_n \\ &\quad= X^{-a_n} \int_{-\infty}^{\infty} \Gamma(-a_n + \mathrm{i}\varepsilon_n \xi_n)\, \mathrm{e}^{\mathrm{i}\xi_n(\varepsilon_n \log X - 2\pi\alpha_n L)}\, \mathrm{d}\xi_n \\ &\quad= 2\pi X^{-a_n}\, \mathrm{e}^{\varepsilon_n a_n(\varepsilon_n \log X - 2\pi\alpha_n L)} (\exp(-\mathrm{e}^{-\varepsilon_n(\varepsilon_n \log X - 2\pi\alpha_n L)}) - 1) \\ &\quad= 2\pi\, \mathrm{e}^{-2\pi a_n \varepsilon_n \alpha_n L} (\exp(-\mathrm{e}^{-\log X + 2\pi\varepsilon_n \alpha_n L}) - 1).\end{aligned}$$

Thus, writing

$$R_n(z) = \mathrm{e}^{-2\pi\varepsilon_n a_n z L} (\exp(-\mathrm{e}^{-\log X + 2\pi\varepsilon_n z L}) - 1), \tag{3.10}$$

for $1 \le n \le N$ and $z \in \mathbb{C}$, we have

$$\hat{\mathcal{R}}(\boldsymbol{\alpha}) = (2\pi)^N L^{N-1} \prod_{n=1}^{N} R_n(\alpha_n). \tag{3.11}$$

Next, we calculate $\hat{r}(\boldsymbol{\alpha})$. From the definitions of $r$ and $w$ in (3.8) and (2.2), we see that

$$\begin{aligned}\hat{r}(\boldsymbol{\alpha}) &= \int_{\mathbb{R}^{N-1}} r(\mathbf{u})\, \mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{\alpha}\cdot\mathbf{u}}\, \mathrm{d}\mathbf{u} \\ &= \int_{\mathbb{R}^{N-1}} \mathcal{R}(\mathbf{u}) \prod_{n=1}^{N-1} \left(1 + \frac{u_n^2}{4L^2}\right) \mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{\alpha}\cdot\mathbf{u}}\, \mathrm{d}\mathbf{u}\end{aligned}$$

$$= \int_{\mathbb{R}^{N-1}} \prod_{n=1}^{N-1} \left(1 - \frac{1}{16\pi^2 L^2} \frac{\partial^2}{\partial \alpha_n^2}\right) \mathcal{R}(\mathbf{u})\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{\alpha}\cdot\mathbf{u}}\, \mathrm{d}\mathbf{u}$$
$$= \prod_{n=1}^{N-1} \left(1 - \frac{1}{16\pi^2 L^2} \frac{\partial^2}{\partial \alpha_n^2}\right) \int_{\mathbb{R}^{N-1}} \mathcal{R}(\mathbf{u})\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{\alpha}\cdot\mathbf{u}}\, \mathrm{d}\mathbf{u}$$
$$= \prod_{n=1}^{N-1} \left(1 - \frac{1}{16\pi^2 L^2} \frac{\partial^2}{\partial \alpha_n^2}\right) \hat{\mathcal{R}}(\boldsymbol{\alpha}).$$

The inversion of the differential operators and integrals may be justified by the exponential decay of $\mathcal{R}(\mathbf{u})$ and the fact that the partial derivatives of $\mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{\alpha}\cdot\mathbf{u}}$ with respect to the $\alpha_n$'s, up to and including those of the second order, are continuous. The following lemma shows that $\mathcal{R}(\mathbf{u})$ does indeed decay exponentially.

LEMMA 3.3

$$\mathcal{R}(\mathbf{u}) \ll L^N \exp\left(-\frac{1}{NL} \sum_{n<N} |u_n|\right)$$

*uniformly for* $\mathbf{u} \in \mathbb{R}^{N-1}$.

*Proof*. Using (3.7) and Stirling's formula, and recalling that $u_N = 0$ and $a_n \gg 1/\log T$ for $1 \le n \le N$, we see that

$$\mathcal{R}(L\mathbf{u}) = \int_{-\infty}^{\infty} X^{i\sum_{n<N} \varepsilon_n (u_n - t)} \prod_{n=1}^{N} \Gamma(-a_n + \mathrm{i}\varepsilon_n(u_n - t))\, \mathrm{d}t$$
$$\ll \int_{-\infty}^{\infty} \prod_{n=1}^{N} (L\, \mathrm{e}^{-|u_n - t|})\, \mathrm{d}t = L^N \int_{-\infty}^{\infty} \exp\left(-\sum_{n\le N} |u_n - t|\right) \mathrm{d}t.$$

Now

$$\sum_{n\le N} |u_n - t| \ge |t| + \frac{1}{N} \sum_{n<N} (|u_n| - |t|) = \frac{|t|}{N} + \frac{1}{N} \sum_{n<N} |u_n|.$$

Hence,

$$\mathcal{R}(L\mathbf{u}) \ll L^N \exp\left(-\frac{1}{N} \sum_{n<N} |u_n|\right),$$

which implies the result. □

It follows from (3.10) and (3.11) that $\hat{\mathcal{R}}(\boldsymbol{\alpha})$ is analytic in the $N-1$ variables $\alpha_1, \ldots, \alpha_{N-1}$ regarded as complex variables. Thus, by Cauchy's integral formula we have

$$\hat{r}(\boldsymbol{\alpha}) = \frac{1}{(2\pi \mathrm{i})^{N-1}} \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_{N-1}} \prod_{n=1}^{N-1} \left(\frac{1}{z_n - \alpha_n} - \frac{1}{8\pi^2 L^2 (z_n - \alpha_n)^3}\right)$$
$$\times\, \hat{\mathcal{R}}(z_1, \ldots, z_{N-1})\, \mathrm{d}z_{N-1} \cdots \mathrm{d}z_1,$$

where $\mathcal{C}_n$, $1 \le n \le N-1$, is the positively oriented circle $|z_n - \alpha_n| = 1/(10L)$. Finally, writing $z_N = -\sum_{n<N} z_n$ and using (3.11), we find that

$$\hat{r}(\boldsymbol{\alpha}) = 2\pi \left(\frac{\log T}{2\pi \mathrm{i}}\right)^{N-1} \int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_{N-1}} \prod_{n=1}^{N-1} \left(\frac{1}{z_n - \alpha_n} - \frac{1}{8\pi^2 L^2 (z_n - \alpha_n)^3}\right) \times \prod_{n=1}^{N} R_n(z_n)\, \mathrm{d}z_{N-1} \cdots \mathrm{d}z_1. \tag{3.12}$$

### 3.4. *Estimates for $R_n$ and $\hat{r}$*

In order to complete the proof of Theorem 2.1, we need several estimates for $R_n$ and $\hat{r}$. These are provided by the next few lemmas.

LEMMA 3.4 *Let $\alpha_n \in \mathbb{R}$ and $z \in \mathbb{C}$. If $|z - \alpha_n| \le 1/(10L)$, then*

$$R_n(z) = \begin{cases} -\mathrm{e}^{-2\pi\varepsilon_n\alpha_n a_n L}(1 + O(X\,\mathrm{e}^{-2\pi\varepsilon_n\alpha_n L} + L^{-1})) & \text{if } \varepsilon_n\alpha_n > \dfrac{\log X}{\log T}, \\ \asymp X^{-1}\,\mathrm{e}^{2\pi\varepsilon_n\alpha_n(1-a_n)L} & \text{if } \varepsilon_n\alpha_n \le \dfrac{\log X}{\log T}. \end{cases} \tag{3.13}$$

*In particular,*

$$R_n(z) \ll R_n(\alpha_n), \tag{3.14}$$

*for $|z - \alpha_n| \le 1/(10L)$.*

*Proof*. The bound in (3.14) follows easily from (3.13). To prove (3.13), let $z = \alpha_n + w/L$ with $w = \rho\,\mathrm{e}^{\mathrm{i}\theta}$, $0 \le \rho \le 1/10$ and $0 \le \theta < 2\pi$. By definition,

$$R_n(z) = \mathrm{e}^{-2\pi\varepsilon_n a_n z L}(\exp(-\mathrm{e}^{-\log X + 2\pi\varepsilon_n z L}) - 1). \tag{3.15}$$

Since $a_n \asymp 1/L$, the first factor is

$$\mathrm{e}^{-2\pi\varepsilon_n a_n z L} = \mathrm{e}^{-2\pi\varepsilon_n\alpha_n a_n L}(1 + O(L^{-1})). \tag{3.16}$$

Suppose that $\varepsilon_n\alpha_n > \log X/\log T$. Then

$$\begin{aligned} |\exp(-\mathrm{e}^{-\log X + 2\pi\varepsilon_n z L})| &= |\exp(-\mathrm{e}^{-\log X + 2\pi\varepsilon_n\alpha_n L + 2\pi\varepsilon_n w})| \\ &= \exp(-\mathrm{e}^{-\log X + 2\pi\varepsilon_n\alpha_n L}\,\Re\,\mathrm{e}^{2\pi\varepsilon_n w}). \end{aligned}$$

Now

$$\Re\mathrm{e}^{2\pi\varepsilon_n w} = \mathrm{e}^{2\pi\varepsilon_n\rho\cos\theta}\cos(2\pi\varepsilon_n\rho\sin\theta) \ge \mathrm{e}^{-2\pi/10}\cos(2\pi/10) > 0.4316 > \cdots > \tfrac{2}{5}.$$

Hence,

$$|\exp(-\mathrm{e}^{-\log X + 2\pi\varepsilon_n z L})| < \exp(-2\,\mathrm{e}^{2\pi\varepsilon_n\alpha_n L}/5X) \ll X\,\mathrm{e}^{-2\pi\varepsilon_n\alpha_n L},$$

the last estimate following from the inequality $\mathrm{e}^{-y} \ll 1/y$ for $y \ge 1$. Using this bound and (3.16) in (3.15), we obtain the first estimate in (3.13).

Suppose next that $\varepsilon_n\alpha_n \le \log X/\log T$. Then $|X^{-1}\,\mathrm{e}^{2\pi\varepsilon_n zL}| \le \mathrm{e}^{2\pi/10} < 2$. Thus, by the estimate $\mathrm{e}^z - 1 \asymp |z|$ for $|z| < 2$, we have

$$-1 + \exp(-X^{-1}\,\mathrm{e}^{2\pi\varepsilon_n zL}) \asymp |X^{-1}\,\mathrm{e}^{2\pi\varepsilon_n zL}| \asymp X^{-1}\,\mathrm{e}^{2\pi\varepsilon_n\alpha_n L}.$$

Combining this and (3.16) in (3.15), we obtain the second estimate in (3.13). □

LEMMA 3.5 *There exist positive constants* $A_n$ *such that*

$$R_n(u) \ll \mathrm{e}^{-A_n|u|} \quad (1 \le n \le N). \tag{3.17}$$

*Moreover, if* $\epsilon = \log\log T/(2\log T)$, *then*

$$\max_{u\in[-\epsilon,\epsilon]} |R_n(u)| \ll (\log T)^{-5/6}. \tag{3.18}$$

*Proof*. We take $z = \alpha_n$ and set $u = \varepsilon_n\alpha_n$ in (3.13) for $R_n(z)$. Since $a_n \asymp 1/\log T$ and $a_n > 0$, there exists a constant $A_n > 0$ such that $a_n \ge A_n/\log T$. Therefore, from the first estimate in (3.13) we have $R_n(\varepsilon_n u) \ll \mathrm{e}^{-A_n u}$ if $u > \log X/\log T$. Hence,

$$R_n(u) \ll \mathrm{e}^{-A_n|u|} \quad \text{if } |u| > \log X/\log T. \tag{3.19}$$

Similarly, from the second estimate in (3.13) we see that

$$R_n(u) \ll 1 \quad \text{if } |u| \le \log X/\log T.$$

Thus, by this and (3.19), (3.17) holds for all $u \in \mathbb{R}$.

To obtain (3.18), note that $X = (\log T)^{4/3}$, so we have $\epsilon = \log\log T/(2\log T) < \log X/\log T$. We may therefore use the second bound in (3.13), from which we see that

$$\max_{u\in[-\epsilon,\epsilon]} |R_n(u)| \ll X^{-1}\,\mathrm{e}^{\epsilon(1-a_n)\log T} \ll X^{-1}\,\mathrm{e}^{(\log\log T)/2}$$
$$\ll (\log T)^{1/2}(\log T)^{-4/3} = (\log T)^{-5/6}.$$

□

LEMMA 3.6 *We have*

$$\hat{r}(\boldsymbol{\alpha}) \ll L^{N-1}\prod_{n=1}^{N}|R_n(\alpha_n)| \tag{3.20}$$

*uniformly for* $\boldsymbol{\alpha} \in \mathbb{R}^{N-1}$. *Moreover, any of the terms in the product may be deleted. Also, if* $\boldsymbol{\alpha} \in U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)$, *then*

$$\hat{r}(\boldsymbol{\alpha}) = (-1)^N 2\pi T^{-\sum_{n\le N} a_n\varepsilon_n\alpha_n}\log^{N-1} T\left\{1 + O\left(X\sum_{n=1}^{N}\mathrm{e}^{-2\pi L\varepsilon_n\alpha_n} + L^{-1}\right)\right\}. \tag{3.21}$$

*Proof*. The bound (3.20) follows from (3.12) and (3.14). We may delete any term in the product because (3.17) implies that $R_n(\alpha_n) \ll 1$. We obtain the second assertion by using (3.13) in (3.12). □

### 3.5. *Completion of the Proof of Theorem* 2.1

According to (3.9), we have

$$I\left(\frac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T\right) = (-1)^N N(T) \int_{\mathbb{R}^{N-1}} F(\boldsymbol{\alpha}; T)\hat{r}(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} + O(TL^{N-1/3}).$$

To estimate the integral, we decompose it into three pieces, $I_1$, $I_2$ and $I_3$, where $I_1$ is the portion of the integral over $\eta(S, \epsilon)$, $I_2$ is over $U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)$ and $I_3$ is over $\mathbb{R}^{N-1} \setminus (U_{N,\boldsymbol{\varepsilon}}(\log X/\log T) \cup \eta(S, \epsilon))$. Thus,

$$I(\tfrac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T) = (-1)^N N(T)(I_1 + I_2 + I_3) + O(TL^{N-1/3}). \tag{3.22}$$

First consider $I_1$. By (3.20)

$$\begin{aligned} I_1 = \int_{\eta(S,\epsilon)} F(\boldsymbol{\alpha}; T)\hat{r}(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} &\ll L^{N-1} \int_{\eta(S,\epsilon)} |F(\boldsymbol{\alpha}; T)| \prod_{n=1}^{N} |R_n(\alpha_n)|\, \mathrm{d}\boldsymbol{\alpha} \\ &\ll L^{N-1} \sum_{m=1}^{N} \int_{\eta(S_m,\epsilon)} |F(\boldsymbol{\alpha}; T)| \prod_{n=1}^{N} |R_n(\alpha_n)|\, \mathrm{d}\boldsymbol{\alpha}. \end{aligned} \tag{3.23}$$

To estimate the integral over $\eta(S_m, \epsilon)$, for $m = 1, \ldots, N-1$, let $\mathbf{k} = (k_1, \ldots, k_{N-1}) \in \mathbb{Z}^{N-1}$ and let $\tilde{\mathbf{k}}_m$ be the $(N-2)$-tuple obtained from $\mathbf{k}$ by deleting $k_m$. Furthermore, let $I_k = (k, k+1]$ and

$$I(\tilde{\mathbf{k}}_m) = I_{k_1} \times \cdots \times I_{k_{m-1}} \times [-\epsilon, \epsilon] \times I_{k_{m+1}} \times \cdots \times I_{k_{N-1}}.$$

By the comment after (3.20), we may delete the $N$th term of the product on the last line of (3.23). Hence,

$$\begin{aligned} \int_{\eta(S_m,\epsilon)} |F(\boldsymbol{\alpha}; T)| \prod_{n=1}^{N} |R_n(\alpha_n)|\, \mathrm{d}\boldsymbol{\alpha} &\ll \sum_{\tilde{\mathbf{k}}_m} \int_{I(\tilde{\mathbf{k}}_m)} |F(\boldsymbol{\alpha}; T)| \prod_{n=1}^{N-1} |R_n(\alpha_n)|\, \mathrm{d}\boldsymbol{\alpha} \\ &\ll \max_{\alpha_m \in [-\epsilon,\epsilon]} |R_m(\alpha_m)| \sum_{\tilde{\mathbf{k}}_m} \left( \prod_{\substack{1 \le n < N \\ n \ne m}} \max_{\alpha_n \in I_{k_n}} |R_n(\alpha_n)| \right) \\ &\quad \times \int_{I(\tilde{\mathbf{k}}_m)} |F(\boldsymbol{\alpha}; T)|\, \mathrm{d}\boldsymbol{\alpha}. \end{aligned}$$

By (3.3), the integral is $\ll 1$ uniformly in $\tilde{\mathbf{k}}_m$. Thus, by (3.17) and (3.18), this is

$$\ll \max_{\alpha_m \in [-\epsilon,\epsilon]} |R_m(\alpha_m)| \prod_{\substack{1\le n<N\\ n\neq m}} \left( \sum_{k_n=-\infty}^{\infty} \max_{\alpha_n \in I_{k_n}} |R_n(\alpha_n)| \right)$$

$$\ll (\log T)^{-5/6} \prod_{\substack{1\le n<N\\ n\neq m}} \left( \sum_{k_n=-\infty}^{\infty} \mathrm{e}^{-A_n|k_n|} \right) \ll (\log T)^{-5/6}. \tag{3.24}$$

The integral over $\eta(S_N,\epsilon)$ is handled similarly. We let $\mathbf{k} = (k_1,\ldots,k_{N-1})$ and $I_k = (k, k+1]$ be as before, and write

$$I(\mathbf{k}) = I_{k_1} \times I_{k_2} \times \cdots \times I_{k_{N-1}}.$$

Then

$$\begin{aligned}\int_{\eta(S_N,\epsilon)} |F(\boldsymbol{\alpha};T)| \prod_{n=1}^{N} |R_n(\alpha_n)|\,\mathrm{d}\boldsymbol{\alpha} &= \sum_{\mathbf{k}} \int_{I(\mathbf{k})\cap\eta(S_N,\epsilon)} |F(\boldsymbol{\alpha};T)| \prod_{n=1}^{N} |R_n(\alpha_n)|\,\mathrm{d}\boldsymbol{\alpha}\\ &\ll \max_{\alpha_N\in[-\epsilon,\epsilon]} |R_N(\alpha_N)| \sum_{\mathbf{k}} \int_{I(\mathbf{k})} |F(\boldsymbol{\alpha};T)| \prod_{n=1}^{N-1} |R_n(\alpha_n)|\,\mathrm{d}\boldsymbol{\alpha}\\ &\ll \max_{\alpha_N\in[-\epsilon,\epsilon]} |R_N(\alpha_N)| \sum_{\mathbf{k}} \left( \prod_{1\le n<N} \max_{\alpha_n\in I_{k_n}} |R_n(\alpha_n)| \right)\\ &\quad \times \int_{I(\mathbf{k})} |F(\boldsymbol{\alpha};T)|\,\mathrm{d}\boldsymbol{\alpha}.\end{aligned}$$

As before, by (3.3) and Lemma 3.5, this is $\ll (\log T)^{-5/6}$. Combining this with (3.24) in (3.23), we see that

$$I_1 \ll L^{N-11/6}. \tag{3.25}$$

Next, we estimate

$$I_2 = \int_{U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)} F(\boldsymbol{\alpha};T)\hat{r}(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha}.$$

Since $U_{N,\boldsymbol{\varepsilon}}(\log X/\log T) \cap \eta(S,\epsilon) = \emptyset$, by Hypothesis LC, we may replace $F(\boldsymbol{\alpha};T)$ by $F^*(\boldsymbol{\alpha};T)$. Using this and our estimate for $\hat{r}$ from (3.21), we then find that

$$\begin{aligned} I_2 &= (-1)^N 2\pi \log^{N-1} T \int_{U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)} F^*(\boldsymbol{\alpha};T) T^{-\sum_{n\le N} a_n\varepsilon_n\alpha_n}\,\mathrm{d}\boldsymbol{\alpha}\\ &\quad + O\left( XL^{N-1} \sum_{m=1}^{N} \int_{U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)} |F^*(\boldsymbol{\alpha};T)|\,\mathrm{e}^{-2\pi L\varepsilon_m\alpha_m} T^{-\sum_{n\le N} a_n\varepsilon_n\alpha_n}\,\mathrm{d}\boldsymbol{\alpha} \right)\\ &\quad + O\left( L^{N-2} \int_{U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)} |F^*(\boldsymbol{\alpha};T)| T^{-\sum_{n\le N} a_n\varepsilon_n\alpha_n}\,\mathrm{d}\boldsymbol{\alpha} \right)\\ &= I_{21} + I_{22} + I_{23}, \end{aligned} \tag{3.26}$$

say.

In estimating $I_{22}$, we treat only the term $m = 1$ since the others are handled the same way (and satisfy the same bound). We have

$$I_{22} \ll XL^{N-1} \int_{U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)} |F^*(\boldsymbol{\alpha}; T)| \, \mathrm{e}^{-\varepsilon_1 \alpha_1 \log T} \left( \prod_{n=1}^{N} \mathrm{e}^{-a_n \varepsilon_n \alpha_n \log T} \right) \mathrm{d}\boldsymbol{\alpha}.$$

Now, for each $n$, $a_n \varepsilon_n \alpha_n > 0$, so we may delete the factors corresponding to $n = 1$ and $n = N$ from the product. We then substitute $\lambda_n$ for $\varepsilon_n \alpha_n$ and note that $\varepsilon_n = \varepsilon_n^{-1}$ to see that

$$I_{22} \ll XL^{N-1} \int_{\log X/\log T}^{\infty} \cdots \int_{\log X/\log T}^{\infty} |F^*(\varepsilon_1 \lambda_1, \ldots, \varepsilon_{N-1} \lambda_{N-1}; T)| \, \mathrm{e}^{-\lambda_1 \log T} \left( \prod_{n=2}^{N-1} \mathrm{e}^{-a_n \lambda_n \log T} \right) \mathrm{d}\boldsymbol{\lambda},$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{N-1})$ and $\mathrm{d}\boldsymbol{\lambda} = \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{N-1}$. Now let $K$ be a large positive integer and split this into two pieces, $J_1$ and $J_2$, where $J_1$ is $XL^{N-1}$ times the contribution from the integral over the box $\mathscr{B} = (\log X/\log T, K]^{N-1}$, and $J_2$ is the rest. Then

$$I_{22} \ll J_1 + J_2. \tag{3.27}$$

To treat $J_1$, we observe that, by Hypothesis LC, $F^*(\boldsymbol{\alpha}; T)$ is bounded on $\mathscr{B}$. Also, since $a_n \approx 1/\log T$ and $a_n > 0$ for each $n$, there exist positive constants $A_n$ such that $a_n \geq A_n/\log T$. Therefore,

$$\begin{aligned} J_1 &\ll XL^{N-1} \int_{\log X/\log T}^{K} \cdots \int_{\log X/\log T}^{K} \mathrm{e}^{-\lambda_1 \log T} \left( \prod_{n=2}^{N-1} \mathrm{e}^{-a_n \lambda_n \log T} \right) \mathrm{d}\boldsymbol{\lambda} \\ &\ll XL^{N-1} \left( \int_{\log X/\log T}^{\infty} \mathrm{e}^{-\lambda_1 \log T} \, \mathrm{d}\lambda_1 \right) \prod_{n=2}^{N-1} \left( \int_0^{\infty} \mathrm{e}^{-A_n \lambda_n} \, \mathrm{d}\lambda_n \right) \\ &\ll L^{N-2}. \end{aligned} \tag{3.28}$$

To treat $J_2$, we split $[\log X/\log T, \infty)$ into the intervals $I_0 = (\log X/\log T, 1]$, $I_1 = (1, 2]$, $I_2 = (2, 3], \ldots$. Let $\mathbf{k} = (k_1, k_2, \ldots, k_{N-1})$ denote an $(N-1)$-tuple of non-negative integers and write

$$I(\mathbf{k}) = I_{k_1} \times I_{k_2} \times \cdots \times I_{k_{N-1}}.$$

Then

$$J_2 \ll XL^{N-1} \sum_{\mathbf{k}} \int_{I(\mathbf{k})} |F^*(\varepsilon_1 \lambda_1, \ldots, \varepsilon_{N-1} \lambda_{N-1}; T)| \left( \mathrm{e}^{-\lambda_1 \log T} \prod_{n=2}^{N-1} \mathrm{e}^{-A_n \lambda_n} \right) \mathrm{d}\boldsymbol{\lambda},$$

where the sum is over all tuples $\mathbf{k}$ with at least one component $k_n \geq K$. The expression in parentheses is

$$\leq C(k_1) \prod_{n=2}^{N-1} e^{-A_n k_n} \quad \text{with } C(k_1) = \begin{cases} T^{-k_1} & \text{if } k_1 \geq 1, \\ X^{-1} & \text{if } k_1 = 0. \end{cases}$$

Hence,

$$J_2 \ll XL^{N-1} \sum_{\mathbf{k}} \left( C(k_1) \prod_{n=2}^{N-1} \mathrm{e}^{-A_n k_n} \int_{I(\mathbf{k})} |F^*(\varepsilon_1 \lambda_1, \ldots, \varepsilon_{N-1}\lambda_{N-1}; T)| \, \mathrm{d}\boldsymbol{\lambda} \right).$$

By Hypothesis AC, the integral is $\ll 1$ uniformly in $\mathbf{k}$. Therefore,

$$J_2 \ll XL^{N-1} \sum_{\mathbf{k}} \left( C(k_1) \prod_{n=2}^{N-1} \mathrm{e}^{-A_n k_n} \right).$$

This is bounded by a sum of $N-1$ terms of the form

$$XL^{N-1} \left( \sum_{k_1} C(k_1) \right) \prod_{n=2}^{N-1} \left( \sum_{k_n} \mathrm{e}^{-A_n k_n} \right),$$

in each of which exactly one of the components is summed beginning at $K$, and all the others at 0. Consider, for example, the term for which the second component begins at $K$. It contributes

$$\begin{aligned} &\ll XL^{N-1} \left( \sum_{k_1 \geq 0} C(k_1) \right) \left( \sum_{k_2 \geq K} \mathrm{e}^{-A_2 k_2} \right) \prod_{n=3}^{N-1} \left( \sum_{k_n \geq 0} \mathrm{e}^{-A_n k_n} \right) \\ &\ll XL^{N-1} X^{-1} \mathrm{e}^{-A_2 K} \\ &\ll L^{N-1} \mathrm{e}^{-A_2 K}. \end{aligned} \tag{3.29}$$

The other terms are similar except that, if it is the first component that begins at $K$, the bound is even smaller. Adding the estimates for the $N-1$ terms together, we find that

$$J_2 \ll L^{N-1} \sum_{n=1}^{N-1} \mathrm{e}^{-A_2 K} \ll L^{N-1} \mathrm{e}^{-AK},$$

where $A = \min\{A_1, \ldots, A_{N-1}\}$. Since we may take $K$ to be arbitrarily large, we see that

$$J_2 = o(L^{N-1}).$$

Using this and the bound $J_1 \ll L^{N-2}$ from (3.28) in (3.27), we obtain

$$I_{22} = o(L^{N-1}).$$

The estimation of $I_{23}$ is similar but easier, and leads to

$$I_{23} \ll L^{N-2}.$$

Combining our estimates for $I_{22}$ and $I_{23}$ in (3.26), we finally find that

$$I_2 = (-1)^N 2\pi (\log T)^{N-1} \int_{U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)} F^*(\boldsymbol{\alpha}; T) T^{-\sum_{n\le N} a_n \epsilon_n \alpha_n} \,\mathrm{d}\boldsymbol{\alpha} + o(L^{N-1}). \tag{3.30}$$

Next, we show that $I_3$ is relatively small. The range of integration in $I_3$ excludes the set $\eta(S,\epsilon)$, and so, as in the case of $I_2$, we may replace $F(\boldsymbol{\alpha}; T)$ by $F^*(\boldsymbol{\alpha}; T)$. Writing $H_n = \{\boldsymbol{\alpha} : \epsilon_n \alpha_n \le \log X/\log T\}$ for $1 \le n \le N$, we see that $\mathbb{R}^{N-1} \setminus U_{N,\boldsymbol{\varepsilon}}(\log X/\log T) = \bigcup_{n=1}^N H_n$. Thus,

$$I_3 = \int_{\mathbb{R}^{N-1}\setminus(U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)\cup\eta(S,\epsilon))} F^*(\boldsymbol{\alpha}; T)\hat{r}(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} \ll \sum_{n=1}^{N} \int_{H_n} |F^*(\boldsymbol{\alpha}; T)\hat{r}(\boldsymbol{\alpha})|\,\mathrm{d}\boldsymbol{\alpha}. \tag{3.31}$$

We will only estimate the integral over $H_1$ as the others are handled the same way.

The argument proceeds along the lines of that for $I_{22}$. By the first part of Lemma 3.6 and the second bound in (3.13), we see that

$$\int_{H_1} |F^*(\boldsymbol{\alpha}; T)\hat{r}(\boldsymbol{\alpha})|\,\mathrm{d}\boldsymbol{\alpha} \ll L^{N-1}X^{-1} \int_{H_1} |F^*(\boldsymbol{\alpha}; T)| \left( \mathrm{e}^{\varepsilon_1\alpha_1(1-a_1)\log T} \prod_{n=2}^{N-1} |R_n(\alpha_n)| \right) \mathrm{d}\boldsymbol{\alpha}.$$

Setting $\lambda_1 = \varepsilon_1\alpha_1$ and $\lambda_n = \alpha_n$ for $2 \le n \le N-1$, and writing $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{N-1})$, we see that this is

$$\ll L^{N-1}X^{-1} \int_{-\infty}^{\log X/\log T} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |F^*(\varepsilon_1\lambda_1, \ldots, \lambda_{N-1}; T)| \left( \mathrm{e}^{\lambda_1(1-a_1)\log T} \prod_{n=2}^{N-1} |R_n(\lambda_n)| \right) \mathrm{d}\boldsymbol{\lambda}.$$

Next, we let $K$ be a large positive integer and split this into two parts, $\tilde{J}_1$ and $\tilde{J}_2$, where $\tilde{J}_1$ is $L^{N-1}X^{-1}$ times the contribution from the integral over the box $\tilde{\mathscr{B}} = (-K, \log X/\log T] \times [-K, K]^{N-2}$, and $\tilde{J}_2$ is the rest. Thus, we have

$$\int_{H_1} |F^*(\boldsymbol{\alpha}; T)\hat{r}(\boldsymbol{\alpha})|\,\mathrm{d}\boldsymbol{\alpha} \ll \tilde{J}_1 + \tilde{J}_2. \tag{3.32}$$

By Hypothesis LC, $F^*(\boldsymbol{\alpha}; T)$ is bounded on $\tilde{\mathscr{B}}$, so

$$\begin{aligned} \tilde{J}_1 &\ll L^{N-1}X^{-1} \int_{-K}^{\log X/\log T} \cdots \int_{-K}^{K} \left( \mathrm{e}^{\lambda_1(1-a_1)\log T} \prod_{n=2}^{N-1} |R_n(\lambda_n)| \right) \mathrm{d}\boldsymbol{\lambda} \\ &\ll L^{N-1}X^{-1} \left( \int_{-\infty}^{\log X/\log T} \mathrm{e}^{\lambda_1(1-a_1)\log T}\,\mathrm{d}\lambda_1 \right) \prod_{n=2}^{N-1} \left( \int_{-\infty}^{\infty} |R_n(\lambda_n)|\,\mathrm{d}\lambda_n \right) \\ &\ll L^{N-2}, \end{aligned} \tag{3.33}$$

where the last line follows from the second because, by (3.17), the integrals over $(-\infty, \infty)$ are bounded.

To estimate $\tilde{J}_2$, we split $(-\infty, \log X/\log T]$ into intervals $\tilde{I}_0 = (-1, \log X/\log T]$, $\tilde{I}_{-1} = (-2, -1]$, $\tilde{I}_{-2} = (-3, -2]$, ..., and we split $(-\infty, \infty)$ into intervals $I_k = (k, k+1]$ with $k = 0, \pm 1, \pm 2, \ldots$. Then, setting

$$I(\mathbf{k}) = \tilde{I}_{k_1} \times I_{k_2} \times \cdots \times I_{k_{N-1}},$$

we see that

$$\tilde{J}_2 \ll L^{N-1} X^{-1} \sum_{\mathbf{k}} \int_{I(\mathbf{k})} |F^*(\varepsilon_1 \lambda_1, \lambda_2, \ldots, \lambda_{N-1}; T)| \left( \mathrm{e}^{\lambda_1 (1-a_1) \log T} \prod_{n=2}^{N-1} |R_n(\lambda_n)| \right) \mathrm{d}\boldsymbol{\lambda},$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_{N-1})$ runs over $(N-1)$-tuples of integers such that $k_1 \leq 0$ and, in addition, at least one component $k_n$ (possibly $k_1$) has $|k_n| \geq K$. By (3.17), when $\boldsymbol{\lambda} \in I(\mathbf{k})$, the expression in parentheses is

$$\ll \tilde{C}(k_1) \prod_{n=2}^{N-1} \mathrm{e}^{-A_n |k_n|} \quad \text{with } \tilde{C}(k_1) = \begin{cases} T^{-|k_1|/2} & \text{if } k_1 < 0, \\ X & \text{if } k_1 = 0, \\ 0 & \text{if } k_1 > 0. \end{cases}$$

Hence, by Hypothesis AC,

$$\begin{aligned} \tilde{J}_2 &\ll L^{N-1} X^{-1} \sum_{\mathbf{k}} \left( \tilde{C}(k_1) \prod_{n=2}^{N-1} \mathrm{e}^{-A_n |k_n|} \int_{I(\mathbf{k})} |F^*(\varepsilon_1 \lambda_1, \ldots, \varepsilon_{N-1} \lambda_{N-1}; T)| \, \mathrm{d}\boldsymbol{\lambda} \right) \\ &\ll L^{N-1} X^{-1} \sum_{\mathbf{k}} \left( \tilde{C}(k_1) \prod_{n=2}^{N-1} \mathrm{e}^{-A_n |k_n|} \right). \end{aligned}$$

This is bounded by a sum of $N$ terms

$$L^{N-1} X^{-1} \left( \sum_{k_1} \tilde{C}(k_1) \right) \prod_{n=2}^{N-1} \left( \sum_{k_n} \mathrm{e}^{-A_n |k_n|} \right),$$

in each of which one component, say $k_n$, is summed over $|k_n| \geq K$, and the others are summed over $(-\infty, \infty)$ (here we use the fact that $\tilde{C}(k_1) = 0$ when $k_1 > 0$). For example, the term of this type for which $|k_2| \geq K$ contributes

$$\begin{aligned} &\ll L^{N-1} X^{-1} \left( \sum_{k_1 \leq 0} \tilde{C}(k_1) \right) \left( \sum_{|k_2| \geq K} \mathrm{e}^{-A_2 |k_2|} \right) \prod_{n=3}^{N-1} \left( \sum_{k_n = -\infty}^{\infty} \mathrm{e}^{-A_n |k_n|} \right) \\ &\ll L^{N-1} X^{-1} X \, \mathrm{e}^{-A_2 K} \\ &\ll L^{N-1} \mathrm{e}^{-A_2 K}. \end{aligned}$$

The other terms are similar except that, if the first component is the one that begins at $K$, the bound is even smaller. Thus, adding the estimates together, we see that

$$\tilde{J}_2 \ll L^{N-1} \mathrm{e}^{-AK},$$

where $A = \min\{A_1, \dots, A_{N-1}\}$. Since we may take $K$ to be arbitrarily large, we obtain

$$\tilde{J}_2 = o(L^{N-1}).$$

Inserting our estimates for $\tilde{J}_1$ and $\tilde{J}_2$ into (3.32), we now find that

$$\int_{H_1} |F^*(\boldsymbol{\alpha}; T)\hat{r}(\boldsymbol{\alpha})| \,\mathrm{d}\boldsymbol{\alpha} = o(L^{N-1}).$$

The same estimate holds for the other terms on the right-hand side of (3.31), so we see that

$$I_3 = o(L^{N-1}). \tag{3.34}$$

We now combine (3.25), (3.30) and (3.34) in (3.22) and obtain

$$\begin{aligned} I\left(\frac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T\right) &= T \log^N T \int_{U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)} F^*(\boldsymbol{\alpha}; T) T^{-\sum_{n\le N} a_n\varepsilon_n\alpha_n} \,\mathrm{d}\boldsymbol{\alpha} \\ &\quad + N(T)(O(L^{N-11/6}) + o(L^{N-1})) + O(T\, L^{N-1/3}) \\ &= T \log^N T \int_{U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)} F^*(\boldsymbol{\alpha}; T) T^{-\sum_{n\le N} a_n\varepsilon_n\alpha_n} \,\mathrm{d}\boldsymbol{\alpha} + o(T \log^N T). \end{aligned}$$

This is almost the assertion of Theorem 2.1; the difference is that the integral in the statement of Theorem 2.1 is over $U_{N,\boldsymbol{\varepsilon}}(0)$ rather than over $U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)$. However, using our hypotheses that $F^*(\boldsymbol{\alpha}; T)$ is bounded pointwise on $[-K, K]^{N-1}$ for each fixed $K > 0$, and bounded on average on unit cubes in $\mathbb{R}^{N-1}$, one easily sees that the integral over $(U_{N,\boldsymbol{\varepsilon}}(0) \setminus U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)) \cap [-K, K]^{N-1}$ is no greater than $O(\log X/\log T)$, and the integral over the rest of $U_{N,\boldsymbol{\varepsilon}}(0) \setminus U_{N,\boldsymbol{\varepsilon}}(\log X/\log T)$ is $O(\mathrm{e}^{-AK})$ for some positive constant $A$. Theorem 2.1 therefore follows.

### 3.6. *Proof of Corollary* 2.1

By Theorem 2.1, we have

$$I\left(\frac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T\right) \ll T \log^N T \int_{U_{N,\boldsymbol{\varepsilon}}(0)} |F^*(\boldsymbol{\alpha}; T)| T^{-\sum_{n\le N} a_n\varepsilon_n\alpha_n} \,\mathrm{d}\boldsymbol{\alpha} + o(T \log^N T),$$

and it suffices to show that the integral is bounded. Discarding the term $n = N$ in the exponent of the integrand and writing $a_n = A_n/\log T$ and $\lambda_n = \alpha_n\varepsilon_n$, $1 \le n \le N-1$, we see that the integral is

$$\begin{aligned} &\ll \int_0^\infty \cdots \int_0^\infty |F^*(\varepsilon_1\lambda_1, \dots, \varepsilon_{N-1}\lambda_{N-1}; T)|\, \mathrm{e}^{-\sum_{n<N} A_n\lambda_n} \,\mathrm{d}\boldsymbol{\lambda} \\ &\ll \sum_{\mathbf{k}} \int_{I(\mathbf{k})} |F^*(\varepsilon_1\lambda_1, \dots, \varepsilon_{N-1}\lambda_{N-1}; T)|\, \mathrm{e}^{-\sum_{n<N} A_n\lambda_n} \,\mathrm{d}\boldsymbol{\lambda}, \end{aligned}$$

where $I(\mathbf{k}) = (k_1, k_1+1] \times (k_2, k_2+1] \times \ldots \times (k_{N-1}, k_{N-1}+1]$ and the sum is over those $\mathbf{k}$ whose components are all non-negative. By Hypothesis AC, this is

$$\ll \prod_{n=1}^{N-1} \left( \sum_{k_n=0}^{\infty} \mathrm{e}^{-A_n k_n} \right) \ll 1.$$

Hence,

$$\int_{U_{N,\boldsymbol{\varepsilon}}(0)} |F^*(\boldsymbol{\alpha}; T)| T^{-\sum_{n \le N} a_n \varepsilon_n \alpha_n} \, \mathrm{d}\boldsymbol{\alpha} \ll 1.$$

The corollary now follows.

## 4. Proof of Theorem 2.2

In this section, we prove Theorem 2.2. First we require a lemma.

LEMMA 4.1 *Assume RH, let $\Lambda_{\mathbf{b}}$ with $\mathbf{b} = (b_1, \ldots, b_L)$ be as in (2.8) and suppose that $|b_l| < \frac{1}{10}$ with $\Re b_l > 0$. Then, for $\frac{1}{2} \le \sigma_0 \le \frac{9}{10}$,*

$$\Psi_{\mathbf{b}}(x) = (-1)^N {\sum_{n \le x}}' \Lambda_{\mathbf{b}}(n) = R_{\mathbf{b}}(x) + \frac{1}{2\pi \mathrm{i}} \int_{\sigma_0 - \mathrm{i}\infty}^{\sigma_0 + \mathrm{i}\infty} \prod_{l=1}^{L} \frac{\zeta'}{\zeta}(s + b_l) \frac{x^s}{s} \, \mathrm{d}s,$$

*where $R_{\mathbf{b}}(x)$ is the sum of the residues of*

$$\prod_{l=1}^{L} \frac{\zeta'}{\zeta}(s + b_l) \frac{x^s}{s},$$

*at the points $s = 1 - b_l$.*

*Proof*. The method of proof is standard, so we only sketch it. If $a > 1$ and $T$ is large, then, by Perron's formula [**12**], one has

$$\Psi_{\mathbf{b}}(x) = \frac{1}{2\pi \mathrm{i}} \int_{a - \mathrm{i}T}^{a + \mathrm{i}T} \prod_{l=1}^{L} \frac{\zeta'}{\zeta}(s + b_l) \frac{x^s}{s} \, \mathrm{d}s + E(x, T),$$

where $E(x, T)$ is a small error term that tends to zero as $T \to \infty$. The integrand has no poles in the half-plane $\Re s \ge \frac{1}{2}$ except for the simple poles from the factors $\zeta'/\zeta(s + b_l)$ at $s = 1 - b_l,\ 1 \le l \le L$. Pulling the contour left to $\Re s = \sigma_0$, we see that the sum of the residues inside the resulting rectangle is $R_{\mathbf{b}}(x)$. Moreover, the contribution to the integral from the top and bottom edges of the rectangle tends to zero as $T \to \infty$. The result follows. □

Now let $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ with $a_n > 0$ and $a_n \approx 1/\log T$ for $1 \le n \le N$. Let $1 \le J < N$ and write $\mathbf{a}_J = (a_1, a_2, \ldots, a_J)$ and $\mathbf{a}'_J = (a_{J+1}, a_{J+2}, \ldots, a_N)$. We also write $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N)$ with $\varepsilon_n = +1$ for $1 \le n \le J$, and $\varepsilon_n = -1$ for $J + 1 \le n \le N$.

Recalling that $\Delta_{\mathbf{a}_J}(x) = (-1)^N \sum'_{n \le x} \Lambda_{\mathbf{a}_J}(n) - R_{\mathbf{a}_J}(x)$, we see, from Lemma 4.1, that

$$\frac{\Delta_{\mathbf{a}_J}(\mathrm{e}^{\tau+\delta}) - \Delta_{\mathbf{a}_J}(\mathrm{e}^{\tau})}{\mathrm{e}^{\sigma\tau}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \prod_{j=1}^{J} \frac{\zeta'}{\zeta}(\sigma + a_j + \mathrm{i}t) \left(\frac{\mathrm{e}^{\delta(\sigma+\mathrm{i}t)} - 1}{\sigma + \mathrm{i}t}\right) \mathrm{e}^{-2\pi\mathrm{i}t(-\tau/2\pi)}\,\mathrm{d}t,$$

for $\frac{1}{2} \le \sigma \le \frac{9}{10}$. This expresses the left-hand side as a Fourier transform. We use this with Plancherel's formula in the form

$$\int_{-\infty}^{\infty} \hat{f}(\tau)\hat{g}(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} f(t)g(-t)\,\mathrm{d}t,$$

where

$$\hat{f}(\tau) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-2\pi\mathrm{i}t\tau}\,\mathrm{d}t,$$

and similarly for $\hat{g}$. We then obtain

$$\begin{aligned}\int_{-\infty}^{\infty} &(\Delta_{\mathbf{a}_J}(\mathrm{e}^{\tau+\delta}) - \Delta_{\mathbf{a}_J}(\mathrm{e}^{\tau}))(\Delta_{\mathbf{a}'_J}(\mathrm{e}^{\tau+\delta}) - \Delta_{\mathbf{a}'_J}(\mathrm{e}^{\tau}))\,\mathrm{e}^{-2\sigma\tau}\,\mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \prod_{n=1}^{N} \frac{\zeta'}{\zeta}(\sigma + a_n + \mathrm{i}\varepsilon_n t) \left|\frac{\mathrm{e}^{\delta(\sigma+\mathrm{i}t)} - 1}{\sigma + \mathrm{i}t}\right|^2 \mathrm{d}t. \qquad (4.1)\end{aligned}$$

Next, we set $\mathrm{e}^{\delta} = 1 + 1/T$ and $\sigma = \frac{1}{2} + c$ with $0 \le c \le \frac{4}{10}$. Then we see that

$$\frac{\mathrm{e}^{\delta(\sigma+\mathrm{i}t)} - 1}{\sigma + \mathrm{i}t} = 2\,\mathrm{e}^{\mathrm{i}\delta t/2} \frac{\sin(\delta t/2)}{t}\left(1 + O\left(\frac{1}{1+|t|}\right)\right) + O\left(\frac{1}{(1+|t|)T}\right).$$

Replacing $\mathrm{e}^{\tau}$ by $x$ and using the estimate

$$\frac{\zeta'}{\zeta}(\sigma + \mathrm{i}t) \ll \frac{\log(2+|t|)}{\sigma - 1/2}, \qquad (4.2)$$

which is valid for $\sigma > \frac{1}{2}$ on RH, we then find that

$$\begin{aligned}\int_{0}^{\infty} &\left(\Delta_{\mathbf{a}_J}\left(x + \frac{x}{T}\right) - \Delta_{\mathbf{a}_J}(x)\right)\left(\Delta_{\mathbf{a}'_J}\left(x + \frac{x}{T}\right) - \Delta_{\mathbf{a}'_J}(x)\right) x^{-2-2c}\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty} \prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2} + c + a_n + \mathrm{i}\varepsilon_n t\right)\left(\frac{\sin(\delta t/2)}{t}\right)^2 \mathrm{d}t + O\left(\frac{\log^{N+1} T}{T^2 \prod_{1 \le n \le N}(c + a_n)}\right). \qquad (4.3)\end{aligned}$$

The error term here is

$$\ll \frac{\log^{2N+1} T}{T^2},$$

where the implied constant depends on $A_1, \ldots, A_N$. The estimate (4.2) also allows us to replace $\delta = T^{-1} + O(T^{-2})$ by $T^{-1}$ with the same error.

It remains to show that in (4.3) we may remove the portion of the integral over the interval [0, 1] with an acceptable error term. To do this, first observe that, for $x \in [0, 1]$, $\Delta_{\mathbf{a}_J}(x + x/T) - \Delta_{\mathbf{a}_J}(x) = R_{\mathbf{a}_J}(x) - R_{\mathbf{a}_J}(x + x/T)$ is the sum of the residues of

$$\frac{x^s}{s}\left(1 - \left(1 + \frac{1}{T}\right)^s\right)\prod_{j=1}^{J}\frac{\zeta'}{\zeta}(s + a_j),$$

at the points $s = 1 - a_j$, $1 \le j \le J$. Letting $B = \max\{A_1, A_2, \ldots, A_N\}$ and applying Cauchy's integral formula, we find that this is

$$= \frac{1}{2\pi \mathrm{i}}\int_{|s-1|=2B/\log T}\frac{x^s}{s}\left(1 - \left(1 + \frac{1}{T}\right)^s\right)\prod_{j=1}^{J}\frac{\zeta'}{\zeta}(s + a_j)\,\mathrm{d}s$$
$$\ll \frac{x^{1-2B/\log T}}{T}\left(\frac{\log T}{B}\right)^{J-1},$$

where the circle in the integral is positively oriented. Thus,

$$\int_0^1\left(\Delta_{\mathbf{a}_J}\left(x + \frac{x}{T}\right) - \Delta_{\mathbf{a}_J}(x)\right)\left(\Delta_{\mathbf{a}'_J}\left(x + \frac{x}{T}\right) - \Delta_{\mathbf{a}'_J}(x)\right)x^{-2-2c}\,\mathrm{d}x$$
$$\ll \int_0^1 \frac{x^{-2(C+B)/\log T}}{T^2}\left(\frac{\log T}{B}\right)^{N-2}\mathrm{d}x \ll \frac{1}{T^2}\left(\frac{\log T}{B}\right)^{N-2},$$

where $c = C\log T$. Theorem 2.2 now follows.

## 5. Proof of Theorem 2.3

### 5.1. *Lemmas for the proof of Theorem* 2.3

To prove Theorem 2.3, we appeal to modified versions of two lemmas in [6]. These concern the equivalence under certain conditions of

$$\int_0^T g(t, \eta)\,\mathrm{d}t \sim T, \tag{5.1}$$

as $T \to \infty$, and

$$\int_0^\infty g(t, \eta)\left(\frac{\sin \kappa t}{t}\right)^2\mathrm{d}t \sim \frac{\pi}{2}\kappa, \tag{5.2}$$

as $\kappa \to 0+$.

LEMMA 5.1 *Let $g(t, \eta)$ be a continuous function of $t$ and $\eta$ for $t \ge 0$ and $\eta \ge 2$. Suppose that $g(t, \eta) \ll \log^N(t + 2)$ and that $\int_0^T |g(t, \eta)|^2\,\mathrm{d}t \ll T$ holds for $\eta\log^{-N-1}\eta \le T \le \eta\log^{N+1}\eta$. If (5.1) holds uniformly for $\eta\log^{-N-1}\eta \le T \le \eta\log^{N+1}\eta$, then (5.2) holds for $\eta \approx 1/\kappa$.*

The proof is similar to Goldston [6, Proof of Lemma 2]. A minor difference is that Goldston assumes $g(t,\eta)\geq 0$. Our assumption that $\int_0^T |g(t,\eta)|^2\,\mathrm{d}t \ll T$ allows us to get around this.

Our next lemma is similar to [6, Lemma 3].

LEMMA 5.2 *Suppose that $g(t,\eta)$ satisfies the same hypotheses as in Lemma 5.1. If (5.2) holds uniformly for $\eta^{-1}\log^{-N-1}\eta \leq \kappa \leq \eta^{-1}\log^{N+1}\eta$, then (5.1) holds for $\eta \approx T$.*

*Proof*. The method closely follows Goldston [6], except for a slight difference at the end. This difference arises because we are not assuming that $g(t,\eta)$ is non-negative.

Suppose that $K(x)$ is an even $C^2$ function that is integrable on $\mathbb{R}$, with $K(x)$ and $K'(x)$ vanishing as $x\to\infty$ and with $K''(x) \ll 1/(1+|x|)^3$. Integrating by parts twice, we find

$$\hat{K}(t) = \frac{1}{2}\int_{-\infty}^{\infty} K''(x)\left(\frac{\sin \pi t x}{\pi t}\right)^2 \mathrm{d}x,$$

where $\hat{K}(t)$ is the Fourier transform of $K(x)$. Thus, for any function $G(t,\eta)$, we have

$$G(t,\eta)\hat{K}(t/T) = \frac{T^2}{2}G(t,\eta)\int_{-\infty}^{\infty} K''(x)\left(\frac{\sin(\pi t x/T)}{\pi t}\right)^2 \mathrm{d}x.$$

If $\int_0^T |G(t,\eta)|\,\mathrm{d}t \ll T^{1+\epsilon}$, then

$$\begin{aligned}\int_0^{\infty} G(t,\eta)\hat{K}(t/T)\,\mathrm{d}t &= \frac{T^2}{2}\int_{-\infty}^{\infty} K''(x)\int_0^{\infty} G(t,\eta)\left(\frac{\sin(\pi t x/T)}{\pi t}\right)^2 \mathrm{d}t\,\mathrm{d}x\\ &= T^2\int_0^{\infty} K''(x) G_1(\pi x/T,\eta)\,\mathrm{d}x, \qquad (5.3)\end{aligned}$$

where

$$G_1(\kappa,\eta) = \int_0^{\infty} G(t,\eta)\left(\frac{\sin(\kappa t)}{\pi t}\right)^2 \mathrm{d}t. \qquad (5.4)$$

Here changing the order of integration is justified by absolute convergence.

Now let

$$K(x) = \frac{\sin(2\pi x)+\sin(2\pi(1+\delta)x)}{2\pi x(1-4\delta^2x^2)}.$$

Then $K(x)$ satisfies the assumptions above, and

$$\hat{K}(t) = \begin{cases} 1 & \text{if } |t|\leq 1,\\ \cos^2(\pi(|t|-1)/(2\delta)) & \text{if } 1\leq |t|\leq 1+\delta,\\ 0 & \text{if } |t|\geq 1+\delta.\end{cases}$$

With this choice of $K$, the left-hand side of (5.3) is approximately $\int_0^T G(t,\eta)\,\mathrm{d}t$, so it is tempting to choose $G(t,\eta)=g(t,\eta)$. This would not work well, however, because on the right-hand side one needs to integrate against $K''(x)$. If instead, we choose

$$G(t,\eta) = g(t,\eta) - 1,$$

we need only show that the right-hand side of (5.3) is small.

From (5.4), the definition of $G(t,\eta)$, and our assumption that $g(t,\eta) \ll \log^N(t+2)$, we see that

$$G_1(\kappa,\eta) \ll \int_0^{\kappa^{-1}} \log^N(t+2)\kappa^2\,\mathrm{d}t + \int_{\kappa^{-1}}^{\infty} \log^N(t+2)t^{-2}\,\mathrm{d}t \ll \kappa\log^N(\kappa^{-1}+2).$$

Using this to estimate the tails, we see that the integral on the right-hand side of (5.3) equals

$$T^2\int_{\log^{-N-1/2}T}^{\log^{N+1/2}T} K''(x)G_1(\pi x/T,\eta)\,\mathrm{d}x + O(T/(\log T)^{1/2}).$$

By (5.4), (5.2) and the definition of $G(t,\eta)$, for $\eta \asymp T$, we have $G_1(\pi x/T,\eta) = o(x/T)$ as $x/T \to 0^+$. Thus, the last expression is

$$o\left(T^2\int_{\log^{-N-1/2}T}^{\log^{N+1/2}T} \frac{1}{1+x^3}\frac{x}{T}\,\mathrm{d}x\right) + O\left(\frac{T}{(\log T)^{1/2}}\right) = o(T).$$

We now see that

$$\int_0^{\infty} G(t,\eta)\hat{K}(t/T)\,\mathrm{d}t = o(T),$$

and therefore that

$$\int_0^{\infty} g(t,\eta)\hat{K}(t/T)\,\mathrm{d}t = \left(1+\frac{\delta}{2}\right)T + o(T).$$

On the other hand, by the Cauchy–Schwarz inequality and our assumption that $\int_0^T |g(t,\eta)|^2\,\mathrm{d}t \ll T$, we find that

$$\begin{aligned}\int_0^{\infty} g(t,\eta)\hat{K}(t/T)\,\mathrm{d}t &= \int_0^T g(t,\eta)\,\mathrm{d}t + O\left(\int_T^{(1+\delta)T} |g(t,\eta)|\,\mathrm{d}t\right)\\ &= \int_0^T g(t,\eta)\,\mathrm{d}t + O\left(\delta T\left(\int_0^{(1+\delta)T} |g(t,\eta)|^2\right)\,\mathrm{d}t\right)^{1/2}\\ &= \int_0^T g(t,\eta)\,\mathrm{d}t + O(T(\delta(1+\delta))^{1/2}).\end{aligned}$$

The lemma follows on taking $\delta$ small. □

LEMMA 5.3 *Assume RH, Hypotheses AC and LC. Let $B_1,\ldots,B_N$ be fixed positive real numbers. Suppose that $\eta\log^{-N-2}\eta \le T \le \eta\log^{N+2}\eta$. Then we have*

$$\begin{aligned}\int_0^T \prod_{n=1}^N \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log\eta}+\mathrm{i}\varepsilon_n t\right)\,\mathrm{d}t = \int_0^T \prod_{n=1}^N \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\mathrm{d}t\\ + O(T\log^{N-1}T\ \log\log T) \qquad (5.5)\end{aligned}$$

*and*

$$\int_0^\infty \prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log\eta}+\mathrm{i}\varepsilon_n t\right)\left(\frac{\sin t/2T}{t}\right)^2 \mathrm{d}t$$
$$= \int_0^\infty \prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\left(\frac{\sin t/2T}{t}\right)^2 \mathrm{d}t + O(T^{-1}\log^{N-1}T\log\log T). \qquad (5.6)$$

*Proof*. To prove (5.5), it is enough to show that

$$\int_0^T \prod_{n=1}^{N_1-1} \frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log\eta}+\mathrm{i}\varepsilon_n t\right)\prod_{n=N_1}^{N} \frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\mathrm{d}t$$
$$-\int_0^T \prod_{n=1}^{N_1} \frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log\eta}+\mathrm{i}\varepsilon_n t\right)\prod_{n=N_1+1}^{N} \frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\mathrm{d}t$$
$$= O(T\log^{N-1}T\ \log\log T), \qquad (5.7)$$

for $1\le N_1\le N$. By Cauchy's integral formula, we have

$$\frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log\eta}+\mathrm{i}\varepsilon_n t\right)-\frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)$$
$$=\int_{B_n/\log T}^{B_n/\log\eta}\frac{\mathrm{d}}{\mathrm{d}u}\frac{\zeta'}{\zeta}\left(\frac12+u+\mathrm{i}\varepsilon_n t\right)\mathrm{d}u$$
$$=\int_{B_n/\log T}^{B_n/\log\eta}\left(\frac{1}{2\pi\mathrm{i}}\int_{|z-u|=B_n/(2\log T)}\frac{\zeta'}{\zeta}\left(\frac12+z+\mathrm{i}\varepsilon_n t\right)\frac{\mathrm{d}z}{(z-u)^2}\right)\mathrm{d}u.$$

Since $\log T=\log\eta+O(\log\log\eta)$, we have $\Re z\asymp(\log T)^{-1}$ for all $z$ on the circle of integration. Thus, the difference on the left-hand side of (5.7) is

$$=\int_0^T \prod_{n=1}^{N_1-1} \frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log\eta}+\mathrm{i}\varepsilon_n t\right)\prod_{n=N_1+1}^{N} \frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)$$
$$\times\int_{B_{N_1}/\log T}^{B_{N_1}/\log\eta}\left(\frac{1}{2\pi\mathrm{i}}\int_{|z-u|=B_{N_1}/2\log T}\frac{\zeta'}{\zeta}\left(\frac12+z+\mathrm{i}\varepsilon_{N_1} t\right)\frac{\mathrm{d}z}{(z-u)^2}\mathrm{d}u\right)\mathrm{d}t$$
$$=\int_{B_{N_1}/\log T}^{B_{N_1}/\log\eta}\left(\frac{1}{2\pi\mathrm{i}}\int_{|z-u|=B_{N_1}/2\log T}\left(\int_0^T\prod_{n=1}^{N_1-1} \frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log\eta}+\mathrm{i}\varepsilon_n t\right)\right.\right.$$
$$\left.\left.\times\prod_{n=N_1+1}^{N} \frac{\zeta'}{\zeta}\left(\frac12+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\frac{\zeta'}{\zeta}\left(\frac12+z+\mathrm{i}\varepsilon_{N_1} t\right)\mathrm{d}t\right)\frac{\mathrm{d}z}{(z-u)^2}\right)\mathrm{d}u.$$

Applying the Cauchy–Schwarz inequality twice, we see that the integral with respect to $t$ is

$$\leq \left(\int_0^T \left|\prod_{n=1}^{N_1-1} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log \eta}+\mathrm{i}\varepsilon_n t\right)\right|^4 \mathrm{d}t\right)^{1/4} \times \left(\int_0^T \left|\prod_{n=N_1+1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\right|^4 \mathrm{d}t\right)^{1/4}$$
$$\times \left(\int_0^T \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+z+\mathrm{i}\varepsilon_{N_1} t\right)\right|^2 \mathrm{d}t\right)^{1/2}$$
$$\ll T \log^N T,$$

by Corollary 2.1. The left-hand side of (5.7) is therefore

$$\ll T \log^N T \left|\frac{1}{\log \eta} - \frac{1}{\log T}\right| \log T \ll T \log^{N-1} T \log\log T.$$

Now we prove (5.6). First, arguing as above, we find that

$$\int_0^T \left(\prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log \eta}+\mathrm{i}\varepsilon_n t\right) - \prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\right)\left(\frac{\sin t/2T}{t}\right)^2 \mathrm{d}t$$
$$\ll T^{-1} \log^{N-1} T \log\log T \tag{5.8}$$

and

$$\int_X^{2X} \left(\prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log \eta}+\mathrm{i}\varepsilon_n t\right) - \prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\right)\left(\frac{\sin t/2T}{t}\right)^2 \mathrm{d}t$$
$$\ll X^{-1} \log^{N-1} T \log\log T, \tag{5.9}$$

for $T \leq X \leq T \log^{N+1} T$. We decompose the interval $(0, T\log^{N+1} T]$ into a union of subintervals $[0, T], (T, 2T], (2T, 4T], \ldots, (2^L T, T\log^{N+1} T]$, say. Then, using the estimates in (5.8) and (5.9), we see that

$$\int_0^{T\log^{N+1} T} \left(\prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log \eta}+\mathrm{i}\varepsilon_n t\right) - \prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\right)\left(\frac{\sin t/2T}{t}\right)^2 \mathrm{d}t$$
$$\ll T^{-1} \log^{N-1} T \log\log T.$$

Also, by (4.2),

$$\int_{T\log^{N+1} T}^{\infty} \prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\left(\frac{\sin t/2T}{t}\right)^2 \mathrm{d}t$$
$$\ll \int_{T\log^{N+1} T}^{\infty} \frac{(\log t \log T)^N}{t^2}\,\mathrm{d}t \ll T^{-1} \log^{N-1} T,$$

and similarly

$$\int_{T\log^{N+1}T}^{\infty}\prod_{n=1}^{N}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log\eta}+\mathrm{i}\varepsilon_n t\right)\left(\frac{\sin t/2T}{t}\right)^2\,\mathrm{d}t \ll T^{-1}\log^{N-1}T.$$

Combining these estimates, we obtain

$$\int_0^{\infty}\prod_{n=1}^{N}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log\eta}+\mathrm{i}\varepsilon_n t\right)\left(\frac{\sin t/2T}{t}\right)^2\,\mathrm{d}t$$
$$=\int_0^{\infty}\prod_{n=1}^{N}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{B_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\left(\frac{\sin t/2T}{t}\right)^2\,\mathrm{d}t+O(T^{-1}\log^{N-1}T\log\log T),$$

which is (5.6). □

### 5.2. *Completion of the proof of Theorem* 2.3

We apply the lemmas of the last section to the function

$$g(t,\eta)=2\Re\prod_{n=1}^{N}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{C+A_n}{\log\eta}+\mathrm{i}\varepsilon_n t\right)\Big/(f(C,\mathbf{A},J)\log^N\eta),$$

but first we must check that the hypotheses of the lemmas are satisfied. First, by (4.2), we see that $g(t,\eta)\ll\log^N(t+2)$. Moreover, by Corollary 2.1, for $\eta\log^{-N-1}\eta\le T\le\eta\log^{N+1}\eta$ we have

$$\int_0^T|g(t,\eta)|^2\,\mathrm{d}t\ll T.$$

Thus, $g(t,\eta)$ satisfies the requirements of Lemmas 5.1 and 5.2.

Next, we restate the asymptotic formulas in (2.10) in terms of $g(t,\eta)$. First, since

$$I_{\pm}\left(\frac{1}{2}+\frac{C}{\log T},\frac{\mathbf{A}}{\log T},\boldsymbol{\varepsilon};\ T\right)=\int_0^T 2\Re\prod_{n=1}^{N}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{C+A_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\mathrm{d}t,$$

the estimate

$$I_{\pm}\left(\frac{1}{2}+\frac{C}{\log T},\frac{\mathbf{A}}{\log T},\boldsymbol{\varepsilon};\ T\right)\sim f(C,\mathbf{A},J)T\log^N T$$

is equivalent to

$$\int_0^T g(t,T)\,\mathrm{d}t\sim T. \tag{5.10}$$

Second, by Theorem 2.2, the assertion that

$$P\left(\frac{C}{\log T},\frac{\mathbf{A}}{\log T},J;\ T\right)\sim f(C,\mathbf{A},J)\frac{\log^N T}{2T} \tag{5.11}$$

is equivalent to

$$\int_0^\infty 2\Re \prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{C+A_n}{\log T}+\mathrm{i}\varepsilon_n t\right)\left(\frac{\sin t/2T}{t}\right)^2 \mathrm{d}t \sim \frac{\pi}{2} f(C,\mathbf{A},J)\frac{\log^N T}{2T}.$$

Therefore, in terms of $g(t,\eta)$, (5.11) is the same as

$$\int_0^\infty g(t,T)\left(\frac{\sin t/2T}{t}\right)^2 \mathrm{d}t \sim \frac{\pi}{2}\frac{1}{2T}. \tag{5.12}$$

Thus, Theorem 2.3 says that if one of (5.10) and (5.12) holds, then so does the other.

Suppose now that (5.10) holds. By Lemma 5.3, we see that (5.1) then holds uniformly for $\eta \log^{-N-1}\eta \le T \le \eta \log^{N+1}\eta$. Thus, by Lemma 5.1, we have (5.12). Next, suppose that (5.12) holds. Again, by Lemma 5.3, we see that (5.2) holds uniformly for $\eta^{-1}\log^{-N-1}\eta \le \kappa \le \eta^{-1}\log^{N+1}\eta$. Thus, by Lemma 5.2, we have (5.10). This concludes the proof of Theorem 2.3.

## 6. The first few cases of *I*

Assuming that to leading order the zeros of the zeta-function have the same correlations as the eigenvalues of large unitary matrices, one could, in principle, evaluate $I(\frac{1}{2},\mathbf{a},\boldsymbol{\varepsilon};T)$ for any particular $\mathbf{a}$ and $\boldsymbol{\varepsilon}$. This just involves an application of Theorem 2.1 followed by a computation of the integral involving $F(\boldsymbol{\alpha};T)$. However, this calculation seems quite difficult in general, even with a computer-algebra package. For example, see [**4**], where the following formula was proved conditionally, but only for $a=b$:

$$\frac{1}{T}\int_0^T \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+a+\mathrm{i}t\right)\right|^2 \frac{\zeta'}{\zeta}\left(\frac{1}{2}+b+\mathrm{i}t\right)\mathrm{d}t \sim \log T\frac{T^{-(a+b)}}{(a+b)^2}.$$

A different approach, assuming the random matrix conjectures for the zeros of the zeta-function, is available from the papers of Conrey *et al.* [**2**, **3**] in which they give precise conjectural formulas for integrals of the form

$$\int_0^T \frac{\prod_{j=1}^{J}\zeta(1/2+a_j+\mathrm{i}t)\prod_{k=1}^{K}\zeta(1/2+b_k-\mathrm{i}t)}{\prod_{l=1}^{L}\zeta(1/2+u_l+\mathrm{i}t)\prod_{m=1}^{M}\zeta(1/2+v_m-\mathrm{i}t)}\,\mathrm{d}t,$$

where $J+K=L+M=N$. By differentiating these formulae with respect to the appropriate variables $a_j$, $b_k$, $u_l$ and $v_m$, and then setting certain variables equal to each other, one can obtain any $I(\frac{1}{2},\mathbf{a},\boldsymbol{\varepsilon};T)$. The (complicated) general formula has recently been worked out by Conrey and Snaith [**1**].

As an illustration, we record the conjecture for $I(\frac{1}{2},\mathbf{a},\boldsymbol{\varepsilon};T)$ in the cases where $N=J+K=4$. We shall write $\mathbf{a}=(a,b,c,d)$ instead of $\mathbf{a}=(a_1,a_2,a_3,a_4)$ as we did previously. We assume that $a,b,c,d$ are positive and are $\approx 1/\log T$.

The case of two pluses and two minuses is

$$\frac{1}{T}\int_0^T \frac{\zeta'}{\zeta}\left(\frac{1}{2}+a+\mathrm{i}t\right)\frac{\zeta'}{\zeta}\left(\frac{1}{2}+b+\mathrm{i}t\right)\frac{\zeta'}{\zeta}\left(\frac{1}{2}+c-\mathrm{i}t\right)\frac{\zeta'}{\zeta}\left(\frac{1}{2}+d-\mathrm{i}t\right)\mathrm{d}t$$
$$\sim \frac{1}{(b+c)^2(a+d)^2}+\frac{1}{(a+c)^2(b+d)^2}-\frac{T^{-a-c}}{(a-b)(a+c)^2(c-d)}-\frac{T^{-a-c}}{(a+c)^2(b+c)(c-d)}$$
$$-\frac{T^{-a-c}}{(a-b)(a+c)^2(a+d)}-\frac{T^{-a-c}}{(a+c)^2(b+c)(a+d)}-\frac{T^{-a-c}}{(a+c)^2(b+d)^2}$$
$$-\frac{2T^{-a-d}}{(a-b)(a+c)(a+d)^2}-\frac{2T^{-a-d}}{(b+c)^2(a+d)^2}+\frac{2T^{-a-d}}{(a-b)(c-d)(a+d)^2}$$
$$-\frac{2T^{-a-d}}{(a+c)(a+d)^2(b+d)}-\frac{2T^{-a-d}}{(a+d)^2(b+d)(-c+d)}-\frac{T^{-b-d}}{(a+c)^2(b+d)^2}$$
$$+\frac{T^{-b-d}}{(a-b)(b+c)(b+d)^2}-\frac{T^{-b-d}}{(b+c)(a+d)(b+d)^2}+\frac{T^{-b-d}}{(a-b)(b+d)^2(-c+d)}$$
$$-\frac{T^{-b-d}}{(a+d)(b+d)^2(-c+d)}+\frac{(a-b)^2(c-d)^2T^{-a-b-c-d}}{(a+c)^2(b+c)^2(a+d)^2(b+d)^2}.$$

In particular, we have

$$\frac{1}{T}\int_0^T \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+a+\mathrm{i}t\right)\right|^4 \mathrm{d}t \sim \frac{T^{-2a}(T^{2a}-2a^2\log^2 T-1)}{8a^4}.$$

For the case of one plus and three minuses, we have

$$\frac{1}{T}\int_0^T \frac{\zeta'}{\zeta}\left(\frac{1}{2}+a+\mathrm{i}t\right)\frac{\zeta'}{\zeta}\left(\frac{1}{2}+b-\mathrm{i}t\right)\frac{\zeta'}{\zeta}\left(\frac{1}{2}+c-\mathrm{i}t\right)\frac{\zeta'}{\zeta}\left(\frac{1}{2}+d-\mathrm{i}t\right)\mathrm{d}t$$
$$\sim -\frac{T^{-a-b}}{(a+b)^2(b-c),(b-d)}-\frac{T^{-a-b}}{(a+b)^2(a+c)(b-d)}-\frac{T^{-a-b}}{(a+b)^2(b-c)(a+d)}$$
$$-\frac{T^{-a-b}}{(a+b)^2(a+c)(a+d)}-\frac{T^{-a-c}}{(a+b)(a+c)^2(c-d)}+\frac{T^{-a-c}}{(b-c)(a+c)^2(c-d)}$$
$$-\frac{T^{-a-c}}{(a+b)(a+c)^2(a+d)}+\frac{T^{-a-c}}{(b-c)(a+c)^2(a+d)}-\frac{T^{-a-d}}{(a+b)(a+c)(a+d)^2}$$
$$+\frac{T^{-a-d}}{(a+c)(b-d)(a+d)^2}+\frac{T^{-a-d}}{(a+b)(c-d)(a+d)^2}+\frac{T^{-a-d}}{(b-d)(a+d)^2(-c+d)}.$$

As special cases of this, we find that

$$\frac{1}{T}\int_0^T \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+a+\mathrm{i}t\right)\right|^2\frac{\zeta'}{\zeta}\left(\frac{1}{2}+b+\mathrm{i}t\right)^2 \mathrm{d}t \sim \frac{T^{-2a-b}(2a(T^a-T^b)-(a^2-b^2)T^a\log T)}{2a(a^2-b^2)^2}$$

and

$$\frac{1}{T}\int_0^T \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+a+\mathrm{i}t\right)\right|^2 \frac{\zeta'}{\zeta}\left(\frac{1}{2}+a+\mathrm{i}t\right)^2 \mathrm{d}t \sim -\frac{T^{-2a}\log T\,(a\log T-1)}{8a^3}.$$

## Funding

D.W.F. and S.M.G. were supported by the American Institute of Mathematics and the National Science Foundation Focused Research Group grant DMS 0244660. Work of S.M.G. was also supported by National Science Foundation grants DMS 0201457 and DMS 0653809.

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