Quaternion-Kähler manifolds and Lie groups — a survey

$E\otimes H$

Simon Salamon
London – Oberwolfach
4 November 2013

Based on talks in 2010 at the conferences:
Symmetric spaces and their generalisations, Levico
String theory, wall crossing, and QK geometry, IHP
And more recent observations.

Contents

1. Quaternionic symmetric spaces

Classical compact ones of real dimension $4n$:

$$\mathbb{HP}^n = \frac{Sp(n+1)}{Sp(n)\times Sp(1)}$$

$$\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n)\times U(2))}$$

$$\mathbb{G}\mathrm{r}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n)\times SO(4)}.$$

Exceptional ones of real dimensions $8, 28, 40, 64, 112$:

$$\frac{G_2}{SO(4)}, \qquad \frac{F_4}{Sp(3)Sp(1)},$$

$$\frac{E_6}{SU(6)Sp(1)}, \qquad \frac{E_7}{Spin(12)\,Sp(1)}, \qquad \frac{E_8}{E_7\,Sp(1)}.$$

$\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$ and $\mathbb{G}\mathrm{r}_4(\mathbb{R}^6)$ are also Kähler manifolds. The others have $b_2 = 0$, and cannot admit an almost complex structure (Gauduchon-Moroianu-Semmelmann).

Wolf explained the series in 1965. Given a compact simple Lie algebra $\mathfrak{g}$, choose a highest root subalgebra $\mathfrak{su}(2)=\mathfrak{sp}(1)$. Then

$$H = K\,Sp(1) = \{g \in G : \mathrm{Ad}(g)(\mathfrak{su}(2)) = \mathfrak{su}(2)\}.$$

Moreover,

$$M = \frac{G}{K\,Sp(1)} = \frac{G}{H}$$

is symmetric and $G/KU(1)$ is a holomorphic contact manifold, the 'adjoint variety'.

If G is centreless, $K \subseteq Sp(n)_r$ and

$$H \subseteq Sp(n)_r \times_{\mathbb{Z}_2} Sp(1)_l = Sp(n)Sp(1) \subseteq SO(4n).$$

Definition. A *QK manifold* is a Riemannian manifold of dimension $4n$, with $n \geqslant 2$, whose holonomy group H equals $Sp(n)Sp(1)$ or a subgroup thereof.

The isotropy representations of these spaces have special merit, and crop up in different fields.

For each Wolf space $G/KSp(1)$, we get a symplectic representation $K \to \mathrm{End}(\mathbb{C}^{2n})$.

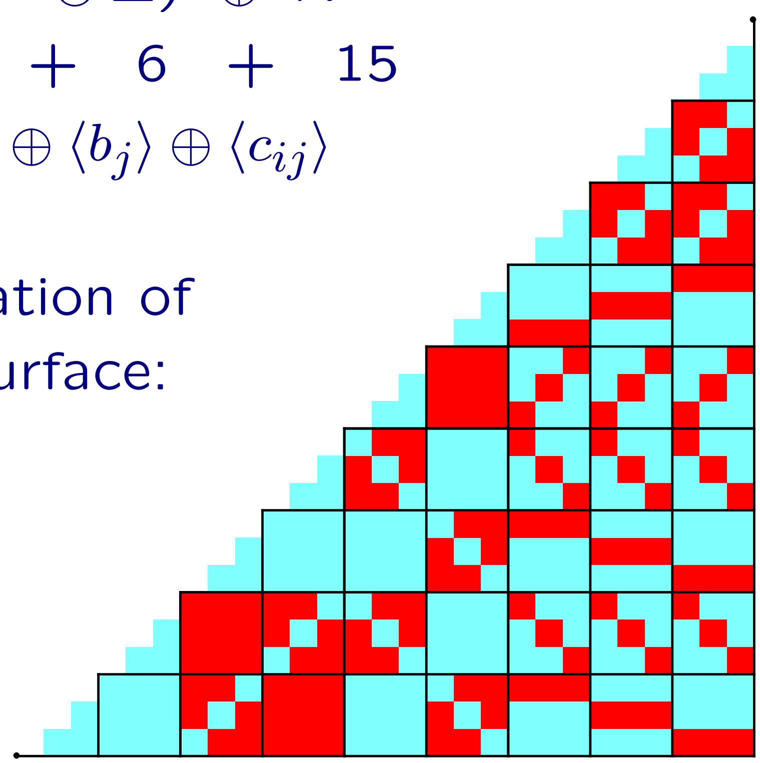
Example. Consider $\mathfrak{e}_6 = \mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus \mathfrak{m}$, where

$$\mathfrak{m}_c = \Lambda^{3,0} \otimes \Sigma = \mathbb{C}^{40}$$

is the tangent space and $\Sigma = \mathbb{C}^2$. But E_6 also acts on

$$\begin{aligned} \mathbb{C}^{27} &= (\Lambda^{1,0} \otimes \Sigma) \oplus \Lambda^{0,2} \\ &= \quad 6 \quad + \quad 6 \quad + \quad 15 \\ &= \langle a_i \rangle \oplus \langle b_j \rangle \oplus \langle c_{ij} \rangle \end{aligned}$$

giving Schläfli's configuration of the 27 lines on a cubic surface:



Theorems. (Alekseevsky 1968-70) All compact QK homogeneous spaces arise from Wolf's construction. There exist homogeneous non-symmetric QK spaces with $s<0$ (amplified by Cortés).

(i) QK does not imply Kähler!

(ii) If $H \subsetneq Sp(n)Sp(1)$ then M must be symmetric.

(iii) One normally excludes the HK case $H \subseteq Sp(n)$.

(iv) M should be *self-dual* and *Einstein* when $n=1$.

Any QK curvature tensor R belongs to

$$S^2(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \cong S^2\mathfrak{sp}(n) \oplus \mathfrak{sp}(n)\mathfrak{sp}(1) \oplus S^2\mathfrak{sp}(1).$$

Most summands are Bianchi-inconsistent, and

$$R = R_{\mathsf{HK}} \oplus sR_0, \quad R_{\mathsf{HK}} \in S^4E \subset S^2\mathfrak{sp}(n).$$

Corollary. M is necessarily Einstein. It is locally HK iff the scalar curvature s vanishes.

QK really means 'nearly HK' because of the analogy

Calabi-Yau	$\longleftrightarrow$	nearly-Kähler	(e.g. S^6)
hyperkähler	$\longleftrightarrow$	quaternion-Kähler	(e.g. $\mathbb{HP}^n$)

2. The twistor space

Let M be QK. Its complexified tangent space is

$$(T_m M)_c = E \otimes H, \qquad E = \mathbb{C}^n, \quad H = \mathbb{C}^2.$$

The reduction to $Sp(n)Sp(1)$ equips $T_m M$ with a 2-sphere

$$Z_m = \{aI + bJ + cK : a^2 + b^2 + c^2 = 1\}$$

of almost complex structures, where $IJ = K = -JI$. We have

$$\mathrm{End}(TM) \supset Z \cong \mathbb{P}(H).$$

Equivalently, Z is a subbundle of the rank 3 vector bundle with fibre

$$V_m = \{a\,\omega_1 + b\,\omega_2 + c\,\omega_3 : a, b, c \in \mathbb{R}\} \subset \Lambda^2 T^*_m M.$$

Theorem. The tautological almost complex structure on Z determined by the (Levi-Civita) horizotal distribution is integrable. So Z is a complex manifold (generalizing the AHS construction in dimension 4).

Twistor space exists over any *quaternionic manifold*, one with a $GL(n,\mathbb{H})Sp(1)$-structure and torsion-free connection (Bérard Bergery).

Over any quaternionic manifold, we can choose a *local* basis I,J,K with I integrable and $IJ=K=-JI$. This makes QK manifolds very close to being complex and (if $s>0$) Kähler.

There is the notion of *instanton* over a quaternionic manifold M^{4n}, namely a bundle (F,∇) with 'self-dual' curvature, which lifts holomorphically over Z.

Examples. If $F=\mathbb{H}$, removing the zero section,

$$\mathscr{U}=H^*/\mathbb{Z}_2 \quad \text{(fibre } \mathbb{RP}^3\times\mathbb{R}^+ \text{)}$$

has an $\mathbb{H}^*$-invariant hypercomplex structure (Swann).

If M is QK (in particular, Einstein) then E is an instanton, and $TM\cong E\otimes H$ is quaternionic, but not (?) itself QK.

A host of associated bundles can be constructed over a quaternionic manifold M:

$$
\begin{array}{ccccc}
\mathscr{S}^{4n+3} & \hookrightarrow & \mathscr{U}^{4n+4} & & \\
\downarrow & & & \searrow \mu & \\
Z^{4n+2} & & \hookrightarrow & & V^{4n+3} \\
& \searrow & & \swarrow & \\
& & M^{4n} & &
\end{array}
$$

Z is the twistor space with fibre $\mathbb{CP}^1 \cong S^2$.

V is the span of I, J, K, fibre $\mathbb{R}^3 = \mathfrak{sp}(1)$.

$\mathscr{U} = H/\mathbb{Z}_2$ is hyper-complex; it has both HK and QK metrics if M is QK with $s > 0$.

$\mathscr{S}$ has fibre $SO(3)$; it is 3-Sasakian if M is QK>0 and can be smooth even if M is an orbifold (Galicki).

3. Fano contact manifolds

When M^{4n} is a Wolf space, its twistor space

$$Z=\frac{G}{KU(1)}\stackrel{\pi}{\longrightarrow}\frac{G}{KSp(1)}=M.$$

is an adjoint orbit in $\mathfrak{g}$, polarized by a holomorphic line bundle L. Each fibre $\pi^{-1}(m)$ is a rational curve $\mathbb{CP}^1$ with normal bundle $2n\mathcal{O}(1)$ (whereas $L|_{\mathbb{CP}^1}\cong\mathcal{O}(2)$).

Wolf pointed out that Z has a holomorphic contact structure $\theta\in H^0(Z,\Omega^1(L))$, so

$$0\;\neq\;\theta\wedge(d\theta)^n\in H^0(Z,\mathcal{O}(\boldsymbol{\kappa}\otimes L^{n+1}),$$

and $\overline{\boldsymbol{\kappa}}\cong L^{n+1}$. There is a holomorphic short exact sequence

$$0\to D\to TZ\stackrel{\theta}{\longrightarrow}L\to 0$$

of vector bundles, in which D is horizontal. In fact, $D\cong L^{1/2}\otimes\pi^*E$.

Example. $\mathbb{CP}^{2n+1}(\to\mathbb{HP}^n)$ has $L=\mathcal{O}(2)$, but in general Z is Fano of index $n+1$.

The twistor dictionary in general:

M QK, $s \neq 0$	Z complex contact
point	vertical rational curve
complex structure	holomorphic section
Killing field X	$s \in H^0(Z, \mathcal{O}(L))$
Dirac operator	$\overline{\partial}$ on $\Lambda^{0,*} \otimes \mathcal{O}(-n)$
$s > 0$	Z Kähler-Einstein
$s > 0$, compact	Z contact Fano
minimal 2-sphere	contact rational curve
$b_2(M) + 1$	$= b_2(Z)$

Interpretation of solutions to linear field equations as elements of Čech cohomology is the essence of the Penrose programme.

Big questions.

(i) Is every contact Fano manifold homogeneous? (True under additional assumptions: Beauville et al.)

(ii) Is every positive QK manifold (meaning complete with $s > 0$) symmetric?

4. Twistor configurations

In the Penrose fibration

$$\begin{array}{ccc} \mathbb{CP}^3 & \supset & \mathbb{CP}^3 \setminus \mathbb{CP}^1 \to \mathbb{CP}^1 \\ \downarrow & & \downarrow \\ S^4 = \mathbb{HP}^1 & \supset & \mathbb{H} = \mathbb{R}^4, \end{array}$$

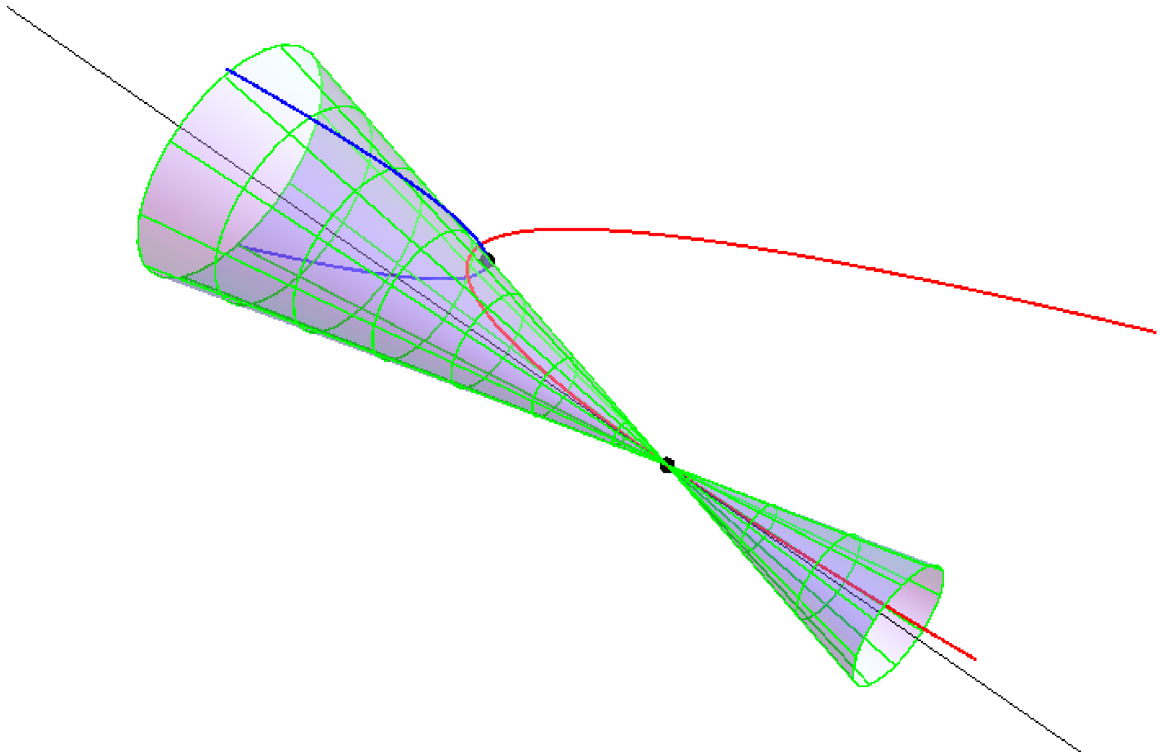
conformal geometry is encoded into holomorphic data invariant by j (the antipodal map on each fibre S^2). A holomorphic section over $U \subset \mathbb{H}$ is the same as an *orthogonal complex structure* on U.

Applications. (i) Any OCS over $\mathbb{R}^4 \setminus \{p_1, \dots, p_n\}$ is conformally constant.

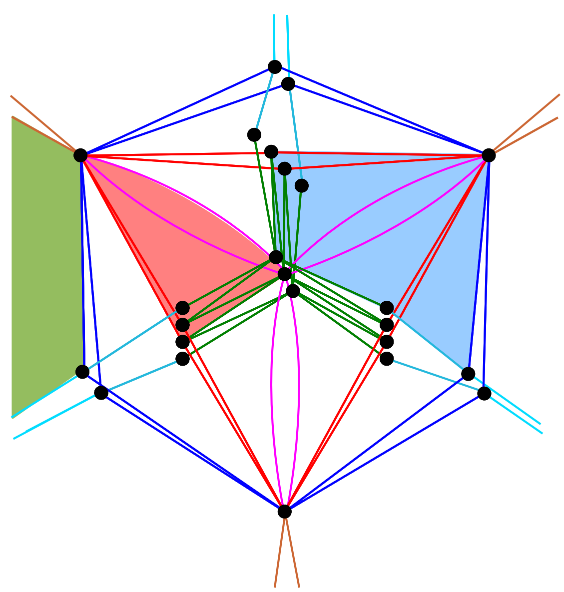
(ii) This is false for $\mathbb{R}^6 = \mathbb{R}^4 \times \mathbb{R}^2$, which inherits an OCS from $\mathbb{CP}^3$!

(iii) A smooth quadric in $\mathbb{CP}^3$ has at most 2 twistor lines, unless j-invariant. A smooth cubic surface in $\mathbb{CP}^3$ has at most 5 twistor lines (out of the 27).

Recent illustrations of the 4-dimensional theory:



The function $\mathbb{H} \ni q \mapsto q^2 + qi$ (Gentili-S-Stoppato)



Discriminants of cubic surfaces (Armstrong-S)

An example exploiting $\mathbb{CP}^7 \to \mathbb{HP}^3$ (Hoggar 1998).

Consider 3 finite groups acting projectively on $\mathbb{H}^4$:

- V_1, multiplication by $1, i, j, k \in Sp(1)_r$
- V_2, double sign changes of the coordinates
- V_3, double transpositions of the coordinates

The product

$$A = V_1 \times V_2 \times V_3$$

acts as $(\mathbb{Z}_2)^6$ on $\mathbb{CP}^7$. Fix unit quaternions

$$p = \tfrac{1}{2}(1 + i + j - k), \quad q = \tfrac{1}{2}(1 + i - j - k).$$

Proposition. The orbit $A \cdot [0, p, q, j]$ is a SIC-POVM: it consists of 64 points mutually equidistant in $\mathbb{CP}^7$ projecting to ? points in $\mathbb{HP}^3$.

Such SIC-POVM's of $(n+1)^2$ points are conjectured to exist in $\mathbb{CP}^n$ for all n (Zauner 1999).

5. QK reduction

For an $Sp(n)Sp(1)$-structure, the space of 2-forms is

$$(\Lambda^2 T_m^* M)_c \cong S^2E \oplus S^2H \oplus (\Lambda_0^2 E \otimes S^2 H),$$

where $V = S^2H = \mathfrak{sp}(1)$ is locally spanned by $\omega_1, \omega_2, \omega_3$. There is an invariant 4-form

$$\Omega = \sum_{r=1}^{3} \omega_r \wedge \omega_r.$$

Lemma. If $n \geqslant 3$ the condition $d\Omega = 0$ implies that $\nabla\Omega = 0$ and M is QK (Swann).

Locally, QK metrics (with $s > 0$, $s = 0$ or $s < 0$) can be constructed from the quotient construction.

Suppose that M^{4n} is a QK manifold with an isometric $U(1)$ action generating a Killing vector field X such that $\mathscr{L}_X \Omega \equiv 0$. Define a 2-form

$$\mu = \pi(dX^\flat) = \sum_{r=1}^{3} \mu_r \omega_r \in \Gamma(M, V).$$

Then

$$\begin{aligned} sX \lrcorner\, \Omega &= d\mu, \\ sX \lrcorner\, \mu &= df, \end{aligned}$$

where $f = \frac{1}{2}\|\mu\|^2$. The triple f, μ, Ω gives rise to an *equivariantly closed* 4-form

$$\Omega_X = fx^2 - \mu sx + \Omega s^2.$$

The 2-form μ determines a section

$$s_\mu \in H^0(Z, \mathcal{O}(L))$$

whose zero set consists of OCS's $\pm J_\mu$ on $M \setminus \{f{=}0\}$.

Theorem. If $U(1)$ acts freely then $f^{-1}(0)/U(1)$ has a natural QK structure (Galicki-Lawson).

This extends naturally to an isometric action by a Lie group G, and is a version of the HyperKähLeR quotient construction.

G2Q structures develop ideas from Atiyah-Witten's paper on M-theory.

Example. The diagonal action of $S^1 = U(1) \subset Sp(3)$ on $\mathbb{H}^3$ gives rise to an $SU(3)$-equivariant picture

$$\begin{array}{ccc} f^{-1}(0) = S^5 & \subset & \mathbb{HP}^2 \setminus \mathbb{CP}^2 \\ \downarrow & & \downarrow \\ \mathbb{CP}^2 & \longleftarrow & \Lambda^2_- T^*\mathbb{CP}^2 = X. \end{array}$$

The 7-dimensional space X admits a complete metric with holonomy G_2.

Theorem: Let M^8 be QK with a free S^1 action. Then

(i) $M \setminus \{f = 0\}$ has an explicit Kähler metric (Haydys).

(ii) $f^{-1}(c)/S^1$ has half-flat structures (GNS).

(iii) M/S^1 has a G_2-structure with $d\varphi \equiv 0$.

The 3-form φ is a modification of $X \lrcorner\, \Omega$, and gives G_2 holonomy when the gradients of f and $\|X\|$ are parallel.

6. Nilpotency

If M^{4n} is a QK manifold with an isometry group G of dimension ℓ, then

$$\mathfrak{g}_c \cong H^0(Z, \mathcal{O}(L)).$$

The morphism

$$\begin{array}{rcl} \Phi : Z & \dashrightarrow & \mathbb{P}(\mathfrak{g}_c^*) \\ z & \mapsto & [s_1(z), \ldots, s_\ell(z)], \end{array}$$

is a contact moment map for the action of G_c.

Dichotomy. If $\wp \in S^k\mathfrak{g}^*$ is G-invariant, either

(a) $\wp \notin \ker(S^k\mathfrak{g}_c^* \to H^0(Z, \mathcal{O}(L^k))^*)$, or

(b) $\Phi(Z)$ lies in the zero set of $\wp$.

In (a), the image of $\wp$ vanishes on k local sections of $Z \to M$ each of which determines a *G-invariant* OCS of type $aI + bJ + cK$. If these are absent, (b) asserts that $\Phi(Z)$ lies inside the *nilpotent variety* in $\mathbb{P}(\mathfrak{g}_c^*)$.

Any nilpotent orbit in $\mathfrak{g}_c^*$ arises from an $\mathfrak{su}(2)\subset\mathfrak{g}$:

$$\mathscr{U} = (\mathrm{Ad}\, G_c)(e) \subset \mathfrak{g}_c, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C}).$$

Such orbit admits a HK metric (Kronheimer), but only if $\mathscr{U}$ is minimal is $\mathscr{U}/\mathbb{C}^*$ compact. In this case, it is the twistor space Z that fibres over $M = G/K\,Sp(1)$.

The fundamental 3-form $\langle [X,Y],\, Z\rangle$ on a compact Lie algebra $\mathfrak{g}$ defines a function $f:\mathbb{G}\mathrm{r}_3(\mathfrak{g})\to\mathbb{R}$ for which

(i) $V\in\mathbb{G}\mathrm{r}_3(\mathfrak{g})$ is critical iff V is a subalgebra;

(ii) f is maximal on the Wolf space of minimal $\mathfrak{su}(2)$'s;

(iii) one can compute $\mathrm{Hess}(f)$ at any critical V.

Example. For $G=SU(3)$, we have:

$$\mathfrak{su}(2) \;\subset\; \mathfrak{su}(3), \qquad G(V) = \frac{\dot{S}U(3)}{U(2)} = \mathbb{CP}^2$$

$$\mathfrak{so}(3) \;\subset\; \mathfrak{su}(3), \qquad G(V) = \frac{\dot{S}U(3)}{SO(3)} = L^5.$$

In the second case, $\mathfrak{su}(3)_c \cong \Sigma^2\oplus\Sigma^4$, and

$$T_V\mathbb{G}\mathrm{r}_3(\mathfrak{g}) \;\cong\; V\otimes V^\perp \;\cong\; \Sigma^2\otimes\Sigma^4 \;\cong\; \underset{+}{\Sigma^2}\oplus\underset{0}{\Sigma^4}\oplus\underset{-}{\Sigma^6}$$

$$T_V \mathbb{G}\mathrm{r}_3(\mathfrak{g}) \;\cong\; V \otimes V^{\perp} \;\cong\; \Sigma^2 \otimes \Sigma^4 \;\cong\; \underset{+}{\Sigma^2} \oplus \underset{0}{\Sigma^4} \oplus \underset{-}{\Sigma^6}$$

The associated *unstable manifold* M^8 is the union of L^5 and the upward flow lines of the vector field grad f. It is diffeomorphic to a rank 3 vector bundle over L^5 with fibre Σ^2, and

$$T_c M^8 \cong \Sigma^2 \oplus \Sigma^4 \cong \Sigma^3 \otimes H.$$

In fact, M^8 is locally symmetric:

$$\frac{G_2}{SO(4)} \setminus \mathbb{CP}^2 \;\xrightarrow{3:1}\; M^8.$$

Theorem. For G compact simple, f is a Morse-Bott function on $\mathbb{G}\mathrm{r}_3(\mathfrak{g})$. The unstable manifold determined by an $\mathfrak{su}(2)$ is QK with twistor space $\mathbb{P}(\mathscr{U})$, where $\mathscr{U}$ is the associated nilpotent orbit.

Problem. Describe the resulting metrics explicitly.

7. Index theory

Let M^{4n} be a Wolf space or a positive QK manifold with a Lie group G of isometries. Its virtual $Spin(4n)$ representation is

$$\Delta_+ - \Delta_- = \Lambda_0^n(E-H) = \bigoplus_{p+q=n} (-1)^p R^{p,q},$$

where $R^{p,q} = \Lambda_0^p E \otimes S^q H$.

The coupled Dirac operator

$$\Gamma(M, \Delta_+ \otimes R^{p,q}) \longrightarrow \Gamma(M, \Delta_- \otimes R^{p,q})$$

has index $i^{p,q} = \int_M \mathrm{ch}(R^{p,q})\hat{A}(M)$. The following is a G*-equivariant statement*:

Theorem. $$(-1)^p i^{p,q} = \begin{cases} 0 & \text{if } p+q<n, \\ b_{2p-2}+b_{2p} & \text{if } p+q=n, \\ \dim G & p=0,\ q=n+2. \end{cases}$$

Thus $i^{p,n-p}$ arises from cohomology on which G acts trivially (an example of Witten rigidity).
Whilst $i^{0,n+2}$ arises from the adjoint representation.

Index theory gives a linear constraint on the Betti numbers and estimates on the isometry group, in terms of characteristic classes including $u \in H^4(M,\mathbb{Z})$ that represents Ω.

Example. If $\dim M = 8$ then $b_2 + 1 = b_4$. The map $S^2(H^2) \to H^4$ then restricts the value of b_2. Moreover

$$\dim G = 5 + \textstyle\int_M u^2.$$

If $b_4 = 1$ then

$$\dim G = \begin{cases} 5+16 & = \dim Sp(3), \\ 5+9 & = \dim G_2, \\ 5+4 & = \dim Sp(1)^3, \\ 5+1 & = \dim SO(4), \end{cases}$$

corresponding to

$$\mathbb{HP}^2 = \frac{Sp(3)}{Sp(2)\times Sp(1)}, \quad \frac{G_2}{SO(4)}, \quad \frac{\mathbb{HP}^2}{(\mathbb{Z}_2)^2}, \quad ?$$

Only the first two spaces are non-singular.

8. Towards a classification

Let M^{4n} be a compact positive QK manifold.

The odd Betti numbers b_{2p+1} of M^{4n} all vanish.

Theorem. If $b_2(M) > 0$ then M is isometric to the Wolf space $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$ (LeBrun-S, Wiśniewski).

If $b_2(Z) > 1$ there exists a family of rational curves on Z transverse to the fibres over M, and a Fano contraction

$$Z \longrightarrow \mathbb{CP}^{n+1}$$

with *its* fibres tangent to the contact distribution D. This in fact forces $Z = \mathbb{P}(T^*\mathbb{CP}^{n+1})$.

Dichotomy. Ignoring $\mathbb{HP}^n$, a QK manifold M is *spin* iff n is even.

Let M^{4n} be a compact positive QK manifold.

If n is even, M is spin and $\hat{A}(M)=0$ because $s>0$.

Theorem. A positive QK manifold M^8 is isometric to a Wolf space (Poon-S).

An attempt to push this to dimension 12 and 20 using elliptic genera needs the assumption $\hat{A}=0$ (Herrera, corrected by Amann-Dessai).

Theorem. If $b_4=1$ and $3\leqslant n\leqslant 6$ then $M\cong\mathbb{HP}^n$ (S, Amann).

All exceptional Wolf spaces have $b_4=1$, including $G_2/SO(4)$ $(n{=}2)$ and $F_4/Sp(3)Sp(1)$ $(n{=}7)$.

Question. Does a positive QK manifold M^{4n} necessarily admit an isometry group of positive dimension? Yes, at least if $n\leqslant 4$ (or if $n=5$ and $\hat{A}=0$). But, if n is odd, must the $\hat{A}$ genus vanish?

9. Poincaré polynomials

Given an oriented compact manifold M, consider

$$P(t) = 1 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots$$

and suppose $\chi = P(-1) \neq 0$. Then

$$\log P(-1+t) = \log \chi - d\,t + \phi\,t^2 + \cdots$$

where $d = \dim M$, and $16\phi = 4P''(-1)/\chi - d^2$. By construction, this coefficient is additive for products:

$$\phi(M \times N) = \phi(M) + \phi(N).$$

Theorems.

(i) If M^{4n} is compact HK then $\chi = 0$ or $\phi = -\frac{5}{6}n$.

(ii) If $M^d = G/H$ is an irreducible compact symmetric space of type ADE or any Hermitian symmetric space, $\phi = \frac{1}{12}(\mathrm{cox}(\mathfrak{g}) - 2)d$ (Fino-S).

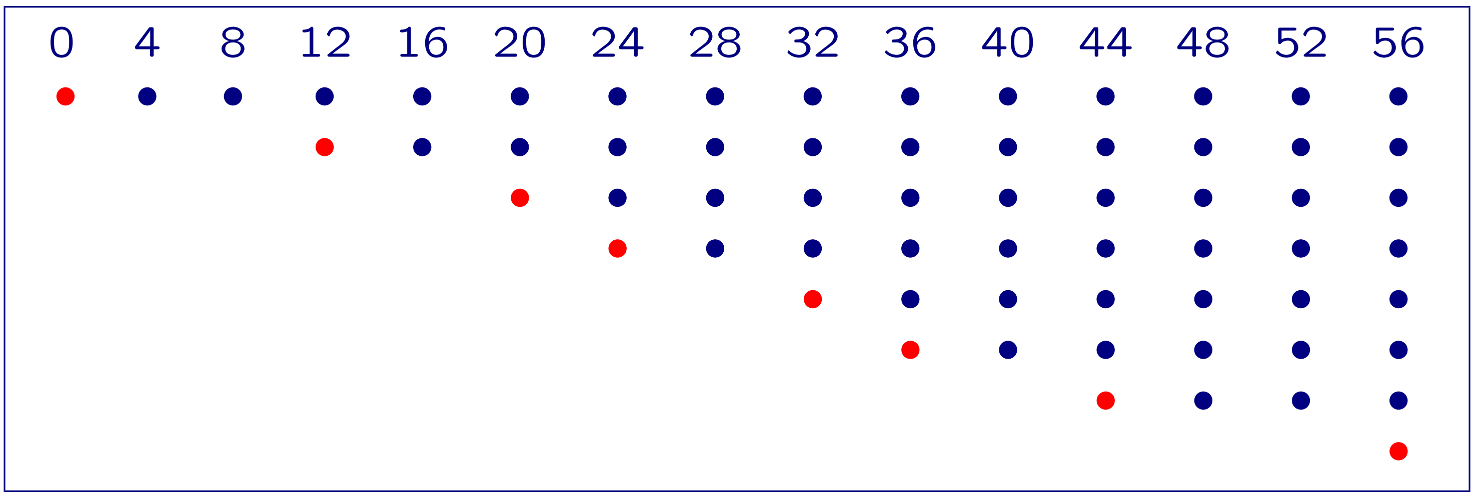
(iii) If M^{4n} is an ADE Wolf space then $\phi = \frac{1}{3}n^2$.

Example. The signature of an ADE Wolf space equals its rank: $b_{2n}^{+} = b_{2n} = r$, and χ equals the number $\frac{1}{2}(\dim G - r)$ of positive roots.

$E_8/E_7Sp(1)$ has 8 primitive coho classes $\sigma_k \in H^{4k}(M, \mathbb{R})$

$$H^{56}(M, \mathbb{R}) = \left\langle \sigma_k \cup u^{14-k} : k = 0, 3, 5, 6, 8, 9, 11, 14 \right\rangle,$$

exhibiting 'secondary Poincaré duality' about $k = 7$:



Question. Is the intersection form $S^2(H^{56}(M, \mathbb{Z})) \to \mathbb{Z}$ diagonalizable or the E_8 lattice? (Hirzebruch-Sladowy)

PS (Herrera, Weingart) The quaternionic volume is:

$$\begin{aligned} \int_M u^{28} &= 2^3.3^2.5^2.7.31.37.41.43.47.53 \\ &= \frac{5!\,9!\,57!}{19!23!29!} = 63468758442600 \end{aligned}$$