Quotienting a metric with holonomy $G_2$

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Simons Collaboration in Geometry, Analysis, and Physics

Rauischholzhausen, 29 August 2018
There is the concept of a non-degenerate 3-form on $\mathbb{R}^7$, but it can be positive or negative.

The former ($\varphi$, varying smoothly) defines a $G_2$ structure on $M^7$, an underlying Riemannian metric $h$, and a 4-form $\ast \varphi$.

$$\text{Hol}(h) \subseteq G_2 \iff \nabla \varphi = 0 \iff \begin{cases} d\varphi = 0 \\ d\ast \varphi = 0 \end{cases}$$

In this case, $h$ is Ricci-flat.

$G_2$ manifolds are analogues of Calabi-Yau 3-folds. Many compact manifolds admitting such metrics are known, but not (of course) the exact metrics themselves.
**G₂ metrics with symmetry**

If \((N^6, k)\) is nearly-Kähler (weak holonomy \(SU(3)\)) then

- \(dr^2 + r^2 k\) has holonomy \(G₂\) on \(\mathbb{R}^+ \times N\)
- \(dr^2 + (\sin r)^2 k\) has weak holonomy \(G₂\) on \((0, \pi) \times N\)

We can take

\[ N = S^3 \times S^3, \quad \mathbb{C}P^3, \quad F = SU(3)/T^2, \]

with isometry groups \(SU(2)^3, SO(5) \simeq Sp(2), SU(3)\).

NK metrics with a co-homogeneous one action by \(SU(2)^2\) exist on both \(S^3 \times S^3\) and \(S^6\) [Foscolo-Haskins-Nordström 2016].

Complete \(G₂\) metrics with \(SU(2)^2 \times U(1)\) symmetry (so rank 3) are also known [FHN 2018].
A nilpotent example

An ansatz has been described for $G_2$ metrics with a $T^3$ action [Madsen-Swann 2018].

**Simplest example.** Rather than a NK space, take a nilmanifold $N^6$ based on the Lie algebra $(0, 0, 0, 23, 31, 12)$, so there is a basis $(e_i)$ of 1-forms so that $de_4 = e_2 \wedge e_3$ etc. Then $N$ has an $SU(3)$ structure that can be evolved into a metric

$$
\mu^2(e_1^2 + e_2^2 + e_3^2) + \frac{1}{\mu}(e_4^2 + e_5^2 + e_6^2) + \mu^3 d\mu^2.
$$

with holonomy equal to $G_2$ on $M = (0, \infty) \times N$.

Here, $\mu$ is one component of a moment map $M \to \mathbb{R}^4$ that arises from the toric theory.
G₂ and physics

“...all of physics has a completely geometric origin in M theory on a singular $G₂$ manifold” [Acharya 2016]

- string theories are modelled on 6 hidden dimensions in space
- the only known compact Ricci-flat 6-manifolds have special holonomy $SU(3)$, thus the importance of Calabi-Yau spaces
- M theory unifies the five supersymmetric string theories by adding an 11th dimension
- $G₂$ manifolds provide suitable models, and are expected to come with circle fibrations
- singularities of codimension 4 and 7 are needed to produce Yang-Mills fields and particles
From the theory of ALE spaces, $\mathbb{R}^+ \times \mathbb{C}P^3_{n,n,1,1}$ is conjectured to carry a metric with holonomy $G_2$. This is true when $n = 1$, and this lecture will focus on an $S^1$ quotient of $\mathbb{R}^+ \times \mathbb{C}P^3$ that resembles $\mathbb{R}^6$ with two singular $\mathbb{R}^3$'s meeting at the origin.
Suppose that $U(1)$ acts on a (non-compact) manifold with a $G_2$ holonomy metric $h$, and that $\mathcal{L}_X \varphi = 0$. Set

\[
\begin{align*}
1/t &= \|X\| = h(X, X)^{1/2} \quad \text{measures orbit size} \\
\eta &= t^2 X \perp h \quad \text{dual 1-form with } \eta(X) = 1 \\
F &= d\eta \quad \text{so } X \perp F = 0 \text{ and } dF = 0 \\
\sigma &= X \perp \varphi \quad \text{so } d\sigma = 0.
\end{align*}
\]

Then

\[
\begin{align*}
\varphi &= \eta \wedge \sigma + t^{3/2} \psi^+ \\
\star \varphi &= \eta \wedge (t^{1/2} \psi^-) + \frac{1}{2} (t \sigma)^2.
\end{align*}
\]

Here, $\Psi = \psi^+ + i \psi^-$ is an induced $(3, 0)$-form for the $SU(3)$ structures on the base, and $F = d\eta$ is the curvature 2-form.
Today’s example: \( S^1 \) acting on \( S^4 \)

\[
\begin{array}{c}
S^7 \\
\downarrow \\
\mathbb{C}P^3 \\
\downarrow \\
S^4
\end{array} \quad \begin{array}{c}
\mathbb{R}^8 \\
\downarrow \\
\mathbb{R}^+ \times \mathbb{C}P^3 \\
\downarrow \\
\mathbb{R}^3 \times \mathbb{R}^3
\end{array} = \begin{array}{c}
\mathbb{R}^4 \times \mathbb{R}^4 \\
\downarrow \\
\mathbb{R}^3 \cup \mathbb{R}^3
\end{array}

\[
\begin{array}{c}
\rightarrow \\
S^4/S^1 = \mathbb{D}^3
\end{array}
\]

\( Q \) is induced from the action of \( SO(2) \) on \( S^4 \subset \mathbb{R}^2 \oplus \mathbb{R}^3 \).

The action fixes two 2-spheres in \( \mathbb{C}P^3 \), giving \( \mathbb{R}^3 \cup \mathbb{R}^3 \) in \( \mathbb{R}^6 \).
Consider
\[ C = \mathbb{R}^+ \times \mathbb{C}P^3 \]
\[ \downarrow Q \]
\[ \mathbb{R}^6 \setminus 0 = \mathcal{M} \]

We seek explicit descriptions of:

- the NK structure on $\mathbb{C}P^3$ and the $G_2$ 3-form $\varphi$ on $C$
- the 2-torus action on $\mathbb{R}^8$ and the map $Q : C \to M$
- the metric $g$ induced on $M$ from the $G_2$ metric $h$ on $C$
- the symplectic form $\sigma$ on $M$ and the curvature $F$ of $Q$
- subvarieties of $\mathcal{M}$ on which $Q$ or $g$ is flat.
Hopf maps

Let $e = \sum_{i=0}^{7} dx_i^2$ be the Euclidean metric, and $R = \sum x_i^2$.

Right multiplication by $Sp(1)$ gives Killing vector fields

\begin{align*}
Y_1 &= x_1 \partial_0 - x_0 \partial_1 - x_3 \partial_2 + x_2 \partial_3 + x_5 \partial_4 - x_4 \partial_5 - x_7 \partial_6 + x_6 \partial_7 \\
Y_2 &= x_2 \partial_0 - x_0 \partial_2 - \\
Y_3 &= x_3 \partial_0 - x_0 \partial_3 - \\
\end{align*}

$tangent to the fibres of

\begin{align*}
S^7 &
\downarrow 1 \\
\mathbb{C}P^3 &
\downarrow 2, 3 \\
S^4 &
\end{align*}
Set $\alpha_i = Y_i \bot e$, so for example

$$\alpha_1 = x_1 dx_0 - x_0 dx_1 - \cdots - x_7 dx_6 + x_6 dx_7$$

$$-d\alpha_1 = 2(dx_{01} - dx_{23} + dx_{45} - dx_{67})$$

Each 1-form $\hat{\alpha}_i = \alpha_i / R$ is invariant by $\mathbb{R}^*$, and the 2-forms

$$\begin{cases} 
\tau_1 = d\hat{\alpha}_1 - 2\hat{\alpha}_{23} \\
\tau_2 = d\hat{\alpha}_2 - 2\hat{\alpha}_{31} \\
\tau_3 = d\hat{\alpha}_3 - 2\hat{\alpha}_{12}
\end{cases}$$

pass to $S^4$, where they form a basis of ASD forms.

Moreover, $d\hat{\alpha}_1 = \tau_1 + 2\hat{\alpha}_{23}$ is a Kähler form on $(\mathbb{CP}^3, J_1)$. 
Lemma. The NK structure of \((\mathbb{CP}^3, J_2)\) is given by

\[
\omega = \tau_1 - \hat{\alpha}_{23} = d\hat{\alpha}_1 - 3\hat{\alpha}_{23}
\]

\[
\Upsilon = (\hat{\alpha}_2 - i\hat{\alpha}_3) \wedge (\tau_2 + i\tau_3)
\]

These forms satisfy the identities

\[
\begin{align*}
d\omega &= 3\text{Im}\Upsilon, \\
d\Upsilon &= 2\omega^2.
\end{align*}
\]

Note. The conical \(G_2\) structure on \(\mathcal{C}\) now has

\[
\varphi = dR \wedge R^2\omega + R^3\text{Im}\Upsilon = d\left(\frac{1}{3}R^3\omega\right),
\]

\[
*\varphi = dR \wedge R^3\text{Re}\Upsilon + \frac{1}{2}(R^2\omega)^2 = d\left(\frac{1}{4}R^4\text{Re}\Upsilon\right)
\]
The $G_2$ metric

Recall that $e = \sum_{i=0}^{7} dx_i^2$ and $R = \sum_{i=0}^{7} x_i^2$.

Proposition. The $G_2$ metric $h$ on $\mathcal{C} = \mathbb{R}^+ \times \mathbb{CP}^3$ pulls back to

$$\frac{1}{2} dR^2 + 2Re - 2\alpha_1^2 - \alpha_2^2 - \alpha_3^2$$

on $\mathbb{R}^8 \setminus 0$.

This is invariant by $Sp(2)$ and we want to “push it down” to $\mathcal{M}$.

Problems.

1. Use the proposition to prove by computer that $g$ is Ricci-flat.
2. Is there a version for weighted $\mathbb{CP}^3_{n,n,1,1}$?

PS. Consider also metrics obtained by changing coefficients of the $\alpha_i$; preserving the degeneracy condition $Y_1 \perp h = 0$. 
Left multiplication by $U(1)$ on $\mathbb{H}^2$ generates

$$X = X_1 = x_1 \partial_0 - x_0 \partial_1 + x_3 \partial_2 - x_2 \partial_3 + \cdots + x_7 \partial_6 - x_6 \partial_7,$$

and one observes that

$$\frac{1}{2}(X + Y_1) = x_1 \partial_0 - x_0 \partial_1 + x_5 \partial_4 - x_4 \partial_5$$
$$\frac{1}{2}(X - Y_1) = x_3 \partial_2 - x_2 \partial_3 + x_7 \partial_6 - x_6 \partial_7.$$

These define standard $U(1)$ actions on each of $\mathbb{R}^4_{0145}$ and $\mathbb{R}^4_{2367}$.

Using hyperkähler moment maps, it follows that

$$\mathbb{R}^+ \times \frac{\mathbb{CP}^3}{U(1)} \cong \frac{\mathbb{R}^4}{U(1)} \times \frac{\mathbb{R}^4}{U(1)} \cong \mathbb{R}^3 \times \mathbb{R}^3,$$

modulo the origin.
Let \( q = x_0 + x_1 i + x_4 j + x_5 k \). The Killing field \( X + Y_1 \) induces a tri-holomorphic action on \( \mathbb{R}^4_{0145} \) with moment map

\[
q \mapsto \bar{q} i q = u_1 i - u_3 j + u_2 k,
\]

invariant by \( q \leadsto e^{it} q \), where

\[
\begin{align*}
  u_1 &= x_1^2 + x_2^2 - x_5^2 - x_6^2 \\
  u_2 &= 2(-x_1 x_6 + x_2 x_5) \\
  u_3 &= 2(x_1 x_5 + x_2 x_6).
\end{align*}
\]

Note that \( u = |u| \) satisfies \( u^2 = u_1^2 + u_2^2 + u_3^2 \).

Using \( \mathbb{R}^4_{2367} \), we define \( \nu_1, \nu_2, \nu_3 \) and \( \nu \) in the same way.
Invariant functions

Our left $U(1)$ in $Sp(2)$ commutes with $SU(2)$ which acts as $SO(3)$ as follows:

- trivially on the first factor of $\mathbb{R}^2 \oplus \mathbb{R}^3 \supset S^4$
- diagonally on the quotient $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$.

The induced $SU(3)$ structure on $\mathbb{R}^6$ can be expressed in terms of $SO(3)$ invariant quantities manufactured from the coordinates

$$(u, v) = (u_1, u_2, u_3, v_1, v_2, v_3),$$

the radii $u, v$ (and $R = u + v$), using scalar and triple products.

The involution $j$ on $\mathbb{CP}^3$ generates an isometry $\varepsilon: u \leftrightarrow v$. 
A first application

We compute the symplectic form

$$\sigma = X \varphi = d\zeta,$$

where $\zeta = \frac{1}{3} R^3 X \cdot (d\hat{\alpha}_1 - 3\hat{\alpha}_{23})$. It can be expressed in terms of

$$\begin{cases}
\mu_1 &= x_1^2 + x_2^2 - x_3^2 - x_4^2 + x_5^2 + x_6^2 - x_7^2 - x_8^2 \\
\mu_2 &= 2(-x_1 x_4 + x_2 x_3 - x_5 x_8 + x_6 x_7) \\
\mu_3 &= 2(x_1 x_3 + x_4 x_2 + x_5 x_7 + x_6 x_8).
\end{cases}$$

where $\mu_j = X \cdot \alpha_j$. Recall that $R = u + v$.

Theorem.

$$\sigma = R\left(\frac{1}{2} du \wedge dv - du \wedge dv\right) + \frac{1}{2} dR \wedge (v \cdot du - u \cdot dv).$$
**Corollary.** The mapping $\mathcal{M} \rightarrow \mathbb{R}^3$ given by $(u, v) \mapsto u + v$ has Lagrangian fibres.

Set $x = u + v$. The result follows from a computation:

$$dx_1 \wedge dx_2 \wedge dx_3 \wedge \sigma = 0.$$

The single subspace $u = v$ is also Lagrangian.

**Problems.**

3. Find Darboux coordinates for $\omega$. In particular, is there a map $y = \phi(u, v)$ so that $\omega = dx \wedge dy$?

4. Describe the reduced twistor fibration $\mathcal{M} \rightarrow S^4/U(1) = \mathbb{D}^3$. Is it given by $(u, v) \mapsto (u + v)/R$?
The curvature 2-form $F$

The connection 1-form $\eta$ equals $t^2 X \perp h$, where $t = 1/\|X\|$.

Proposition.

\[ t^{-2} = 6uv - 2u \cdot v \]
\[ \eta = 2RX \perp e - 2\mu_1 \alpha_1 - \mu_2 \alpha_2 - \mu_3 \alpha_3. \]

The curvature $F = d\eta$ is invariant by $\mathbb{R}^+ \times \mathbb{R}^+$, and so determined by its restriction $\hat{F}$ to $S^2 \times S^2$.

Theorem.

\[ \hat{F} \equiv \frac{1}{4}\{u, du, du\} + d\left(\frac{1}{2} t^2 \{u, v, du\}\right) \]

$\equiv$ means we have to add on terms after applying $\varepsilon$ to the RHS, so it becomes symmetric in $u, v$. 
The induced complex volume form

The space of $(3, 0)$ forms on $\mathcal{M}$ is generated by $\Psi = \psi^+ + i\psi^-$. 

**Theorem.**

$$-t \psi^+ \equiv \frac{1}{2} v(t^{-2} + 4v^2)\{du, du, du\}$$

$$-v(4u^2 + 3uv + u\cdot v)\{dv, du, du\}$$

$$+((u + 2v)\mathbf{v}\cdot d\mathbf{v} + \mathbf{v}\cdot d\mathbf{v}) \wedge \{u, du, du\}$$

$$+((v\mathbf{u}\cdot d\mathbf{v} - u\mathbf{v}\cdot d\mathbf{v}) \wedge \{v, du, du\}.$$ 

$$\frac{1}{2uv} \psi^- \equiv (t^{-2} + 4v^2)\{du, du, dv\}$$

$$+((3 + \frac{u}{v})\mathbf{u}\cdot \mathbf{v} - 3u^2 - 5uv)\{du, dv, dv\}$$

$$+2\mathbf{v}\cdot d\mathbf{v} \wedge \{u, du, du\}$$

$$+2\frac{u}{v}\mathbf{v}\cdot d\mathbf{u} \wedge \{v, dv, dv\}$$

$$+((1 - \frac{u}{v})\mathbf{v}\cdot d\mathbf{v} + (3 + \frac{v}{u})\mathbf{u}\cdot d\mathbf{v}) \wedge \{v, du, du\}. $$

One presumes that $J$ is not integrable, i.e. that $d(t^{1/2}\psi^+) \neq 0!$
The induced metric

The metric \( g \) on \( \mathcal{M} \cong \mathbb{R}^6 \setminus \{0\} \) for which

\[
Q : (\mathcal{C}, h) \longrightarrow (\mathcal{M}, g)
\]

is a Riemannian submersion on an open set of its domain can be computed via \( Q^*g = h - \eta \otimes \eta \).

**Theorem** [Bryant].

\[
g = \frac{1}{2} \left[ (du + dv) \cdot (du + dv) + (du + dv)^2 \right] \\
+ \frac{t^2}{4} \left[ 8(v du - u dv) \cdot (v du - u dv) \\
+ 2(v du + u dv - u \cdot dv - v \cdot du)^2 \\
- (v du - u dv - u \cdot dv + v \cdot du)^2 \right].
\]
Restricting to subspaces

The restriction of $g$ to the quadrant $\{(0, 0, u, 0, 0, v)\}$ with $u, v > 0$ equals

$$g = \left(1 + \frac{v}{4u}\right) du^2 + \frac{3}{2} dudv + \left(1 + \frac{u}{4v}\right) dv^2.$$ 

This has Gaussian curvature $K \equiv 0$. We’ll explain why.

Extend the domain $\mathbb{R}^2$ to $\{u_1 = 0, v_1 = 0\} \cong \mathbb{R}^4$, which contains representatives of each $SO(3)$ orbit. Here we set

$$\begin{cases} 
    u &= (0, u \cos(\theta + \chi), -u \sin(\theta + \chi)) \\
    v &= (0, v \cos(\theta - \chi), v \sin(\theta - \chi)),
\end{cases}$$

with

$$u = R \cos^2(\phi/2), \quad v = R \sin^2(\phi/2),$$

to ensure that $u + v = R$. 
Corollary. The circle bundle is *flat* over $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. The restriction of $g$ to $\mathbb{R}^4$ equals $dR^2 + R^2 \widehat{g}$, where

$$2 \widehat{g} = d\theta^2 + \frac{1}{4} (3 - \cos 2\theta) d\phi^2$$
$$+ \frac{1}{8} (7 + \cos 2\theta + 2 \sin^2 \theta \cos 2\phi) d\chi^2$$
$$+ 2 \cos \phi d\theta d\chi - \frac{1}{2} \sin 2\theta \sin \phi d\phi d\chi.$$

Invariants of the $SO(3)$ action are $u, v, \theta$, since $u \cdot v = uv \cos 2\theta$.

When $\chi = 0$, we obtain a slice $S^1 \times [0, \pi]$ to the $SO(3)$ orbits on which

$$2 \widehat{g} = d\theta^2 + \frac{1}{4} (3 - \cos 2\theta) d\phi^2.$$

Adding an isometric slice with $\chi = \pi/2$ gives
A surface of revolution

The 2-sphere represents $Q^{-1}$ of a single semi-circle:

Top and bottom circles are identified, and $\mathcal{M}$ is foliated by cones over 2-tori of this shape:

The blue plane corresponds to points where $u, v$ are aligned:
The function

\[ \theta \mapsto \frac{1}{\sqrt{2}} \int_{0}^{\pi} f(\theta) \, d\phi = \frac{\pi}{2} \sqrt{\frac{3}{2} - \frac{1}{2} \cos 2\theta} \]

can be interpreted as a measure of the angles between the subspaces \( u = 0 \) and \( v = 0 \), i.e. the two \( \mathbb{R}^3 \)'s whose union is image of the fixed point set in \( C \).

It varies from \( \pi/2 \) to \( \pi/\sqrt{2} \approx 127^\circ \), and the bulge in the surface of revolution reflects the fact that a circle of radius \( R = 1 \) has circumference \( 2\sqrt{2}\pi \) when \( \theta = \pi/2 \) (and \( u, v \) are anti-aligned).
Finally, take $\theta = 0$ and consider
\[
B = \{(0, -u \sin \chi, -u \cos \chi, 0, v \sin \chi, v \cos \chi)\} \subset M
\]
\[
A = Q^{-1}(B) \subset C.
\]

**Theorem.** $A$ is coassociative (i.e. calibrated by $\star \varphi$, so $\varphi|_A = 0$) and projects to a 3-sphere in $S^4$.

Recall that $\star \varphi = d\left(\frac{1}{4} R^4 \text{Re} \Upsilon\right)$. In fact $\Upsilon|_A = 0$. For
\[
A/\mathbb{R}^+ = \{|z_0| = |z_2|, |z_1| = |z_3|, z_0z_1 + z_2z_3 = 0\}
\]
is a real hypersurface $S^1 \times S^2$ of a complex quadric in $\mathbb{CP}^3$. The $S^2$ factor is a horizontal complex curve annihilated by
\[
\alpha_2 - i\alpha_3 = -2i \lambda \mu e^{i(\alpha + \beta)} d\chi.
\]
Further problems

\[ C = \mathbb{R}^+ \times \mathbb{C}P^3 \]

\[ \downarrow Q \]

\[ \mathbb{R}^6 \setminus 0 = \mathcal{M} \]

5. The 2-form \( \sigma \) on \( \mathcal{M} \) is simplest, but \( J \) is intractible. Can one characterize the induced \( SU(3) \) structure and monopole equations, and use it to reconstruct metrics with holonomy \( G_2 \)?

6. \( \mathcal{M} \) is foliated by \( SO(3) \) orbits, but the induced left-invariant metrics vary in a complicated way. Does the projection \( \mathcal{M} \to \mathbb{D}^3 \) (or some other) give a better description of \( g \)?

7. Extend the previous analysis to the \( U(1) \) quotient of \( \Lambda^2 T^*S^4 \) with its complete \( G_2 \) metric.