Kähler configurations of points

Simon Salamon

KING'S College LONDON

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The Hesse configuration

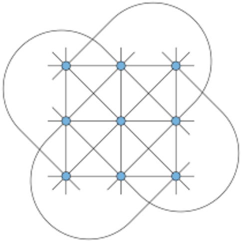
Let $\omega = e^{2\pi i/3}$. Consider the nine points

$[0, 1, -1]$	$[0, 1, -\omega]$	$[0, 1, -\omega^2]$
$[1, 0, -1]$	$[1, 0, -\omega]$	$[1, 0, -\omega^2]$
$[1, -1, 0]$	$[1, -\omega, 0]$	$[1, -\omega^2, 0]$

of $\mathbb{CP}^2$. They are the flexes of the non-singular cubic

$$x^3 + y^3 + z^3 - 3c\,xyz = 0$$

for any $c \in \mathbb{C} \setminus \{1, \omega, \omega^3\}$ and belong in triples on 12 lines:



A 1-parameter family of nine points

The representative vectors all have norm $1/\sqrt{2}$, and any two satisfy $\big|\langle \mathbf{w}, \mathbf{z}\rangle\big| = 1$. It follows that the 9 points are mutually equidistant. The same is true of the set $\mathbb{M}_\theta$ consisting of

$[0, 1, -1]$	$[0, 1, -\omega]$	$[0, 1, -\omega^2]$
$[1, 0, -1]$	$[1, 0, -\omega]$	$[1, 0, -\omega^2]$
$[e^{i\theta}, -1, 0]$	$[e^{i\theta}, -\omega, 0]$	$[e^{i\theta}, -\omega^2, 0]$

Theorem*. Any unordered set of nine mutually equidistant points in $(\mathbb{CP}^2, g)$ is isometric to $\mathbb{M}_\theta$ for some angle θ.

*L. Hughston, S. Salamon, in Advances Math. 2016

Fubini-Study distance

Complex projective space $\mathbb{CP}^{n-1}$ is a compact Kähler manifold. Its Riemannian metric g arises from the standard Hermitian form

$$\langle \mathbf{w}, \mathbf{z} \rangle = \langle \mathbf{w} \,|\, \mathbf{z} \rangle = \sum_{i=1}^{n} \overline{w}_i z_i,$$

on $\mathbb{C}^n$, which is invariant by the unitary group $U(n)$.

The associated distance d satisfies

$$\cos^2\left(\tfrac{1}{2} d([\mathbf{w}], [\mathbf{z}])\right) = \frac{|\langle \mathbf{w}, \mathbf{z} \rangle|^2}{\|\mathbf{w}\|^2 \|\mathbf{z}\|^2} = \frac{\langle \mathbf{w}, \mathbf{z} \rangle \langle \mathbf{z}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle \langle \mathbf{z}, \mathbf{z} \rangle}.$$

The RHS is really a *cross ratio* of four points $[\mathbf{w}], [\mathbf{z}], [\mathbf{w}'], [\mathbf{z}']$.

Moment mapping

The 2-form $\omega_0 = i\sum dz_i \wedge d\bar{z}_i$ on $\mathbb{C}^n$ induces a symplectic form on $\mathbb{CP}^{n-1}$, regarded as a symplectic quotient of $\mathbb{C}^n$ by $\mathrm{U}(1)$. Let

$$\mathscr{H} = \{A \in \mathbb{C}^{n,n} : A = A^*,\ \mathrm{tr}\, A = 1\} \ \simeq\ \mathfrak{su}(n).$$

The mapping $S^{2n-1} \to \mathscr{H}$ for which

$$[\mathbf{z}] \longmapsto \mathbf{z}^*\mathbf{z} = \begin{pmatrix} |z_1|^2 & \bar{z}_1 z_2 & \bar{z}_1 z_3 & \cdots \\ \bar{z}_2 z_1 & |z_2|^2 & \bar{z}_2 z_3 & \cdots \\ \bar{z}_3 z_1 & \bar{z}_3 z_2 & |z_3|^2 & \cdots \\ \cdots & \cdots & \cdots & \end{pmatrix}$$

is $\mathrm{U}(n)$-equivariant and defines an isometric embedding

$$\mathbb{CP}^{n-1} \hookrightarrow \mathscr{H} \simeq \mathbb{R}^{n^2-1}.$$

SIC-POVM

is shorthand for Symmetric Informationally Complete *Positive Operator Valued Measure**. We shall consider such objects in finite-dimensional Hilbert space $\mathbb{C}^n$.

Definition 1. A SIC-POVM is a set of n^2 points $[\mathbf{z}_\alpha]$ in $\mathbb{CP}^{n-1}$ such that (with the normalization $\|\mathbf{z}_\alpha\| = 1$)

$$\left|\langle \mathbf{z}_\alpha, \mathbf{z}_\beta \rangle\right|^2 = \lambda, \quad \alpha \neq \beta, \quad \text{for some fixed } \lambda \in (0,1).$$

This defines a subset $\{P_\alpha = \mathbf{z}_\alpha^* \mathbf{z}_\alpha\}$ of $\mathscr{H}$ and

$$\sum_{\alpha=1}^{n^2} P_\alpha = n\, I, \qquad \operatorname{tr}(P_\alpha P_\beta) = \begin{cases} 1 & \alpha = \beta \\ \lambda & \alpha \neq \beta. \end{cases}$$

*E. B. Davies: *The Quantum Theory of Open Systems*, 1976

Definition 2. A SIC set consists of a regular simplex in $\mathfrak{su}(n)$ whose n^2 vertices lie in the adjoint orbit $\mathbb{CP}^{n-1}$.

In fact, n^2 is the maximum number of mutually equidistant points possible in $\mathbb{CP}^{n-1}$, and in this case λ must equal $\frac{1}{n+1}$.

Proof. Given an equidistant set $\{P_\alpha\}$, set $Q_\beta = P_\beta - \lambda I$. Then

$$\mathrm{tr}(P_\alpha Q_\beta) = \begin{cases} 1 - \lambda & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

So $\{P_\alpha\}$ is linearly independent in $i\,\mathfrak{u}(n)$.

Applying $\mathrm{tr}(Q_\beta \bullet)$ to $I = \sum c_\alpha P_\alpha$, gives $1 - \lambda n = c_\alpha(1 - \lambda)\ \forall \alpha$.

Then $n = n^2 c_\alpha$, and so $n(1 - \lambda n) = 1 - \lambda$.

Example: the 2-sphere

The embedding $\mathbb{CP}^1 = S^2 \hookrightarrow \mathbb{R}^3$ is given by setting

$$\begin{pmatrix} |z_1|^2 & \bar{z}_1 z_2 \\ z_1 \bar{z}_2 & |z_2|^2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1+a & b+ic \\ b-ic & 1-a \end{pmatrix}.$$

One SIC set consists of the points

$$\boxed{\begin{matrix} [1+\sqrt{3},\, 1+i] & [1+i,\, 1+\sqrt{3}] \\ [1+\sqrt{3},\, -1-i] & [-1-i,\, 1+\sqrt{3}] \end{matrix}}$$

This is an orbit of $\langle A, B \rangle$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Any SIC set determines an inscribed tetrahedron and any two are congruent by $SO(3) = \mathrm{SU}(2)/\mathbb{Z}_2$. If $n \geqslant 3$, two SIC sets are *not* in general congruent by $\mathrm{SU}(n)$.

Arbitrary dimensions

Conjecture 1. $\mathbb{CP}^{n-1}$ possesses a SIC set for all n.

Algebraic solutions are known for $n = 2, 3, 4, \ldots, 15, 19, 24, 35, 48$. Extensive numerical verification has been carried out for $n \leqslant 151$.

Conjecture 2*. For all n, there is a SIC set that is an orbit of the (Weyl-)Heisenberg group $H \to (\mathbb{Z}/n\mathbb{Z})^2$.

A vector $\mathbf{z}$ or point $[\mathbf{z}]$ whose orbit $H \cdot [\mathbf{z}]$ is a SIC set is called *fiducial*. An example for $\mathbb{CP}^3$ is

$$[\mathbf{z}] = \Big[-s - i(r+s),\ 1-r+i,\ s+i(s-r),\ 1+r+i\Big].$$

where $r = \sqrt{2}$ and $s = \sqrt{2+\sqrt{5}}$.

*G. Zauner, PhD thesis, Vienna, 1999

Example: a SIC set in $\mathbb{CP}^7$

Using the identification $\mathbb{C}^8 = \mathbb{H}^4$, consider the groups

- V_1, right multiplication by $1, i, j, k \in Sp(1)$
- V_2, double sign changes in $\mathbb{H}^4$
- V_3, double transpositions of the coordinates.

Then $G = V_1 \times V_2 \times V_3 \cong (\mathbb{Z}_2)^6$ is a subgroup of $Sp(4) \subset \mathrm{SU}(8)$.

Fix unit quaternions $p = \frac{1}{2}(1 + i + j - k)$, $q = \frac{1}{2}(1 + i - j - k)$.

Proposition*. $G \cdot [0,\ p,\ q,\ j]$ is a SIC set in $\mathbb{CP}^7$.

This orbit cannot project 'neatly' to $\mathbb{HP}^2$ because no S^2 fibre can contain 4 points.

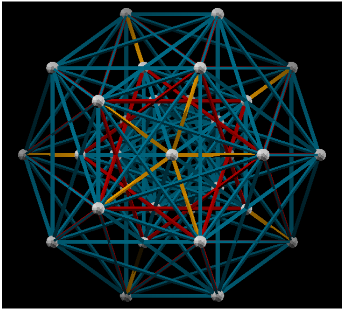
*S. G. Hoggar in Geometria Dedicata, 1998

Analogue: a Gosset polytope

Its 56 vertices form the orbit under $S_8 \times \mathbb{Z}_2$ of

$$(1, 1, 1, -3, -3, 1, 1, 1) \in \mathbb{R}^8.$$

They lie in a subspace $\mathbb{R}^7$ and are the weights for the action of E_7 on $\mathbb{C}^{28}$. All $\binom{56}{2}$ inner products equal -8 or 8, defining 28 mutually equidistant points on $\mathbb{RP}^6$.

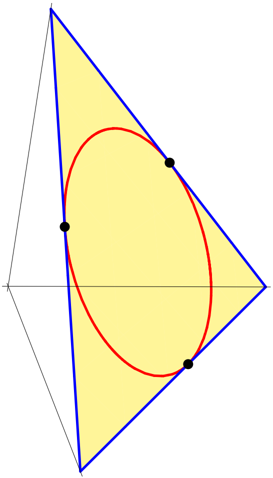


A 2-torus acting on $\mathbb{CP}^2$

Fix $T = \left\{\text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) : \sum \theta_i = 0\right\}$ in $\text{SU}(3)$.

Consider the moment map for the action of T:

$\mu([\mathbf{z}]) = \frac{1}{\|\mathbf{z}\|^2}(|z_1|^2, |z_2|^2, |z_3|^2) = (a^2, b^2, c^2)$.



The image of μ is a 2-simplex whose inscribed circle $\mathscr{C}$ will play a vital role:

Let m_1, m_2, m_3 be the midpoints, and set $C_i = \mu^{-1}(m_i) \subset \mathbb{CP}^2$.

Note that $\mathbb{M}_\theta \subset C_1 \sqcup C_2 \sqcup C_3$.

Correct separation

We set $a = |z_1|/\|\mathbf{z}\|$, $b = |z_2|/\|\mathbf{z}\|$, $c = |z_3|/\|\mathbf{z}\|$.

Then $\mathscr{C}$ is the intersection of the plane $a^2 + b^2 + c^2 = 1$ and the sphere $a^4 + b^4 + c^4 = \frac{1}{2}$.

Lemma. The following are equivalent:

- $\mu([\mathbf{z}]) \in \mathscr{C}$
- $(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 0$
- the three points $\mathbf{z} = [z_1, z_2, z_3]$, $[z_1, \omega z_2, \omega^2 z_3]$, $[z_1, \omega^2 z_2, \omega z_3]$ are the correct distance ($d = 2 \arccos \frac{1}{2} = 2\pi/3$) apart to form part of a SIC set.

The generic point in a SIC set

Let $\mathbb{S}$ be a SIC set in $\mathbb{CP}^2$. Up to the action of $\mathrm{SU}(3)$, we are free to assume that $\mathbb{S}$ contains the two points of C_1 given by

$$\mathbf{z}_1 = \frac{1}{\sqrt{2}}(0,1,-\omega), \quad \mathbf{z}_2 = \frac{1}{\sqrt{2}}(0,1,-\omega^2).$$

Any other point $[\mathbf{z}]$ of $\mathbb{S}$ satisfies $\left|\left\langle \mathbf{z}, \mathbf{z}_j \right\rangle\right|^2 = \frac{1}{4}\|\mathbf{z}\|^2$ for $j = 0, 1$.

Lemma. $\mu([\mathbf{z}]) \in \mathscr{C}$ and we can take

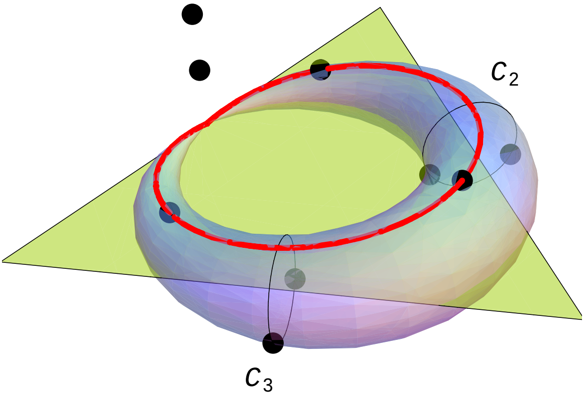
$$[\mathbf{z}] = \mathbf{z}[\sigma, \phi] = \left[e^{i\sigma}\cos\phi,\ \cos(\phi + \tfrac{2\pi}{3}),\ \cos(\phi + \tfrac{4\pi}{3})\right]$$

for some $-\pi < \sigma \leqslant \pi$ and $-\frac{\pi}{2} < \phi \leqslant \frac{\pi}{2}$.

Thus, $[\mathbf{z}]$ lies in a 2-torus, pinched to a point where $\phi = \pi/2$.

The pinched torus $\mathscr{T}$

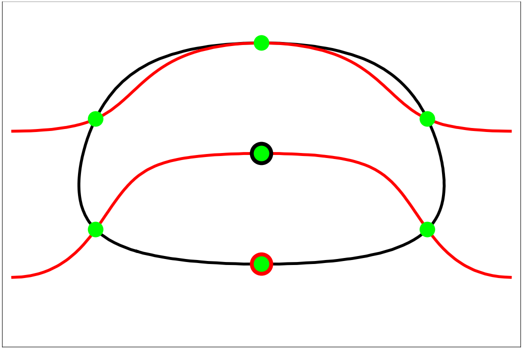
containing $\mathbf{z}[0, \phi + \frac{k\pi}{3}]$ with $k = 0, 1, 2$, and two points in both C_2, C_3, forming a SIC set $\mathbb{L}_\phi$ when $[\mathbf{z}_1], [\mathbf{z}_2]$ are added:



The same thing in a rectangle

Recall that $\mathbf{z}[\sigma, \phi] = \left[e^{i\sigma} \cos\phi,\ \cos(\phi + \frac{2\pi}{3}), \cos(\phi + \frac{4\pi}{3})\right]$.

Lemma. There exists a SIC set $\mathbb{L}_\phi$ containing $[\mathbf{z}_1], [\mathbf{z}_2]$ and $\mathbf{z}[0, \phi]$ for *any* ϕ with $-\frac{\pi}{2} < \phi \leqslant \frac{\pi}{2}$:



Equivalent SIC sets

The remaining six points are then

$$\mathbf{z}[0, \phi + \tfrac{\pi}{3}]$$

$$\mathbf{z}[\tfrac{-2\pi}{3}, \tfrac{\pi}{6}] \qquad \mathbf{z}[\tfrac{2\pi}{3}, \tfrac{\pi}{6}] \qquad \in C_2$$

$$\mathbf{z}[\tfrac{-2\pi}{3}, -\tfrac{\pi}{6}] \qquad \mathbf{z}[\tfrac{2\pi}{3}, -\tfrac{\pi}{6}] \qquad \in C_3$$

$$\mathbf{z}[0, \phi - \tfrac{\pi}{3}]$$

Lemma. $X \cdot \mathbb{L}_\phi = \mathbb{M}_{2\phi+\pi} \subset C_1 \sqcup C_2 \sqcup C_3$, where

$$X = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & \omega & \omega^2 \\ 1 & 1 & 1 \end{pmatrix} \in \mathrm{U}(3).$$

is the normalizer G of H in $\mathrm{U}(n)$, isomorphic (modulo phase) to $\mathrm{SL}(2,\mathbb{Z}_n) \ltimes (\mathbb{Z}_n)^2$.

For $n = 3$, G has order 216 and contains X, since

$$XAX^{-1} = \omega B, \qquad XBX^{-1} = \omega^2 A^{-1} B^{-1}.$$

Moreover, $X^3 = i\omega^2 I$.

Conjecture 3. A fiducial vector $\mathbf{z}$ can always be found in an eigenspace of the unitary matrix X where

$$X_{ij} = \frac{1}{\sqrt{n}} \tau^{2ij+j^2}, \quad \tau = e^{(n+1)\pi i/n}.$$

For $n = 5$ see: G. Horrocks, D. Mumford, in Topology 1973

Trigonometry

Let $\mathbb{S}$ be a SIC set containing $[\mathbf{z}_1], [\mathbf{z}_2]$, and

$$[\mathbf{z}_3] = \mathbf{z}[\sigma, \phi], \quad [\mathbf{z}_4] = \mathbf{z}[\tau, \psi], \quad [\mathbf{z}_5] = \mathbf{z}[\upsilon, \chi] \quad \in \mathscr{T}.$$

Set $x = \tan\phi,\ y = \tan\psi,\ z = \tan\chi$.

Key lemma. $[\mathbf{z}_3]$ and $[\mathbf{z}_4]$ are distance $2\pi/3$ apart iff

$$\begin{aligned}\cos(\sigma - \tau) &= 1 - \frac{9(1 + 2\cos 2(\phi - \psi))}{4(4\cos^2\phi\cos^2\psi + 3\sin 2\phi \sin 2\psi)} \\ &= \frac{-11 + 9x^2 + 9y^2 - 27x^2y^2 - 24xy}{16(1 + 3xy)},\end{aligned}$$

If $A + B + C = 0$ then

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 + 2\cos A \cos B \cos C.$$

Specify their vertices by vertical coordinates x, y, z and set

$$p = x + y + z, \quad q = yz + zx + xy, \quad r = xyz.$$

Corollary. If $\mathbf{z}[\sigma, \phi]$, $\mathbf{z}[\tau, \psi]$, $\mathbf{z}[\upsilon, \chi]$ are all $2\pi/3$ apart then

$$f(x, y, z) = F(p, q, r) = 0,$$

where

$$\begin{gathered} F = 9 - 22p^2 + 9p^4 + 87q - 126p^2q + 27p^4q + 298q^2 - 226p^2q^2 \\ +24p^4q^2 + 414q^3 - 138p^2q^3 + 189q^4 + 27q^5 - 3pr - 50p^3r - 15p^5r \\ +88pqr - 48p^3qr + 234pq^2r + 18p^3q^2r - 144pq^3r + 81pq^4r + 189r^2 \\ -480p^2r^2 - 153p^4r^2 + 1398qr^2 - 306p^2qr^2 + 2736q^2r^2 - 486p^2q^2r^2 \\ +810q^3r^2 + 243q^4r^2 - 558pr^3 - 486p^3r^3 + 2376pqr^3 - 810pq^2r^3 \\ +567r^4 - 162p^2r^4 + 6399qr^4 + 486q^2r^4 + 1701pr^5 + 2187r^6. \end{gathered}$$

cf. $p^2 - 3q = \frac{3}{4}$ in the Euclidean case.

Let $\mathbb{S}$ be a SIC set containing $[\mathbf{z}_1]$, $[\mathbf{z}_2]$ and $[\mathbf{z}_i] = \mathbf{z}[\sigma_i, \phi_i]$ for $i = 3, 4, 5, 6$. Set

$$t = \tan \phi_1, \quad x = \tan \phi_2, \quad y = \tan \phi_3, \quad z = \tan \phi_4,$$

giving 4 equations in 4 unknowns:

$$f(x, y, z) = f(t, y, z) = f(t, x, z) = f(t, x, y) = 0.$$

If a, b, c, d are the elementary symmetric polynomials in t, x, y, z, we can convert this into a system

$$F_1(a, b, c, d) = F_2(a, b, c, d) = F_3(a, b, c, d) = F_4(a, b, c, d) = 0.$$

For example, $F_1 = f(x, y, z) + f(t, y, z) + f(t, x, z) + f(t, x, y)$.

A quotient ideal

Solutions for which $\phi_i = \pm\pi/6$ give rise to $\mathbb{L}_\phi$, and we may assume

$$G = 1 - 3a^2 + 6b + 9b^2 - 18ac - 27c^2 + 18d + 54bd + 81d^2$$

is non-zero. Consider $\mathscr{I} = \langle F_1, F_2, F_3, F_4 \rangle$, and compute

$$\mathscr{I} : \langle G \rangle = \{ r \in \mathbb{R}[a,b,c,d] \; : \; r\, G \in \mathscr{I} \},$$

by finding a Gröbner basis for $\langle uF_1, uF_2, uF_3, uF_4, (1-u)G \rangle$. With approriate orderings, its first element $(/G)$ is

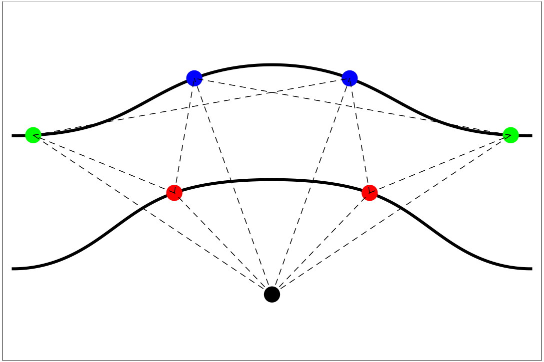
$$-(d-1)^3(3d-1)^3(3+b+3d)(9d-1)^3(1+3b+9d)^3(19+9b+27d).$$

Corollary. If $\mathbb{S}$ is not isometric to $\mathbb{L}_\phi$, one of the following holds:

$$d = 1,\ \tfrac{1}{3},\ \tfrac{1}{9}, \qquad b = -(3d+3),\ -\tfrac{1}{3}(9d+1),\ -\tfrac{1}{9}(27d+19).$$

Cross field passes

The case $9b + 27d + 19 = 0$ gives rise to a 1-parameter set of solutions to the system $F_1 = F_2 = F_3 = F_4 = 0$, such as $(a, b, c, d) = (0, \frac{22}{9}, c, \frac{1}{9})$. However, this gives rise to 9 points of which only 27 of the $\binom{9}{2}$ distances equal $2\pi/3$:



Conclusion

Theorem. Any SIC set in $\mathbb{CP}^2$ is isometric to $\mathbb{L}_\phi$ for some ϕ and therefore to the set

$$\boxed{\begin{array}{ccc} [0,1,-1] & [0,1,-\omega] & [0,1,-\omega^2] \\ [1,0,-1] & [1,0,-\omega] & [1,0,-\omega^2] \\ [e^{2i\phi},1,0] & [e^{2i\phi},\omega,0] & [e^{2i\phi},\omega^2,0] \end{array}}$$

we started with.

Classifying SIC-POVM's in higher dimensions appears beyond the scope of present methods. For $n = 4$, the analogue of $\mathscr{C}$ consists of two parallel circles, isolated points of which generate 64 SIC sets in $\mathbb{CP}^3$.

More references

M. Appleby: J. Math. Phys. 2005

M. Appleby, S. Flammia, G. McConnell, J. Yard: arXiv:1604.06098

S, T. Flammia: J. Phys. A, 2006

J. M. Renes et al: J. Math. Phys. 2004

A. J. Scott, M. Grassl: J. Math. Phys. 2010

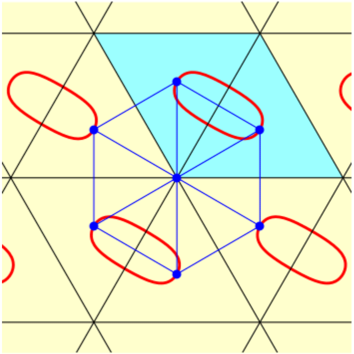
A. J. Scott: arXiv:1703.03993

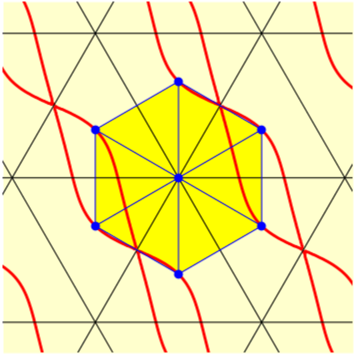
G. Zauner: J. Quant. Inf. 2011

H. Zhu: J. Phys. A 2010

Translation-invariant metrics

Here we see two different fibres $\mu^{-1}(p_i) = \mathbb{R}^2/\mathbb{Z}^2$, for $p_1, p_2 \in \mathscr{C}$.
The red curves are points of distance d from the origin:





The pinched torus $\mathscr{T}$

We may assume that a third (black) point of $\mathbb{S}$ is given by $\mathbf{z}_3 = \mathbf{z}[0, \phi]$, so $[\mathbf{z}_1], [\mathbf{z}_2], [\mathbf{z}_3]$ form an equilateral triangle in $\mathbb{CP}^2$. Most points the correct distance from $[\mathbf{z}_3]$ cannot belong to $\mathbb{S}$: