13 Spherical geometry

Let $\triangle ABC$ be a triangle in the Euclidean plane. From now on, we indicate the interior angles $\angle A = \angle CAB$, $\angle B = \angle ABC$, $\angle C = \angle BCA$ at the vertices merely by A, B, C. The sides of length $a = |BC|$ and $b = |CA|$ then make an angle C. The *cosine rule* states that

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$$

if $C = \pi/2$ it reduces to Pythagoras' theorem. It is easily proved by constructing (say) the altitude AA' of length $a' = h$. (Take BC to be the 'base' of the triangle so that h is the height, and draw the picture.) Now apply Pythagoras to $\triangle AA'B$ and $\triangle AA'C$ to get

$$c^2 = h^2 + (a - b\cos C)^2, \qquad b^2 = h^2 + (b\sin C)^2.$$

The rule follows by eliminating h^2.

The *sine rule* states that

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

It can also be proved using the altitude AA', since

$$b\sin C = h = c\sin B,$$

and the rest follows by symmetry.

It is important to realize that the sine rule can also be deduced algebraically from the cosine rule. The latter tells us that

$$\left(\frac{\sin C}{c}\right)^2 = \frac{1-\cos^2 C}{c^2} = \frac{4a^2b^2 - (a^2+b^2-c^2)^2}{4a^2b^2c^2}.$$

The numerator on the right-hand side, when expanded, is symmetric in $a, b, c,$ and it follows that we can replace c, C by a, A or b, B on the left. The sine rule follows because $\sin C/c > 0$.

The aim of this section is to prove analogous formulas for spherical triangles.

13.1. Spherical triangles: the vertices and sides

Fix Cartesian coordinates in space, with origin $O = (0,0,0)$. Consider the sphere

$$\mathscr{S} = \{P \ : \ d(O,P) = 1\} = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$$

centred at O. The word 'sphere' in geometry refers exclusively to the *surface*, not the inside! The position vector

$$\mathbf{v} = \vec{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

of any point $P \in \mathscr{S}$ is a unit vector, i.e. a vector of norm one: $\|\mathbf{v}\| = 1$.

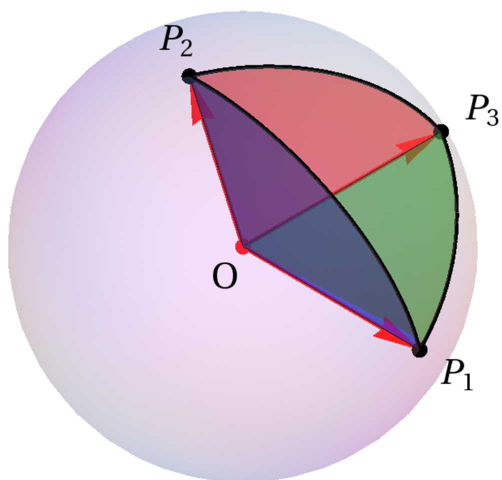
Now suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are three *unit* vectors representing points on $\mathscr{S}$. The corresponding points P_1, P_2, P_3 will be the vertices of a spherical triangle provided the 'unit-column' matrix

$$V = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix},$$

is invertible, or equivalently

$$\begin{aligned} \det V &= (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 = (\mathbf{v}_2 \times \mathbf{v}_3) \cdot \mathbf{v}_1 = (\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_2 \\ &= -(\mathbf{v}_2 \times \mathbf{v}_1) \cdot \mathbf{v}_3 = -(\mathbf{v}_3 \times \mathbf{v}_2) \cdot \mathbf{v}_1 = -(\mathbf{v}_1 \times \mathbf{v}_3) \cdot \mathbf{v}_2 \end{aligned}$$

is non-zero. In this case, P_1, P_2, P_3 are not coplanar, and the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a *basis* of $\mathbb{R}^3$. The 'sides' of the triangle are then the segments of great circles (of radius 1) through the vertices:



Two of our vectors, say $\mathbf{v}_1$ and $\mathbf{v}_2$, generate a (blue) plane Π_3 that passes through O; the intersection $\Pi_3 \cap \mathscr{S}$ is a circle, and the arc from P_1 to P_2 is the side of the triangle opposite P_3. The lengths of the three sides are *equal* to the angles (in radians)

$$\theta_1 = \angle P_2 O P_3, \quad \theta_2 = \angle P_3 O P_1, \quad \theta_3 = \angle P_1 O P_2,$$

whose cosines are

$$c_1 = \cos\theta_1 = \mathbf{v}_2 \cdot \mathbf{v}_3, \quad c_2 = \cos\theta_2 = \mathbf{v}_3 \cdot \mathbf{v}_1, \quad c_3 = \cos\theta_3 = \mathbf{v}_1 \cdot \mathbf{v}_2.$$

These quantities feature in the symmetric matrix

$$V^T V = \begin{pmatrix} 1 & c_3 & c_2 \\ c_3 & 1 & c_1 \\ c_2 & c_1 & 1 \end{pmatrix}.$$

We shall assume that the three angles/lengths are no greater than π.

13.2. Spherical law of cosines[1]

Set $\Delta = \det V \neq 0$, and define

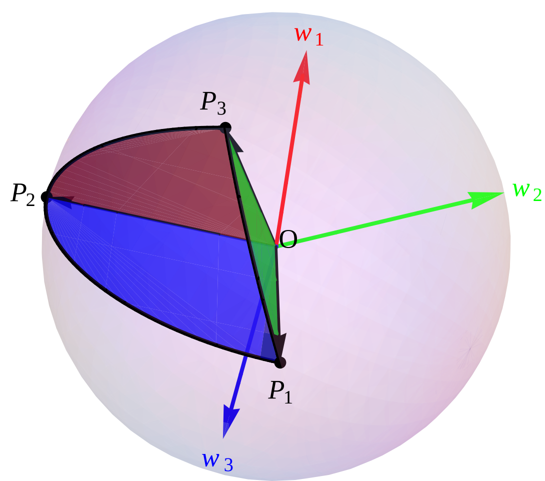
$$\mathbf{w}_1 = \frac{1}{\Delta}\mathbf{v}_2 \times \mathbf{v}_3, \quad \mathbf{w}_2 = \frac{1}{\Delta}\mathbf{v}_3 \times \mathbf{v}_1, \quad \mathbf{w}_3 = \frac{1}{\Delta}\mathbf{v}_1 \times \mathbf{v}_2.$$

Then $\mathbf{w}_1 \cdot \mathbf{v}_1 = 1$, $\mathbf{w}_1 \cdot \mathbf{v}_2 = 0$ and so on, indeed: $\mathbf{w}_i \cdot \mathbf{v}_j = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$

It follows that the inverse of V is the matrix

$$W^T = \begin{pmatrix} \leftarrow \mathbf{w}_1^T \rightarrow \\ \leftarrow \mathbf{w}_2^T \rightarrow \\ \leftarrow \mathbf{w}_3^T \rightarrow \end{pmatrix}.$$

If V is an orthogonal matrix, the original basis is orthonormal and $W = V$. In general, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is the *reciprocal basis* to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.



The vector $\mathbf{w}_1 = \frac{1}{\Delta}\mathbf{v}_1 \times \mathbf{v}_2$ is normal to the plane containing O, P_2, P_3. Moreover, a triple like $\mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1$ has a right-handed orientation, which makes the normal $\mathbf{w}_3$ point 'outwards' from the triangular solid. We previously defined the angle of the spherical triangle at say P_1 to be the angle between the tangents to the two arcs meeting at P_1. But these tangents are both perpendicular to the radial line OP_1, and we are therefore speaking of the angle (defined in §12.3) between the two planes meeting along $\overleftrightarrow{OP_1}$.

The interior angles ϕ_1, ϕ_2, ϕ_3 of the spherical triangle each measure $180^{\rm o}$ minus the angles between the normals pictured overleaf, and so

$$\cos\phi_1 = -\frac{\mathbf{w}_2 \cdot \mathbf{w}_3}{\|\mathbf{w}_2\|\|\mathbf{w}_3\|}, \quad \cos\phi_2 = -\frac{\mathbf{w}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_3\|\|\mathbf{w}_1\|}, \quad \cos\phi_3 = -\frac{\mathbf{w}_1 \cdot \mathbf{w}_2}{\|\mathbf{w}_1\|\|\mathbf{w}_2\|}.$$

[1] The following approach due to W. P. Thurston, 1946–2012

We now apply these calculations to a spherical triangle. Since $V^{-1} = W^T$ and (for any matrix, $(V^{-1})^T = (V^T)^{-1}$), we have $(V^TV)^{-1} = V^{-1}(V^T)^{-1} = W^T(V^{-1})^T = W^TW$, and so

$$W^TW = \frac{1}{\Delta^2}\begin{pmatrix} 1-c_1^2 & c_1c_2-c_3 & c_1c_3-c_2 \\ c_1c_2-c_3 & 1-c_2^2 & c_2c_3-c_1 \\ c_1c_3-c_2 & c_2c_3-c_1 & 1-c_3^2 \end{pmatrix}.$$

Each row of this symmetric matrix is the cross product of the columns of V^TV, and the entries of the matrix are the cofactors of V^TV; this is how they were written down.

It follows that

$$\mathbf{w}_1 \cdot \mathbf{w}_1 = \frac{1}{\Delta^2}(1-c_1^2), \qquad \mathbf{w}_1 \cdot \mathbf{w}_2 = \frac{1}{\Delta^2}(c_1c_2 - c_3).$$

The first equation confirms that $\|\mathbf{w}_1\| = s_3/|\Delta|$, where $s_3 = \sin\theta_3$, something we already know as $\|\mathbf{v}_1 \times \mathbf{v}_2\| = s_3$. The second yields

$$-\cos\phi_3 = \frac{\mathbf{w}_1 \cdot \mathbf{w}_2}{\|\mathbf{w}_1\|\|\mathbf{w}_2\|} = \frac{c_1c_2 - c_3}{s_1s_2}.$$

Rearranging, and writing this out in full,

$$\cos\theta_3 = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi_3.$$

Using A, B, C for the vertices and their interior angles, and a, b, c for the lengths of he opposite sides, we have proved the

Theorem. *The cosine of the length of the third side of a spherical triangle is given by*

$$\boxed{\cos c = \cos a\cos b + \sin a\sin b\cos C}$$

13.3. Applications

If the triangle is very small compared to the unit radius of the sphere (as is the case on the surface of the earth), we may reasonably use approximations

$$\cos x = 1 - \tfrac{1}{2}x^2, \qquad \sin x = x$$

for $x = a, b, c$, given by Taylor's theorem. Then

$$1 - \tfrac{1}{2}c^2 = (1 - \tfrac{1}{2}a^2)(1 - \tfrac{1}{2}b^2) + ab\cos C,$$

and to order 2 we obtain the Euclidean cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

We do not approximate $\cos C$ as there is no assumption that C be small. Pythagoras' theorem is the special case in which $C = \pi/2$ so $\cos C = 0$. The spherical version of 'Pythagoras' is therefore

$$\cos c = \cos a\cos b.$$

The spherical sine rule

$$\boxed{\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}}$$

can be deduced from the cosine rule as we did in the for the Euclidean version

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The fact that the spherical sine rule can also be expressed as

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

suggests that one might be able to interchange the side lengths a, b, c (the θ_i) and vertex angles A, B, C (the ϕ_i) in the cosine formula. This is almost true, because we can switch the roles of the matrices V and W in the proof. But remembering the minus signs in front of $\mathbf{w}_i \cdot \mathbf{w}_j$, gives one overall sign change:

Theorem. *The third angle of a spherical triangle is given by*

$$\boxed{\cos C = -\cos A \cos B + \sin A \sin B \cos c}$$

In all these formulas, we assume that the quantities a, b, c and A, B, C are all less than π. In particular, the mapping $C \mapsto \cos C$ is a *bijection* $[0, \pi) \to (-1, 1]$.

Once one angle C and the lengths a, b of adjacent sides are known, the (first) cosine rule can be used to determine the third side. The same rule (with sides switched) can then be used to find the remaining angles B and C. This tells us that the SAS rule applies to spherical triangles provided we restrict to $k = 1$ to get congruent triangles:

Corollary. *If two spherical triangles have one angle equal and the lengths of the corresponding adjacent sides equal, then all corresponding sides and angles are equal:*

$$a = a',\ b = b',\ C = C' \quad \Rightarrow \quad A = A',\ B = B',\ c = c'.$$

The second cosine rule gives us a property that is *not* true for Euclidean triangles:

Corollary. *The lengths of a spherical triangle are determined by its angles.*

Let us 'grade' spherical geometry according to our initial postulates B1–B5:

B1 fails because lines (meaning, great circles) are not infinite in extent, and the distance between any two points on $\mathscr{S}$ is at most π.

B2 fails because opposite (the correct word is *antipodal*) points lie on infinitely many lines.

B3 is valid because at any point (think of it as the north pole) there is a great circle leaving at any angle.

B4 is valid because we use tangent vector to measure angles.

B5 only works for the scaling factor $k = 1$.

[Not examinable: The failure of B2 is not in itself serious. B2 *will* apply if we merely declare that antipodal points are in fact equal. This gives a new type of plane:

Definition. The *real projective plane* $\mathscr{P}$ is the set of all straight lines passing through the origin O in $\mathbb{R}^3$; these are the *points* of $\mathscr{P}$. The set of all such lines in a given plane through O defines a subset of $\mathscr{P}$, called a *line*.

Each straight line though O intersects $\mathscr{S}$ in two antipodal points, so a single *point* of $\mathscr{P}$ corresponds to a pair of antipodal points of $\mathscr{S}$. A line of $\mathscr{P}$ corresponds to a great circle in $\mathscr{S}$. Since B5 still fails, the parallel postulate is not valid in $\mathscr{P}$. But the situation is rather satisfactory: *any two lines meet in exactly one point*!]

13.4. The area of a spherical triangle

In this course, we have said little about area. But we take it for granted that the area of a sphere of radius 1 is known to be 4π. In view of the last corollary, it is reasonable to suppose that the area of a spherical triangle $\mathscr{T}$ is completely determined by its three angles A, B, C. This means that we can write

$$\text{area}(\mathscr{T}) = f(A, B, C),$$

where f is a positive function whose value depends synmetrically on three variables.

If we divide the solid sphere into segments like those of an orange, we obtain a region on the surface consisting of just two great circles that meet at antipodal points at an equal angle of (say) C. The area of this region will be proportional to C and therefore equal to $(C/2\pi)4\pi = 2C$. We can divide the region into two traingles by choosing a third great circle that cuts the other two, giving us angles $A, \pi - A$ and $B, \pi - B$. Hence the equation

$$f(A, B, C) + f(\pi - A,\ \pi - B,\ C) = 2C$$

must hold for all $A, B, C < \pi$. The obvious guess

$$f(A, B, C) = A + B + C - \pi$$

is in fact the correct formula for f, though the proof of this fact is postponed until the second year:

Theorem (on spherical excess). *The area of a spherical triangle equals the sum of its angles minus* π.

Corollary. *The angles of any spherical triangle add up to a number greater than* π.

Example. An equilateral triangle (meaning $a = b = c$) must have all angles greater than $\pi/3$. We can check this from the cosine rule, which tells us that $A = B = C$ satisfies

$$\cos a = \cos^2 a + (\sin^2 a)\cos A.$$

Set $t = \tan(a/2)$ so that $\sin a = 2t/(1 + t^2)$ and $\cos a = (1 - t^2)/(1 + t^2)$. Then

$$(1 - t^2)(1 + t^2) = (1 - t^2)^2 + 4t^2 \cos A,$$

and $\cos A = \frac{1}{2}(1 - t^2) < \frac{1}{2}$ so $A > \pi/3$.