Quotienting a metric with holonomy G_2

Simon Salamon

with Bobby Acharya and Robert Bryant

SF

Simons Collaboration in
Geometry, Analysis, and Physics

Rauischholzhausen, 29 August 2018

There is the concept of a non-degenerate 3-form on $\mathbb{R}^7$, but it can be *positive* or negative.

The former (φ, varying smoothly) defines a G_2 structure on M^7, an underlying *Riemannian* metric h, and a 4-form $*\varphi$.

$$\mathrm{Hol}(h) \subseteq G_2 \iff \nabla\varphi = 0 \iff \begin{cases} d\varphi = 0 \\ d{*\varphi} = 0 \end{cases}$$

In this case, h is *Ricci-flat*.

G_2 manifolds are analogues of Calabi-Yau 3-folds. Many compact manifolds admitting such metrics are known, but not (of course) the exact metrics themsleves.

G_2 metrics with symmetry

If (N^6, k) is nearly-Kähler (weak holonomy $SU(3)$) then

- $dr^2 + r^2 k$ has holonomy G_2 on $\mathbb{R}^+ \times N$
- $dr^2 + (\sin r)^2 k$ has weak holonomy G_2 on $(0, \pi) \times N$

We can take

$$N = S^3 \times S^3, \qquad \mathbb{CP}^3, \qquad \mathbb{F} = SU(3)/T^2,$$

with isometry groups $SU(2)^3$, $SO(5) \simeq Sp(2)$, $SU(3)$.

NK metrics with a co-homogeneous one action by $SU(2)^2$ exist on both $S^3 \times S^3$ and S^6 [Foscolo-Haskins-Nordström 2016].

Complete G_2 metrics with $SU(2)^2 \times U(1)$ symmetry (so rank 3) are also known [FHN 2018].

A nilpotent example

An ansatz has been described for G_2 metrics with a T^3 action [Madsen-Swann 2018].

Simplest example. Rather than a NK space, take a nilmanifold N^6 based on the Lie algebra $(0,0,0,23,31,12)$, so there is a basis (e_i) of 1-forms so that $de_4 = e_2 \wedge e_3$ etc. Then N has an $SU(3)$ structure that can be evolved into a metric

$$\mu^2(e_1^2 + e_2^2 + e_3^2) + \frac{1}{\mu}(e_4^2 + e_5^2 + e_6^2) + \mu^3 d\mu^2.$$

with holonomy equal to G_2 on $M = (0,\infty) \times N$.

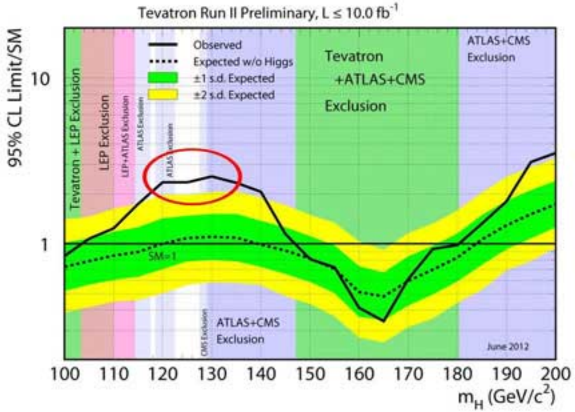
Here, μ is one component of a moment map $M \to \mathbb{R}^4$ that arises from the toric theory.

G_2 and physics

"...all of physics has a completely geometric origin in M theory on a singular G_2 manifold" [Acharya 2016]

- string theories are modelled on 6 hidden dimensons in space
- the only known compact Ricci-flat 6-manifolds have special holonomy $SU(3)$, thus the importance of Calabi-Yau spaces
- M theory unifies the five supersymmetric string theories by adding an 11th dimension
- G_2 manifolds provide suitable models, and are expected to come with circle fibrations
- singularities of codimension 4 and 7 are needed to produce Yang-Mills fields and particles

- From the theory of ALE spaces, $\mathbb{R}^+ \times \mathbb{CP}^3_{n,n,1,1}$ is conjectured to carry a metric with holonomy G_2. This is true when $n = 1$, and this lecture will focus on an S^1 quotient of $\mathbb{R}^+ \times \mathbb{CP}^3$ that resembles $\mathbb{R}^6$ with two singular $\mathbb{R}^3$'s meeting at the origin.



Technicalities

Suppose that $U(1)$ acts on a (non-compact) manifold with a G_2 holonomy metric h, and that $\mathscr{L}_X\varphi = 0$. Set

$$
\begin{aligned}
1/t &= \|X\| = h(X,X)^{1/2} && \text{measures orbit size}\\
\eta &= t^2 X\lrcorner\, h && \text{dual 1-form with } \eta(X) = 1\\
F &= d\eta && \text{so } X\lrcorner\, F = 0 \text{ and } dF = 0\\
\sigma &= X\lrcorner\, \varphi && \text{so } d\sigma = 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
\varphi &= \eta\wedge\sigma + t^{3/2}\psi^+\\
*\varphi &= \eta\wedge(t^{1/2}\psi^-) + \tfrac{1}{2}(t\sigma)^2.
\end{aligned}
$$

Here, $\Psi = \psi^+ + i\psi^-$ is an induced $(3,0)$-form for the $SU(3)$ structures on the base, and $F = d\eta$ is the curvature 2-form.

Today's example: S^1 acting on S^4

$$
\begin{array}{ccccc}
S^7 & & \mathbb{R}^8 & = & \mathbb{R}^4 \times \mathbb{R}^4 \\
\downarrow & & \downarrow & & \\
\mathbb{CP}^3 & & \mathbb{R}^+ \times \mathbb{CP}^3 & & \downarrow\downarrow \\
\downarrow & & & \searrow Q & \\
S^4 & & & & \mathbb{R}^3 \times \mathbb{R}^3 \\
& \searrow & & & \\
& & S^4/S^1 = \mathbb{D}^3 & &
\end{array}
$$

Q is induced from the action of $SO(2)$ on $S^4 \subset \mathbb{R}^2 \oplus \mathbb{R}^3$.

The action fixes two 2-spheres in $\mathbb{CP}^3$, giving $\mathbb{R}^3 \cup \mathbb{R}^3$ in $\mathbb{R}^6$.

Goals

Consider

$$\mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3$$
$$\searrow Q$$
$$\mathbb{R}^6 \setminus 0 = \mathscr{M}$$

We seek explicit descriptions of:

- the NK structure on $\mathbb{CP}^3$ and the G_2 3-form φ on $\mathscr{C}$
- the 2-torus action on $\mathbb{R}^8$ and the map $Q\colon \mathscr{C} \to \mathscr{M}$
- the metric g induced on $\mathscr{M}$ from the G_2 metric h on $\mathscr{C}$
- the symplectic form σ on $\mathscr{M}$ and the curvature F of Q
- subvarieties of $\mathscr{M}$ on which Q or g is flat.

Let $e = \sum_{i=0}^{7} dx_i^2$ be the Euclidean metric, and $R = \sum x_i^2$.

Right multiplication by $Sp(1)$ gives Killing vector fields

$$\begin{aligned}
Y_1 &= x_1\partial_0 - x_0\partial_1 - x_3\partial_2 + x_2\partial_3 + x_5\partial_4 - x_4\partial_5 - x_7\partial_6 + x_6\partial_7 \\
Y_2 &= x_2\partial_0 - x_0\partial_2 - \qquad \cdots \qquad - x_5\partial_7 + x_7\partial_5 \\
Y_3 &= x_3\partial_0 - x_0\partial_3 - \qquad \cdots \qquad - x_6\partial_5 + x_5\partial_6,
\end{aligned}$$

tangent to the fibres of

$$\begin{array}{l} S^7 \\ \downarrow 1 \\ \mathbb{CP}^3 \\ \downarrow 2,3 \\ S^4. \end{array}$$

Linear forms

Set $\alpha_i = Y_i \lrcorner\, e$, so for example

$$\alpha_1 = x_1 dx_0 - x_0 dx_1 - \cdots - x_7 dx_6 + x_6 dx_7$$
$$-d\alpha_1 = 2(dx_{01} - dx_{23} + dx_{45} - dx_{67})$$

Each 1-form $\hat{\alpha}_i = \alpha_i / R$ is invariant by $\mathbb{R}^*$, and the 2-forms

$$\begin{cases} \tau_1 &= d\hat{\alpha}_1 - 2\hat{\alpha}_{23} \\ \tau_2 &= d\hat{\alpha}_2 - 2\hat{\alpha}_{31} \\ \tau_3 &= d\hat{\alpha}_3 - 2\hat{\alpha}_{12} \end{cases}$$

pass to S^4, where they form a basis of ASD forms.

Moreover, $d\hat{\alpha}_1 = \tau_1 + 2\hat{\alpha}_{23}$ is a Kähler form on $(\mathbb{CP}^3, J_1)$.

Lemma. The NK structure of $(\mathbb{CP}^3, J_2)$ is given by

$$\begin{aligned} \omega &= \tau_1 - \hat{\alpha}_{23} = d\hat{\alpha}_1 - 3\hat{\alpha}_{23} \\ \Upsilon &= (\hat{\alpha}_2 - i\hat{\alpha}_3) \wedge (\tau_2 + i\tau_3) \end{aligned}$$

These forms satisfy the identities

$$\begin{cases} d\omega &= 3\,\mathrm{Im}\Upsilon, \\ d\Upsilon &= 2\omega^2. \end{cases}$$

Note. The conical G_2 structure on $\mathscr{C}$ now has

$$\begin{aligned} \varphi &= dR \wedge R^2\omega + R^3\,\mathrm{Im}\Upsilon &= d(\tfrac{1}{3}R^3\omega), \\ *\varphi &= dR \wedge R^3\,\mathrm{Re}\Upsilon + \tfrac{1}{2}(R^2\omega)^2 &= d(\tfrac{1}{4}R^4\,\mathrm{Re}\Upsilon) \end{aligned}$$

The G_2 metric

Recall that $e = \sum_{i=0}^{7} dx_i^2$ and $R = \sum_{i=0}^{7} x_i^2$.

Proposition. The G_2 metric h on $\mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3$ pulls back to

$$\tfrac{1}{2} dR^2 + 2R\,e - 2\alpha_1^2 - \alpha_2^2 - \alpha_3^2$$

on $\mathbb{R}^8 \setminus 0$.

This is invariant by $Sp(2)$ and we want to "push it down" to $\mathscr{M}$.

Problems.

1. Use the proposition to prove by computer that g is Ricci-flat.
2. Is there a version for weighted $\mathbb{CP}^3_{n,n,1,1}$?

PS. Consider also metrics obtained by changing coefficients of the α_i preserving the degeneracy condition $Y_1 \lrcorner\, h = 0$.

A 2-torus action on $\mathbb{R}^8$

Left multiplication by $U(1)$ on $\mathbb{H}^2$ generates

$$X = X_1 = x_1\partial_0 - x_0\partial_1 + x_3\partial_2 - x_2\partial_3 + \cdots + x_7\partial_6 - x_6\partial_7,$$

and one observes that

$$\begin{aligned} \tfrac{1}{2}(X + Y_1) &= x_1\partial_0 - x_0\partial_1 + x_5\partial_4 - x_4\partial_5 \\ \tfrac{1}{2}(X - Y_1) &= x_3\partial_2 - x_2\partial_3 + x_7\partial_6 - x_6\partial_7. \end{aligned}$$

These define standard $U(1)$ actions on each of $\mathbb{R}^4_{0145}$ and $\mathbb{R}^4_{2367}$.

Using hyperkähler moment maps, it follows that

$$\mathbb{R}^+ \times \frac{\mathbb{CP}^3}{U(1)} \cong \frac{\mathbb{R}^4}{U(1)} \times \frac{\mathbb{R}^4}{U(1)} \cong \mathbb{R}^3 \times \mathbb{R}^3,$$

modulo the origin.

Gibbons-Hawking ansatz

Let $q = x_0 + x_1 i + x_4 j + x_5 k$. The Killing field $X + Y_1$ induces a tri-holomorphic action on $\mathbb{R}^4_{0145}$ with moment map

$$q \mapsto \overline{q}\, i\, q = u_1 i - u_3 j + u_2 k,$$

invariant by $q \rightsquigarrow e^{it} q$, where

$$\begin{cases} u_1 = & x_1^2 + x_2^2 - x_5^2 - x_6^2 \\ u_2 = & 2(-x_1 x_6 + x_2 x_5) \\ u_3 = & 2(x_1 x_5 + x_2 x_6). \end{cases}$$

Note that $u = |\mathbf{u}|$ satisfies $u^2 = u_1^2 + u_2^2 + u_3^2$.

Using $\mathbb{R}^4_{2367}$, we define v_1, v_2, v_3 and v in the same way.

Our left $U(1)$ in $Sp(2)$ commutes with $SU(2)$ which acts as $SO(3)$ as follows:

- trivially on the first factor of $\mathbb{R}^2 \oplus \mathbb{R}^3 \supset S^4$
- diagonally on the quotient $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$.

The induced $SU(3)$ structure on $\mathbb{R}^6$ can be expressed in terms of $SO(3)$ invariant quantities manufactured from the coordinates

$$(\mathbf{u}, \mathbf{v}) = (u_1, u_2, u_3, v_1, v_2, v_3),$$

the radii u, v (and $R = u + v$), using scalar and triple products.

The involution j on $\mathbb{CP}^3$ generates an isometry $\varepsilon \colon \mathbf{u} \leftrightarrow \mathbf{v}$.

A first application

We compute the symplectic form

$$\sigma = X \lrcorner \varphi = d\zeta,$$

where $\zeta = \frac{1}{3}R^3 X \lrcorner (d\hat{\alpha}_1 - 3\hat{\alpha}_{23})$. It can be expressed in terms of

$$\begin{cases} \mu_1 & = x_1^2 + x_2^2 - x_3^2 - x_4^2 + x_5^2 + x_6^2 - x_7^2 - x_8^2 \\ \mu_2 & = 2(-x_1x_4 + x_2x_3 - x_5x_8 + x_6x_7) \\ \mu_3 & = 2(x_1x_3 + x_4x_2 + x_5x_7 + x_6x_8). \end{cases}$$

where $\mu_j = X \lrcorner \alpha_j$. Recall that $R = u + v$.

Theorem.

$$\sigma = R(\tfrac{1}{2}du \wedge dv - d\mathbf{u} \mathbin{\mathbb{A}} d\mathbf{v}) + \tfrac{1}{2}dR \wedge (\mathbf{v}\cdot d\mathbf{u} - \mathbf{u}\cdot d\mathbf{v}).$$

Corollary. The mapping $\mathscr{M} \to \mathbb{R}^3$ given by $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} + \mathbf{v}$ has Lagrangian fibres.

Set $\mathbf{x} = \mathbf{u} + \mathbf{v}$. The result follows from a computation:

$$dx_1 \wedge dx_2 \wedge dx_3 \wedge \sigma = 0.$$

The single subspace $\mathbf{u} = \mathbf{v}$ is also Lagrangian.

Problems.

3. Find Darboux coordinates for ω. In particular, is there a there a map $\mathbf{y} = \phi(\mathbf{u}, \mathbf{v})$ so that $\omega = d\mathbf{x} \mathbin{\dot\wedge} d\mathbf{y}$?

4. Describe the reduced twistor fibration $\mathscr{M} \to S^4/U(1) = \mathbb{D}^3$. Is it given by $(\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{u} + \mathbf{v})/R$?

The curvature 2-form F

The connection 1-form η equals $t^2 X \lrcorner h$, where $t = 1/\|X\|$.

Proposition.

$$\begin{aligned} t^{-2} &= 6uv - 2\mathbf{u}\cdot\mathbf{v} \\ \eta &= 2RX \lrcorner e - 2\mu_1\alpha_1 - \mu_2\alpha_2 - \mu_3\alpha_3. \end{aligned}$$

The curvature $F = d\eta$ is invariant by $\mathbb{R}^+ \times \mathbb{R}^+$, and so determined by its restriction $\widehat{F}$ to $S^2 \times S^2$.

Theorem.

$$\widehat{F} \ominus \tfrac{1}{4}\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} + d\Big(\tfrac{1}{2}t^2\{\mathbf{u}, \mathbf{v}, d\mathbf{u}\}\Big)$$

$\ominus$ means we have to add on terms after applying ε to the RHS, so it becomes symmetric in $\mathbf{u}, \mathbf{v}$.

The induced complex volume form

The space of $(3,0)$ forms on $\mathscr{M}$ is generated by $\Psi = \psi^+ + i\psi^-$.

Theorem.

$$
\begin{aligned}
-t\psi^+ \quad \circleddash \quad & \tfrac{1}{2}v(t^{-2}+4v^2)\{d\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} \\
& -v(4u^2+3uv+\mathbf{u}\cdot\mathbf{v})\{d\mathbf{v}, d\mathbf{u}, d\mathbf{u}\} \\
& +((u+2v)\mathbf{v}\cdot d\mathbf{v} + v\mathbf{u}\cdot d\mathbf{v}) \wedge \{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} \\
& +(v\mathbf{u}\cdot d\mathbf{v} - u\mathbf{v}\cdot d\mathbf{v}) \wedge \{\mathbf{v}, d\mathbf{u}, d\mathbf{u}\}.
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{1}{2uv}\psi^- \quad \widehat{\circleddash} \quad & (t^{-2}+4v^2)\{d\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} \\
& +((3+\tfrac{u}{v})\mathbf{u}\cdot\mathbf{v} - \underline{3}u^2 - 5uv)\{d\mathbf{u}, d\mathbf{v}, d\mathbf{v}\} \\
& +2\mathbf{v}\cdot d\mathbf{v} \wedge \{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} \\
& +2\tfrac{u}{v}\mathbf{v}\cdot d\mathbf{u} \wedge \{\mathbf{v}, d\mathbf{v}, d\mathbf{v}\} \\
& +((1-\tfrac{u}{v})\mathbf{v}\cdot d\mathbf{v} + (3+\tfrac{v}{u})\mathbf{u}\cdot d\mathbf{v}) \wedge \{\mathbf{v}, d\mathbf{u}, d\mathbf{u}\}.
\end{aligned}
$$

One presumes that J is not integrable, i.e. that $d(t^{1/2}\psi^+) \neq 0$!

The induced metric

The metric g on $\mathscr{M} \cong \mathbb{R}^6 \setminus 0$ for which

$$Q\colon (\mathscr{C}, h) \longrightarrow (\mathscr{M}, g)$$

is a Riemannian submersion on an open set of its domain can be computed via $Q^*g = h - \eta\otimes\eta$.

Theorem [Bryant].

$$\begin{aligned} g = \frac{1}{2}\Big[&(d\mathbf{u} + d\mathbf{v})\cdot(d\mathbf{u} + d\mathbf{v}) + (du + dv)^2\Big] \\ &+\frac{t^2}{4}\Big[8(v\,d\mathbf{u} - u\,d\mathbf{v})\cdot(v\,d\mathbf{u} - u\,d\mathbf{v}) \\ &\qquad +2(v\,du + u\,dv - \mathbf{u}\cdot d\mathbf{v} - \mathbf{v}\cdot d\mathbf{u})^2 \\ &\qquad - (v\,du - u\,dv - \mathbf{u}\cdot d\mathbf{v} + \mathbf{v}\cdot d\mathbf{u})^2\Big]. \end{aligned}$$

Restricting to subspaces

The restriction of g to the quadrant $\{(0,0,u,0,0,v)\}$ with $u,v>0$ equals

$$g=\left(1+\tfrac{v}{4u}\right)du^2+\tfrac{3}{2}du dv+\left(1+\tfrac{u}{4v}\right)dv^2.$$

This has Gaussian curvature $K\equiv 0$. *We'll explain why.*

Extend the domain $\mathbb{R}^2$ to $\{u_1=0, v_1=0\}\cong\mathbb{R}^4$, which contains representatives of each $SO(3)$ orbit. Here we set

$$\begin{cases} \mathbf{u} &= (0,\ u\cos(\theta+\chi), -u\sin(\theta+\chi)) \\ \mathbf{v} &= (0,\ v\cos(\theta-\chi),\ \ v\sin(\theta-\chi)), \end{cases}$$

with

$$u=R\cos^2(\phi/2), \qquad v=R\sin^2(\phi/2),$$

to ensure that $u+v=R$.

Corollary. The circle bundle is *flat* over $\mathbb{R}^4=\mathbb{R}^2\times\mathbb{R}^2$. The restriction of g to $\mathbb{R}^4$ equals $dR^2+R^2\widehat{g}$, where

$$\begin{aligned} 2\widehat{g} = \quad & d\theta^2+\tfrac{1}{4}(3-\cos 2\theta)d\phi^2 \\ & +\tfrac{1}{8}(7+\cos 2\theta+2\sin^2\theta\,\cos 2\phi)d\chi^2 \\ & +2\cos\phi\,d\theta\,d\chi-\tfrac{1}{2}\sin 2\theta\sin\phi\,d\phi\,d\chi. \end{aligned}$$

Invariants of the $SO(3)$ action are u,v,θ, since $\mathbf{u}\cdot\mathbf{v}=uv\cos 2\theta$.

When $\chi=0$, we obtain a slice $S^1\times[0,\pi]$ to the $SO(3)$ orbits on which

$$2\widehat{g}=d\theta^2+\tfrac{1}{4}(3-\cos 2\theta)d\phi^2.$$

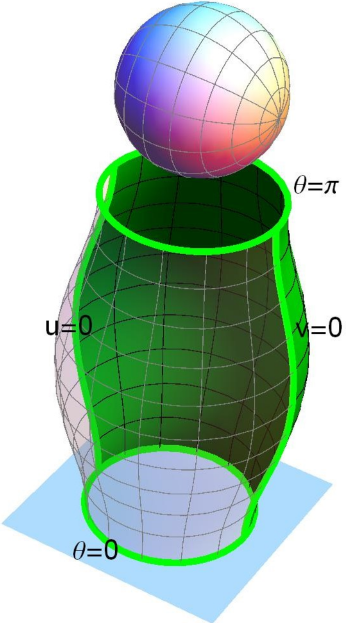
Adding an isometric slice with $\chi=\pi/2$ gives

A surface of revolution

The 2-sphere represents Q^{-1} of a single semi-circle:

Top and bottom circles are identified, and $\mathscr{M}$ is foliated by cones over 2-tori of this shape:

The blue plane corresponds to points where $\mathbf{u}, \mathbf{v}$ are aligned:



The function

$$\theta \mapsto \frac{1}{\sqrt{2}} \int_0^{\pi} f(\theta)\, d\phi = \frac{\pi}{2} \sqrt{\tfrac{3}{2} - \tfrac{1}{2} \cos 2\theta}$$

can be interpreted as a measure of the angles between the subspaces $u = 0$ and $v = 0$, i.e. the two $\mathbb{R}^3$'s whose union is image of the fixed point set in $\mathscr{C}$.

It varies from $\pi/2$ to $\pi/\sqrt{2} \sim 127^{\mathrm{o}}$, and the bulge in the surface of revolution reflects the fact that a circle of radius $R = 1$ has circumference $2\sqrt{2}\pi$ when $\theta = \pi/2$ (and $\mathbf{u}, \mathbf{v}$ are *anti*-aligned).

Finally, take $\theta = 0$ and consider

$$\begin{aligned}\mathscr{B} &= \{(0, -u\sin\chi, -u\cos\chi, 0,\ v\sin\chi,\ v\cos\chi)\} \subset \mathscr{M}\\ \mathscr{A} &= Q^{-1}(\mathscr{B}) \subset \mathscr{C}.\end{aligned}$$

Theorem. $\mathscr{A}$ is coassociative (i.e. calibrated by $*\varphi$, so $\varphi|\mathscr{A} = 0$) and projects to a 3-sphere in S^4.

Recall that $*\varphi = d(\frac{1}{4}R^4\,\mathrm{Re}\Upsilon)$. In fact $\Upsilon\,|\,\mathscr{A} = 0$. For

$$\mathscr{A}/\mathbb{R}^+ = \{|z_0| = |z_2|,\ |z_1| = |z_3|,\ z_0z_1 + z_2z_3 = 0\}$$

is a real hypersurface $S^1 \times S^2$ of a complex quadric in $\mathbb{CP}^3$. The S^2 factor is a horizontal complex curve annihilated by

$$\alpha_2 - i\alpha_3 = -2i\lambda\mu\, e^{i(\alpha+\beta)}\, d\chi.$$

$$
\begin{array}{lll}
\mathscr{C} = \mathbb{R}^{+} \times \mathbb{CP}^{3} & & \\
& \searrow Q & \\
& & \mathbb{R}^{6} \setminus 0 = \mathscr{M}
\end{array}
$$

5. The 2-form σ on $\mathscr{M}$ is simplest, but J is intractible. Can one characterize the induced $SU(3)$ structure and monopole equations, and use it to reconstruct metrics with holonomy G_2?

6. $\mathscr{M}$ is foliated by $SO(3)$ orbits, but the induced left-invariant metrics vary in a complicated way. Does the projection $\mathscr{M} \to \mathbb{D}^3$ (or some other) give a better description of g?

7. Extend the previous analysis to the $U(1)$ quotient of $\Lambda^2_- T^*S^4$ with its *complete* G_2 metric.