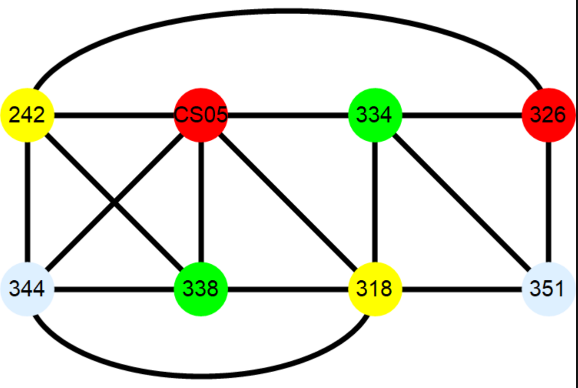
Module Selection

GEOMETRY



Simon Salamon, 6 March 2019

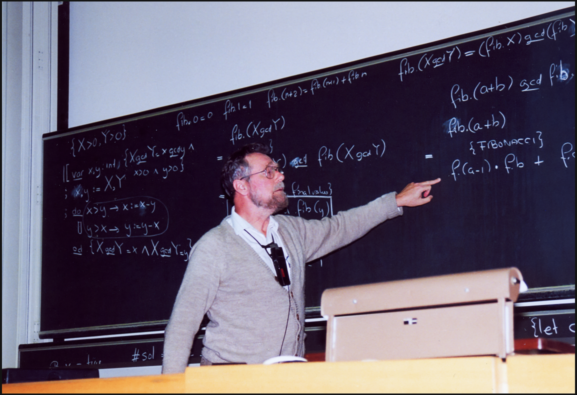
Options in Year 2

251A Discrete Mathematics

223A Geometry of Surfaces

232B Groups and Symmetries

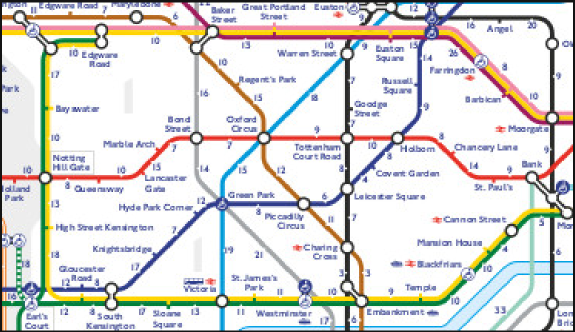
Discrete Mathematics



Dijkstra's shortest path algorithm is the basis of direction finding. Here he was talking about links between Euclid's algorithm and the Fibonacci numbers

				F_4				F_8				F_{12}
0	1	1	2	3	5	8	13	21	34	55	89	144

A weighted graph



…but what exactly do the integers represent?

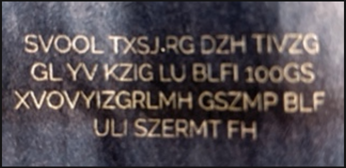
A knight's tour

7	42	9	24	5	40	19	22
10	25	6	41	20	23	4	39
43	8	55	34	51	38	21	18
26	11	58	37	56	33	52	3
59	44	35	54	63	50	17	32
12	27	62	57	36	53	2	49
45	60	29	14	47	64	31	16
28	13	46	61	30	15	48	1

illustrates a Hamiltonian cycle in a graph with 64 vertices

Codes and cryptography

The Queen's message to GCHQ uses a permutation $\pi \in S_{26}$



The RSA algorithm

$$p = 252097800623$$
$$q = 2760727302517$$
$$n = pq = 695973281084403272068091^{*}$$
$$\phi = (p-1)(q-1)$$
$$e = 2017^{*}$$
$$\gcd(e, \phi) = 1$$
$$d = 303992295802035192749689$$
$$ed = 1 \bmod \phi$$
$$\text{message} = 1234$$
$$\text{code} = 1234^{e} \bmod n$$
$$\text{decrypt} = \text{code}^{d} = 1234 \bmod n$$

* these numbers are public, but the prime factors p, q are secret

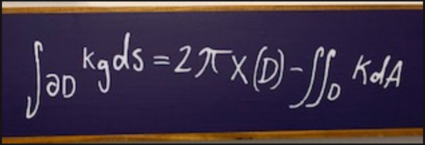
Geometry of Surfaces

taught at KCL by

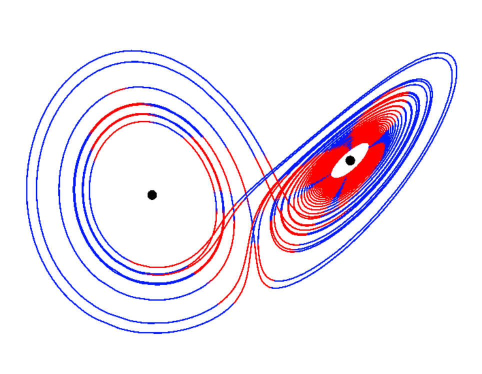
Andrew Pressley

Giuseppe Tinaglia

Jurgen Berndt

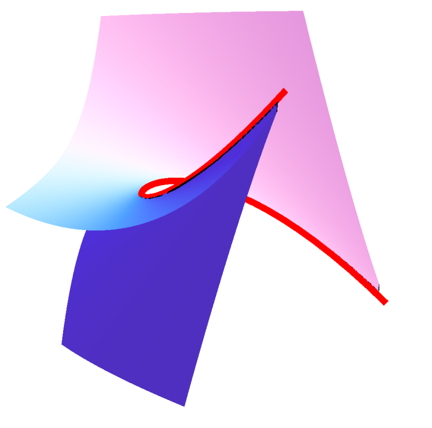


The torsion of a space curve



The torsion of this Lorenz trajectory (computed using the Serret-Frenet equations) flips between positive and negative values because each butterfly wing is roughly planar

Ruled surfaces



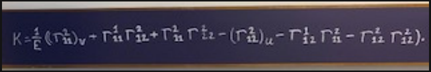
$$x^2y^2 - 4y^3 - 4x^3z + 18xyz - 27z^2 = 0$$

This discriminant surface has zero Gaussian curvature

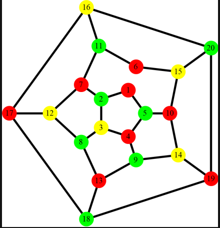
Gauss-Bonnet theorem

$$\int K\,dA = 2\pi\chi = 2\pi(2-2g)$$

for a compact surface in $\mathbb{R}^3$ with g "holes"

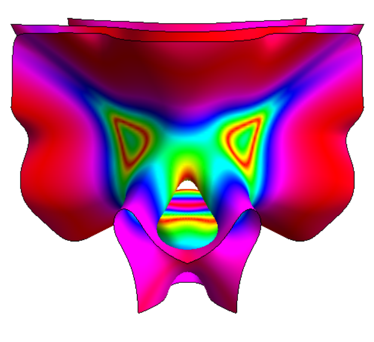


Example: $\chi(S^2) = v - e + f = 2$ and $K \equiv 1$



The ood surface

is not on the syllabus, but was constructed from Serret-Frenet



Geometry options in Year 3

327A Topology

351A Representation of Finite Groups

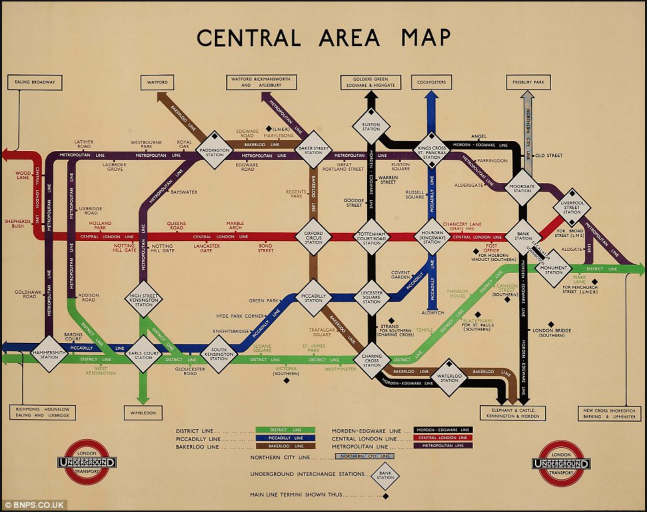
223B Geometry of Surfaces

251B Discrete Mathematics

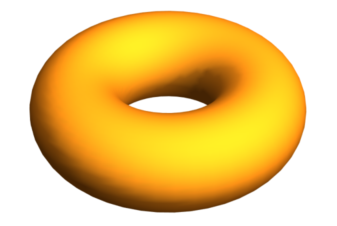
232B Groups and Symmetries

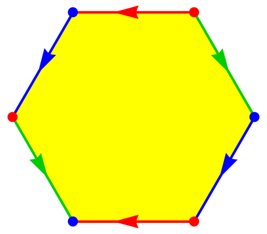
Topology

is geometry without distance!



The torus





Homeomorphism

Topological spaces are defined by specifying their open sets

A mapping $F\colon X \to Y$ is continuous if U open $\Rightarrow f^{-1}(U)$ open

A homeomorphism between X and Y is a bijection $f\colon X \to Y$ for which f and f^{-1} are both continuous. Then X and Y are "topologically the same"

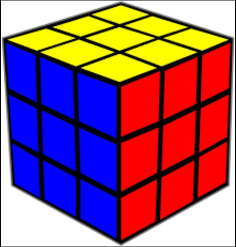
The usual torus (doughnut surface) is homeomorphic to a square (or hexagon) with opposite sides identified

The square is $\mathbb{R}^2/\mathbb{Z}^2$ and the hexagon is $\mathbb{C}/\left\langle 1, e^{2\pi i/3}\right\rangle$. Each has the structure of an abelian group, related to elliptic curves in Number Theory

Representations of Finite Groups

is all about understanding groups from how they act on objects

The symmetry group G of the cube is S_4 of size 24



What subgroup H fixes one vertex?

Each vertex can be identified with a coset of G and $8 = |G|/|H|$

Characters

One can model such actions by a representation, defined as a homomorphism

$$\rho\colon G \longrightarrow \mathrm{Aut}(V)$$

from G to the group of all linear transformations of a vector space V, possibly complex

Its character is the function $\chi : G \longrightarrow \mathbb{C}$ defined by

$$\chi(g) = \mathrm{trace}(\rho(g))$$

so that $\chi(a^{-1}g\,a) = \chi(g)$. Big links with conjugacy classes!

Geometry options in Year 4

MS01 Lie Groups and Lie Algebras

MS18 Manifolds

MS20 Algebraic Geometry

327B Topology

351B Representation of Finite Groups

Lie Groups and Lie Algebras

The group of rotations in $\mathbb{R}^3$ can be described by the group $SO(3)$ of orthogonal 3×3 matrics of determinant 1

This closely related to the group $SU(2)$ of unitary matrices of determinant 1, since there is an isomorphism

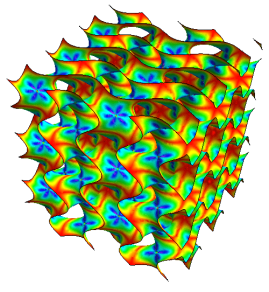
$$f\colon SU(2)/\{I, -I\} \longrightarrow SO(3)$$

Topologically, $SU(2)$ is the 3-dimensional sphere S^3 and $SO(3)$ is real projective space $\mathbb{RP}^3$

A Lie group is both a group and a manifold of a specific dimension (3 above). A Lie algebra is an algebraic [bracket] structure defined on each tangent space

Manifolds

All about studing complicated higher-dimensional "surfaces" using an atlas of coordinate patches

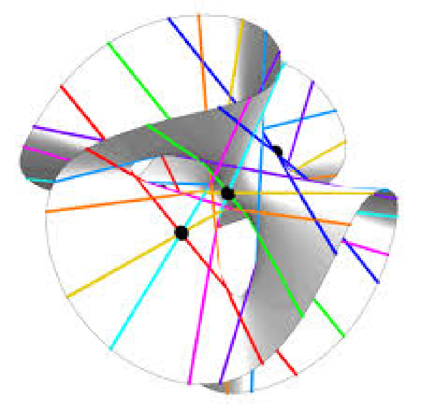


A key notion is that of Riemannian metric, generalizing the first fundamental form of a surface (and Pythagoras' theorem). Another is that of a connection, which allows one to define a directional derivative of a vector field on a manifold:

$$\nabla_j V^i = \frac{\partial V^i}{\partial x^j} + \sum_k \Gamma^i_{jk} V^k$$

Algebraic Geometry

Part of the course deals with intriguing “theories of lines”



27 of them on a cubic surface