Quotients of twistor space

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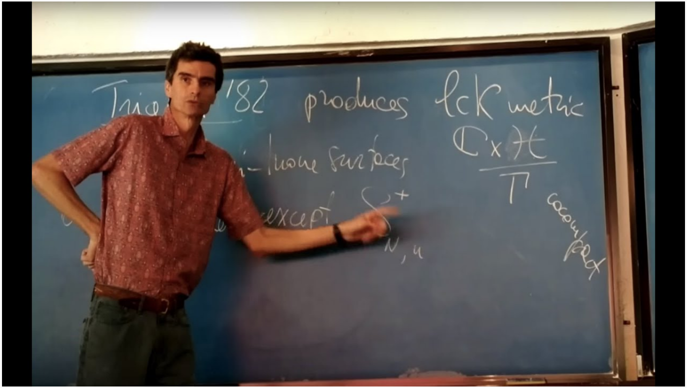
joint work with B. Acharya and R. Bryant



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Themes

Complex geometry

$\Downarrow$

Quadrics

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Ricci-flat metrics in dimensions 4 and 7

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Flat metrics in dimension 2

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Almost-Kähler geometry

Penrose's twistor theory

is based on choosing a real form of the Klein correspondence

$$\begin{array}{ccc} & \mathbb{F}^5 & \\ \swarrow & & \searrow \\ \mathbb{CP}^3 & & \mathbb{G}\mathrm{r}_2(\mathbb{C}^4) = Q^4 \subset \mathbb{CP}^5 \end{array}$$

defined by a reduction

$$SL(4,\mathbb{C}) \supset \begin{cases} SU(2,2) \simeq SO(4,2) & \text{acting on} & S^3 \times S^1 \supset \mathbb{R}^{3,1} \\ SL(2,\mathbb{H}) \simeq SO(5,1) & \text{"} & \mathbb{HP}^1 = S^4 \\ SL(4,\mathbb{R}) \simeq SO(3,3) & \text{"} & \mathbb{RP}^3 \end{cases}$$

Points in a 4-manifold determine projective lines in $\mathbb{CP}^3$.
Points in $\mathbb{CP}^3$ determine α-planes in the 4-quadric.

Hopf geometry

The projection

$$\begin{array}{c} \mathbb{CP}^3 \\ \pi \downarrow \\ \mathbb{HP}^1 = S^4 \end{array}$$

is the prototype fibration of a complex 3-fold over an ASD 4-manifold ($W_+ \equiv 0$) of the sort studied by Pontecorvo et al.

Each space above is a quotient of $\mathbb{H}^{2*} = \mathbb{R}^8 \setminus \{0\}$ (by $\mathbb{C}^*$ or $\mathbb{H}^*$ resp. acting on the right). We'll consider a further quotient by

$$\begin{array}{ccccc} \mathrm{U}(1) & \subset & \mathrm{U}(2) & \subset & \mathrm{Sp}(2) \\ \| & & & & \\ \mathrm{SO}(2) & \subset & \mathrm{SO}(2)\times\mathrm{SO}(3) & \subset & \mathrm{SO}(5), \end{array}$$

defining $$S^1_{\max} \sqcup S^2_{\mathrm{fix}} \quad \subset \quad \mathbb{R}^2 \times \mathbb{R}^3.$$

Real quadrics

An algebraic surface in

$$\mathbb{CP}^3 \quad (\xrightarrow{\pi} S^4)$$

defines an orthogonal complex structure on an open set of S^4 or $\mathbb{R}^4$ [dB-N]. In particular, the quadric

$$Q = \{[z_0, z_1, z_2, z_3] : z_0 z_1 + z_2 z_3 = 0\} \cong S^2 \times S^2$$

(a divisor in $\kappa^{-1/2}$) defines the conformally Kähler structure of

$$S^4 \setminus S^1 \;\cong\; \mathbb{R}^4 \setminus \mathbb{R} \;\cong\; S^2 \times \mathcal{H}.$$

It is invariant by U(2) and the involution j. By contrast,

$$Q' = \{[z_0, z_1, z_2, z_3] : z_0 \bar{z}_3 - z_1 \bar{z}_2 = 0\} \subset \mathbb{CP}^3$$

equals $\pi^{-1}(S^2_{\mathrm{fix}})$ and contains two 2-spheres of fixed points.

Gibbons-Hawking ansatz

$$\begin{array}{lcccc} \mathrm{U}(1)_{\mathrm{sx}} & \text{acts on} & \mathbb{C}^4_{0123} & \text{with weights} & (1,\ 1,\ 1,\ 1) \\ \mathrm{U}(1)_{\mathrm{dx}} & " & " & " & (1,-1,1,-1) \\ \Rightarrow\ T^2 & \text{acts on} & \mathbb{C}^2_{02}\times\mathbb{C}^2_{13} & \text{with weights} & (1,1)\times(1,1). \end{array}$$

Using hyperkähler moment maps (essentially $q \mapsto qi\bar{q}$),

$$\mathbb{R}^+ \times \frac{\mathbb{CP}^3}{\mathrm{U}(1)_{\mathrm{sx}}} = \frac{\mathbb{C}^{4*}}{T^2} \cong \frac{\mathbb{C}^2}{\mathrm{U}(1)} \times \frac{\mathbb{C}^2}{\mathrm{U}(1)} \cong \mathbb{R}^3 \times \mathbb{R}^3.$$

The quotient $\mathbb{R}^6$ has coordinates $(\mathbf{u}, \mathbf{v})$ where

$$u_1 = |z_0|^2 - |z_1|^2, \qquad u_2 - iu_3 = z_0\bar{z}_2,$$

similarly for $\mathbf{v} = (v_1, v_2, v_3)$. The ansatz enables one to recover the flat metrics on $\mathbb{C}^2$ from harmonic functions $1/u$ and $1/v$, where $u = |\mathbf{u}|$ and $v = |\mathbf{v}|$.

Double 3-space

We now have a projection

$$\rho\colon \mathbb{R}^+ \times \mathbb{CP}^3 \longrightarrow \mathbb{R}^6 \ni (\mathbf{u}, \mathbf{v})$$

with circle fibres collapsing along the axes $\mathbb{R}^3 \cup \mathbb{R}^3$. The residual subgroup SO(3) acts diagonally on the $\mathbf{u}$ and $\mathbf{v}$ subspaces.

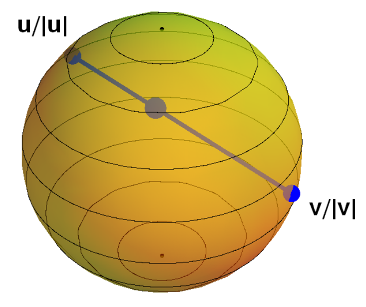
In fact, Q (resp. Q') is the the locus of points in $\mathbb{CP}^3$ for which the circles are horizontal (resp. vertical) w.r.t. the twistor projection $\pi\colon \mathbb{CP}^3 \to S^4$.

Proposition. The "quadrics" Q, Q' correspond to cases in which $\mathbf{u}, \mathbf{v}$ are parallel:

$$\begin{aligned} \mathbb{R}^+ \times Q &= \rho^{-1}\{(\mathbf{u}, \mathbf{v}) : v\mathbf{u} + u\mathbf{v} = 0\} \\ \mathbb{R}^+ \times Q' &= \rho^{-1}\{(\mathbf{u}, \mathbf{v}) : v\mathbf{u} - u\mathbf{v} = 0\} \end{aligned}$$

Reduced twistor fibration

By projecting $\mathbb{R}^5 \to \mathbb{R}^3$, we can identify $S^4/\mathrm{U}(1)_{\mathrm{sx}}$ with the closed unit disc $\overline{D}^3$ whose boundary is S^2_{fix}:



Proposition. The projection $\mathbb{R}^6 \to \overline{D}^3$ is $(\mathbf{u}, \mathbf{v}) \longmapsto \dfrac{\mathbf{u}+\mathbf{v}}{u+v}$.

Note that if $\mathbf{u}$ and $\mathbf{v}$ are aligned then $(\mathbf{u}, \mathbf{v})$ maps into S^2_{fix}.

Eguchi-Hanson metrics

The vector-valued function $\mathbf{u}-\mathbf{v}\colon \mathbb{C}^4 \longrightarrow \mathbb{R}^3$ can be viewed as a hyperkähler moment map for the action of $\mathrm{U}(1)_{\mathrm{dx}}$. Provided $\mathbf{m} \in \mathbb{R}^3$ is non-zero, the hyperkähler quotient

$$\frac{\{\mathbf{z} \in \mathbb{C}^4 : \mathbf{u} - \mathbf{v} = \mathbf{m}\}}{\mathrm{U}(1)_{\mathrm{dx}}} \subset \mathbb{R}^+ \times \mathbb{CP}^3$$

can be identified with $T^*\mathbb{CP}^1$ endowed with its Ricci-flat metric k with holonomy $\mathrm{SU}(2)$. It has a triholomorphic action by $\mathrm{U}(1)_{\mathrm{sx}}$ with moment map $\mathbf{u}$.

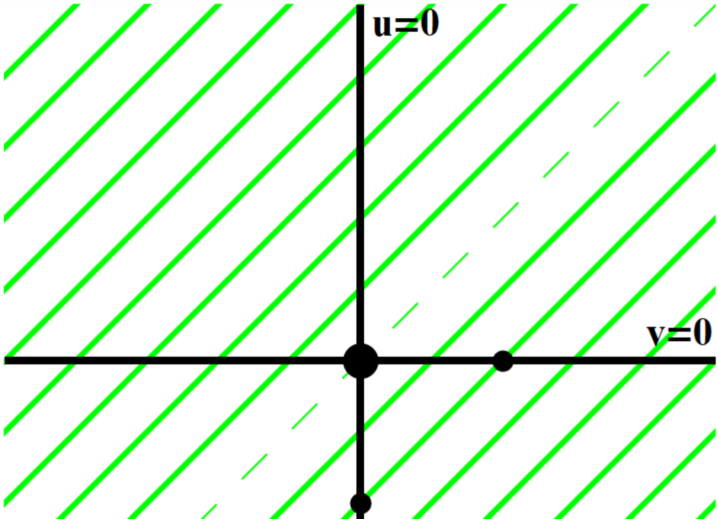
Its image in $\mathbb{R}^6$ is equipped with a harmonic function

$$V = \frac{1}{|\mathbf{u}|} + \frac{1}{|\mathbf{u} - \mathbf{m}|},$$

which can be used to recover $k = V^{-1}\Theta^2 + V g_{\mathrm{euc}}$.

Half-time recap

We have passed from $\mathbb{R}^8$ to $\mathbb{R}^+ \times \mathbb{CP}^3$ and then to $\mathbb{R}^6$. Each line represents an $\mathbb{R}^3$, the base $\mathbf{u} - \mathbf{v} = \mathbf{m}$ of an Eguchi-Hanson space if green, singularitities if black!



G_2 holonomy

If instead, $\mathbb{CP}^3$ is given its nearly-Kähler metric h then the conical metric

$$dr^2 + r^2 h = \frac{dR^2}{4R} + Rh$$

on $\mathbb{R}^+ \times \mathbb{CP}^3$ is Ricci flat with holonomy group G_2. Here $r = \|\mathbf{z}\|$ and $R = r^2 = u + v$.

Lemma. Pulled back to $\mathbb{R}^8$, the associated 3-form φ equals $d(R\tau)$, where

$$\tau = \alpha_1 \wedge dR + \alpha_2 \wedge \alpha_3$$

and the α_i are dual to Killing vectors defined by $SU(2)_{\mathrm{dx}}$.

If the cofactor of τ is replaced by $(1 + R^4)^{1/4}$ then the conical metric becomes a complete metric on $\Lambda^2_+ T^* S^4$ [BS 1989].

Recall the projection

$$\rho\colon \mathbb{R}^+ \times \mathbb{CP}^3 \longrightarrow \mathbb{R}^6 \ni (\mathbf{u}, \mathbf{v})$$

The G_2 structure on the cone will induce an SU(3) structure specified by tensors $(g, \mathrm{J}, \sigma, \Psi)$ on the singular $\mathbb{R}^6$, where σ is a symplectic form and Ψ a $(3,0)$-form [AS].

The G_2 metric is said to relate to Type IIA string theory of $\mathbb{R}^6$, which then acquires a Calabi-Yau metric in which the singular $\mathbb{R}^3$'s are special Lagrangian.

From this point of view,

"details of the induced metric are unimportant"

[Atiyah-Witten]. We decided nonetheless to describe g!

The induced metric

The formula for g simplifies on certain subvarieties of $\mathbb{R}^6$. Consider the negative quadrant

$$\mathscr{L}^2 = \{(\mathbf{u}, \mathbf{v}) = (0, 0, u; 0, 0, -v),\ u, v > 0\} \subset \mathbb{R}^2.$$

Proposition [ABS]. The restriction of g to $\mathscr{L}^2$ equals

$$\left(1 + \frac{v}{2u}\right) du^2 + du\, dv + \left(1 + \frac{u}{2v}\right) dv^2$$

and is flat (i.e. $K \equiv 0$), why?

Note that $\rho^{-1}(\mathscr{L}^2) \subset \mathbb{R}^+ \times Q$. In fact, $\mathscr{L}^2$ is the projection of $\mathbb{R}^+ \times$ a superminimal 2-sphere in $\mathbb{CP}^3$.

Slices

Extend $\mathscr{L}^2$ to

$$\mathscr{L}^3 = \{(0, u\sin\theta, u\cos\theta;\ 0,\ -v\sin\theta, v\cos\theta)\}.$$

Then $\mathbf{u}\cdot\mathbf{v} = uv\cos 2\theta$, so that $\angle(\mathbf{u},\mathbf{v}) = 2\theta$. Set

$$u = R\cos^2(\tfrac{1}{2}\phi), \quad v = R\cos^2(\tfrac{1}{2}\phi)$$

to ensure that $u + v = R$. The orbits of SO(3) on $\mathbb{R}^6$ are parametrized by u, v, θ, so $\mathscr{L}^3$ is a slice to the orbits.

Theorem. The restriction of g to $\mathscr{L}^3$ equals

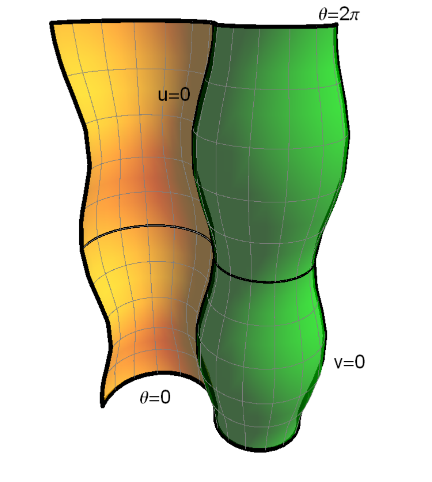
$$dR^2 + \tfrac{1}{2}R^2\big[d\theta^2 + \tfrac{1}{4}(3 - \cos 2\theta)d\phi^2\big].$$

The conical nature of the metric reflects that of $\mathbb{R}^+ \times \mathbb{CP}^3$, and it becomes flat if θ or ϕ is constant.

A surface of revolution

$\mathscr{L}^3$ (green) is adjoined to $P \cdot \mathscr{L}^3$ (yellow) with $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

Together these patches close up topologically to define a torus $\mathscr{T}$ and $\rho^{-1}(\mathscr{T})$ is a cone over $S^1 \times S^2$:



The symplectic form

Recall the projection

$$\rho\colon \mathbb{R}^+ \times \mathbb{CP}^3 \longrightarrow \mathbb{R}^6 \ni (\mathbf{u}, \mathbf{v}),$$

and that $\mathbf{u} - \mathbf{v} = \mathbf{m}$ is the base of an Eguchi-Hanson space.

Define $\sigma = X \lrcorner\, \varphi$ on $\mathbb{R}^6$, where X is the Killing field generating $\mathrm{U}(1)_{\mathrm{sx}}$. Then $d\sigma = 0$.

Theorem [ABS]. The vectors $\mathbf{p} = \mathbf{u} + \mathbf{v}$ and $\mathbf{q} = R(\mathbf{u} - \mathbf{v})$ determine Darboux coordinates on $\mathbb{R}^6$:

$$\sigma = -\tfrac{1}{2} \sum_{i=1}^{3} dp_i \wedge dq_i.$$

Note that this is non-degenerate on $\mathbb{R}^{6*}$. The projections $(\mathbf{u}, \mathbf{v}) \mapsto r\mathbf{u}$ and $(\mathbf{u}, \mathbf{v}) \mapsto r\mathbf{v}$ also have Lagrangian fibres.

Almost Kähler structure

Extend $\mathscr{L}^3$ to the linear subvariety

$$\mathscr{L}^4 = \{(0, u_2, u_3; 0, v_2, v_3),\ uv \neq 0\}.$$

Identifying the (non-integrable) almost complex structure J on $\mathbb{R}^6$ is much harder than finding σ or g, but we have

Theorem. $\mathscr{L}^4$ is a J-holomorphic subvariety w.r.t. the induced SU(3) structure.

By SO(3) invariance, there will be a family of such subvarieties parametrized by $\mathbf{n} \in S^2$ for which $\mathbf{u} \cdot \mathbf{n} = 0 = \mathbf{v} \cdot \mathbf{n}$. Any two will intersect in an integrable holomorphic curve, isomorphic to $\mathscr{L}^2$ (in which $\mathbf{e}_3$ has been replaced by $\mathbf{n}_1 \times \mathbf{n}_2$).

Final remarks

The quotient $\rho\colon \mathbb{R}^+ \times \mathbb{CP}^3 \to \mathbb{R}^6$ has some underlying geometry in common with the situation whereby a metric with holonomy G_2 can be found on the total space of a circle bundle over a 6-manifold with holonomy SU(3).

In an analytic setting studied by Foscolo-Haskins-Nordström, this relationship has proved very fruitful in the construction of many families of complete G_2 metrics on circle bundles over asymptotically conical CY spaces. Such G_2 families

- have controlled geometry of type ALC at infinity,
- collapse to CY spaces with bounded curvature.

There are also examples

- of cohomogeneity one under an action of $SU(2)^2 \times U(1)$,
- asymptotic to the cone over nearly-Kähler $(S^3 \times S^3)/\Gamma$.