Index theory and special geometries

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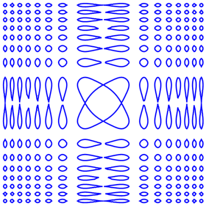

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Index theory and special geometries

for Oussama Hijazi, with admiration

Contents

We describe topological constraints derived from spinors and Dirac operators that can be applied to compact Ricci-flat manifolds and (with variations) to homogeneous and symmetric spaces.

Special geometries: dimension 7

$$\begin{array}{ccc} & & \mathrm{Spin}\,7 \quad \text{acts on} \quad \Delta = \mathbb{R}^8 \\ & \nearrow & \downarrow \\ \mathrm{G}_2 & \subset & \mathrm{SO}(7) \end{array}$$

G_2 is the stabilizer of $\delta \in \Delta$. So a G_2 structure on M^7 is defined by a nowhere zero section of the associated rank 8 vector bundle. Since $\Delta \otimes \Delta \cong \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3$,

$$(\delta \otimes \delta)_0 \in S^2_0\Delta \cong \Lambda^3 \cong \Lambda^4$$

defines stable forms $\varphi \in \Lambda^3$ and $\widehat{\varphi} \in \Lambda^4$.

Theorems. Any oriented spin 7-manifold M^7 admits a smooth G_2 structure with $d\widehat{\varphi} = 0$ [CN]. Such a structure extends to a metric with holonomy $\mathrm{Spin}\,7$ on $M \times (0, t)$ for some t.

Special geometries: dimension 8

$$
\begin{array}{ccccc}
G_2 & \subset & \boxed{\mathrm{Spin}\,7} & \subset & \mathrm{Spin}\,8 \\
 & & \cup & & \\
 & & \mathrm{SU}(4) & & \cup \\
 & & \cup & & \\
 & & \mathrm{Sp}(2) & \subset & \boxed{\mathrm{Sp}(2)\mathrm{Sp}(1)}
\end{array}
$$

$\mathrm{Sp}(2)$	fixes a HK triple $\omega_1, \omega_2, \omega_3$ at T_xM
$\mathrm{Sp}(2)\mathrm{Sp}(1)$	is the stabilizer of $\omega_1^2 + \omega_2^2 + \omega_3^2 = \Omega$
$\mathrm{Spin}\,7$	is the stabilizer of $-\omega_1^2 + \omega_2^2 + \omega_3^2 = \Phi$ [BH]

$d\Phi \equiv 0$ determines a holonomy reduction, but $d\Omega \equiv 0$ does not!

Are there compact 8-manifolds with $\pi_1 = 1$ and $d\Omega \equiv 0$, $\nabla\Omega \neq 0$?

Special geometries: Euler number

Proposition [GG,CCV]. If M^8 (compact and oriented) has a $\mathrm{Spin}\,7$ or an $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure then $e_+ = 0$ where

$$16e_+ = p_1^2 - 4p_2 + 8\chi.$$

For an $\mathrm{SU}(4)$ structure, $TM_c = T^{1,0} \oplus T^{0,1}$ has total Chern class

$$\begin{aligned} 1 - p_1 + p_2 &= (1 + c_2 + c_3 + c_4)(1 + c_2 - c_3 + c_4) \\ &= 1 + 2c_2 + (c_2^2 + 2c_4) \end{aligned}$$

so $8\varepsilon = 8c_4 = 4p_2 - p_1^2$. The argument extends because $\mathrm{SU}(4)$, $\mathrm{Spin}\,7$, $\mathrm{Sp}(2)\mathrm{Sp}(1)$ share a maximal 3-torus.

Regarded as the boundary of a $\mathrm{Spin}\,7$ structure, a G_2 manifold can be assigned an invariant in $\mathbb{Z}/48\mathbb{Z}$ [Crowley-Nordström], and there are at least 24 deformation classes of G_2 structures on an M^7.

Special geometries: triality

$$\begin{array}{lll} \mathrm{Spin}\,8 & \text{acts on} & \Delta = \Delta_+ \oplus \Delta_- \\ \downarrow 2:1 & & \\ \mathrm{SO}(8) & \text{acts on} & T = \Lambda^1 \end{array}$$

Outer automorphisms of $\mathrm{Spin}\,8$ permute T, Δ_+, Δ_-.

Restricting to a maximal 4-torus,

$$\begin{array}{lll} \Lambda^1 & \text{has 8 weights} & \pm x_1, \pm x_2, \pm x_3, \pm x_4 \\ \Delta_+ \oplus \Delta_- & \text{has 16 weights} & \frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4). \end{array}$$

For a reduction to $\mathrm{Spin}\,7$ or $\mathrm{Sp}(2)\mathrm{Sp}(1)$, $\sum x_i = 0$ and $\Lambda^1 \cong \Delta_-$.

By contrast, Δ_+ decomposes as $8 = 1 + 7$ or $3 + 5$, so $e_+ = 0$.

Index theory: Dirac operators

Given a vector bundle V with connection, the coupled Dirac operator

$$\Gamma(M, \Delta_+ \otimes V) \xrightarrow{D_V} \Gamma(M, \Delta_- \otimes V),$$

has index

$$\mathrm{ind}(V) = \dim(\ker D) - \dim(\mathrm{coker}\, D) = \int_M \mathrm{ch}(V)\widehat{A}(M).$$

- $V = \mathbb{C}$ equates the index of D with $\widehat{A}_2 = \frac{1}{5760}(7p_1^2 - 4p_2)$.
- $V = \Delta_+ - \Delta_-$ gives the 2-step de Rham complex, and of course $\mathrm{ind}(V) = \chi$.
- $V = \Delta_+ + \Delta_-$ gives rise to the signature operator, and $\mathrm{ind}(V) = b_4^+ - b_4^- = \frac{1}{45}(7p_2 - p_1^2)$.

Index theory: defining $\widehat{A}$

The tangent bundle of an 8-manifold has Chern character

$$\mathrm{ch}(TM_c) = \sum_1^4 (e^{x_i} + e^{-x_i}) = 8 + 2\sum x_i^2 + \tfrac{1}{12}\sum x_i^4$$

where $p_1 = \sum x_i^2$. Similarly,

$$\mathrm{ch}(\Delta_+ - \Delta_-) = \prod_1^4 (e^{x_i/2} - e^{-x_i/2}) = \varepsilon\,\widehat{A}(M)^{-1},$$

where $\varepsilon = x_1 x_2 x_3 x_4$ is the Euler class and

$$\widehat{A}(M) = \prod_1^4 \frac{x_i/2}{\sinh(x_i/2)} = 1 - \tfrac{1}{24}p_1 + \tfrac{1}{5760}(7p_1^2 - 4p_2) + \cdots$$

Index theory: Betti numbers

Corollary. With a reduction to $\mathrm{Spin}\,7$ or $\mathrm{Sp}(2)\mathrm{Sp}(1)$,

$$24\widehat{A} = \tfrac{1}{2}(3\tau - \chi) = -1 + b_1 - b_2 + b_3 + b^+ - 2b^-.$$

• If M is QK (holonomy $\subseteq \mathrm{Sp}(2)\mathrm{Sp}(1)$) with $R > 0$ then $\widehat{A} = 0$. Also $b_3 = b^- = 0$ so $b_4 = 1 + b_2$, and it is known that M must be one of $\mathbb{HP}^2$, $\mathrm{G}_2/\mathrm{SO}(4)$ or $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4) = \mathbb{G}\mathrm{r}_2(\mathbb{R}^6) = Q^4$.

• If M has holonomy equal to $\mathrm{Spin}\,7$ then $\widehat{A} = 1$, and $b_3 + b^+ = 25 + b_2 + 2b^-$. There are many examples [J,K].

• If M is irreducible HK (holonomy$=\mathrm{Sp}(2)$) then $\widehat{A} = 3$, and $b_3 + b_4 = 46 + 10b_2 \geqslant 76$. Only two (Beauville) examples are known and have $(b_2, b_3, b_4) = (23, 0, 276)$ and $(7, 8, 108)$ [G].

Quaternionic spinors: subbundles

By analogy to $\mathrm{Spin}\,8/\mathrm{Spin}\,7 \cong S^7$,

$$\frac{\mathrm{Spin}\,8}{\mathrm{Sp}(2)\mathrm{Sp}(1)} \cong \frac{\mathrm{SO}(8)}{\mathrm{SO}(5)\times\mathrm{SO}(3)} = \mathbb{G}\mathrm{r}_3(\mathbb{R}^8)$$

Indeed, given an $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure,

$$\begin{aligned} TM_c &\cong E\otimes H \cong \Delta_- \\ \Delta_+ &\cong \Lambda^2_0 E \oplus S^2 H \end{aligned}$$

where $S^2H \cong \langle I, J, K\rangle_c$, so $e_+ = 0$ (cf. [MS]).

A manifold with an $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure will automatically have a $\mathrm{Spin}\,7$ structure, including $\mathbb{HP}^2$ obtained by gluing a ball to a disk bundle in $H \to S^4$. Of course, $\mathbb{HP}^2$ can't have $\mathrm{Spin}\,7$ *holonomy*.

In a similar vein, $\dfrac{\mathrm{G}_2}{\mathrm{SO}(4)} \setminus \mathbb{CP}^2$ fibres over $\mathrm{SU}(3)/\mathrm{SO}(3)$.

Quaternionic spinors: rigidity

Proposition. Let M^8 be QK with $R>0$. Then

$$\mathrm{ind}(S^2H)=1, \quad \mathrm{ind}(TM)=-1-b_2, \quad \mathrm{ind}(\Lambda^2_0 E)=2b_2+1.$$

The modules are trivial representations of the isometry group G. On the other hand,

$$\mathrm{ind}(S^4H) = \dim G = 5+\beta^2 \ \geqslant \ 6,$$

where $\beta=-4c_2(H)\in H^4(M,\mathbb{Z})$ is represented by (a multiple of) Ω.

We want to show that M must be symmetric. Over $\mathbb{HP}^2$, H is the tautological line bundle and $\beta^2=16$. By Mori theory, $b_2>0$ implies $M\cong \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$. So we can assume $b_2=0$ and $b_4=1$. Then $\dim G = 21,\ 14,\ 9,\ 6,$ and the last two cases can be eliminated.

Quaternionic spinors: an instanton

Given an $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure, we have

$$\Delta_+ - \Delta_- = \Lambda^2_0 E - E \otimes H + S^2 H = \Lambda^2_0 (E - H).$$

Now, $E - H$ can't be a genuine vector bundle because:
• any monomorphism $H \to E$ would define a nowwhere zero section of $E \otimes H \cong T\mathbb{HP}^2$ but $\chi = 3$;
• $E - H$ has rank 2, but a calculation shows $c_4(E - H) \neq 0$.

By contrast,

$$c(\Lambda^2_0 E - H) = c(\Delta_+ - S^2 H - H) = 1 - 3\beta,$$

and $\Lambda^2_0 E - H$ *is* a genuine rank 3 vector bundle that pulls back to the Horrocks bundle over $\mathbb{CP}^5$. It belongs to a 27-dimensional moduli space $\mathrm{SL}(3, \mathbb{H})/\mathrm{SU}(3)$ fibring over $\mathrm{SL}(3, \mathbb{H})/\mathrm{Sp}(3)$.

Higher dimensions: another moduli space

Let $\mathscr{N}_g$ denore the moduli space of rank 2 stable bundles over Σ_g with fixed determinant of odd degree. It is a Kähler manifold of dimension $6g-6$ with $\overline{\kappa}=L^2$.

The Verlinde formulae determine the dimensions $h^0(\mathscr{N}_g,\mathscr{O}(L^k))$.

There is a twistor space embedding

$$\begin{array}{ccc} \mathscr{N} & \hookrightarrow & \dfrac{\mathrm{SO}(2g+2)}{\mathrm{U}(g-1)\times\mathrm{SO}(4)} \\ & & \downarrow \\ & & \mathbb{G}\mathrm{r}_4(\mathbb{R}^{2g+2}) \ \ \text{QK} \end{array}$$

which gives

$$T\mathscr{N}_g = Q\otimes W-\psi^2 Q,$$

where $\psi^2=S^2-\Lambda^2$ is an Adams operator [HS], leading to a quick proof of Newstead's conjecture $\beta^g=0$ with $p_1(\mathscr{N})=2(g-1)\beta$.

Higher dimensions: Poincaré polynomial

Let M be a (compact, oriented) manifold of real dimension $2n$, so

$$P(t) = \sum_{i=0}^{2n} b_i t^i$$

and $P(-1) = \chi$. Poincaré duality implies that $P'(-1) = -nP(-1)$.

$$\log \frac{P(-1+t)}{P(-1)} = \log\left(1 - nt + \frac{P''(-1)}{2P(-1)} + \cdots\right) = -nt + \phi\, t^2 + \cdots$$

where $\phi + \frac{1}{2}n^2 = \frac{P''(-1)}{2P(-1)}$. Thus, $\phi(M \times N) = \phi(M) + \phi(N)$, and

$$\mathscr{M}(\lambda) = \{M^{2n} \text{ oriented compact} : \phi(M) = \lambda n\}$$

is closed under products; it's also stable under products with S^2.

Higher dimensions: hyperkähler topology

Theorem [S]. Suppose that M^{4q} is hyperkähler, with $q \geqslant 1$. If $\chi \neq 0$ then $\phi(M) = -\frac{5}{6}q$ so $M \in \mathscr{M}(-\frac{5}{12})$. In all cases,

$$q\chi = 6\sum_{i=0}^{2q-1}(-1)^i(2q-i)^2 b_i.$$

Moreover, the 'odd' Betti numbers of M are all multiples of 4.

Corollary. $24 \,|\, (q\chi)$.

$q = 1 \;\Rightarrow\; 4b_1 + b_2 = 22$
$q = 2 \;\Rightarrow\; 25b_1 + b_3 + b_4 = 46 + 10b_2$
$q = 3 \;\Rightarrow\; 48b_1 + 16b_3 + b_6 = 70 + 30b_2 + 6b_4.$

Symmetric spaces: cosets

A compact Lie group G of rank r has Poincaré polynomial

$$\prod_{i=1}^{r}(1+t^{2m_i+1}),$$

and $\dim G = (h+1)r$ where $h = 2\sum m_i/r$ is the Coxeter number.

If H also has rank r then G/H has Poincaré polynomial

$$\frac{\prod(1-t^{2(m_i+1)})}{\prod(1-t^{2(\overline{m}_i+1)})}$$

and $\chi(G/H) = |W|/|\overline{W}| > 0$. Moreover,

$$\phi(G/H) = \tfrac{1}{6}\sum_{i=1}^{r}(m_i-\overline{m}_i)(m_i+\overline{m}_i-1) \geqslant 0.$$

Symmetric spaces: geography

Consider a compact irreducible inner Riemannian symmetric space $M^{2n} = G/H$. Then $\mathrm{rank}\, G = \mathrm{rank}\, H = r$. Let $h = \frac{1}{r}\dim G - 1$.

If the isometry group G acts effectively, then the isotropy subgroup H coincides with the holonomy group.

Observations [FS]. If either (i) G is of type ADE, or (ii) G/H is Hermitian symmetric then $\phi(M) = \frac{1}{6}(h-2)n$, so

$$M \in \mathscr{M}\left(\tfrac{1}{6}(h-2)\right).$$

If (iii) $M^{4q} = G/H$ is ADE QK then $h = 2q+2$ and $\phi(M) = \frac{1}{3}q^2$.

Symmetric spaces: examples

$$\begin{array}{ll} \mathbb{CP}^{2q+1} & G = \mathrm{SU}(2q+2), \quad h = 2q+2, \quad \phi = \tfrac{1}{3}q(2q+1) \\ \quad \downarrow S^2 & \\ \mathbb{HP}^q & G = \mathrm{Sp}(n+1), \quad h = 2q+2, \quad \phi = \text{ditto} \end{array}$$

$$\frac{\mathrm{E}_6}{\mathrm{SU}(6)\mathrm{SU}(2)}, \ \frac{\mathrm{E}_6}{\mathrm{Spin}\,(10)\mathrm{U}(1)} \quad \in \quad \mathscr{M}(\tfrac{5}{3})$$

$$\frac{\mathrm{E}_7}{\mathrm{SU}(8)}, \ \frac{\mathrm{E}_7}{\mathrm{Spin}\,(12)\mathrm{SU}(2)}, \ \frac{\mathrm{E}_7}{\mathrm{E}_6\,\mathrm{U}(1)} \quad \in \quad \mathscr{M}(\tfrac{8}{3})$$

$$\frac{\mathrm{E}_8}{\mathrm{Spin}\,(16)}, \ \frac{\mathrm{E}_8}{\mathrm{E}_7\,\mathrm{SU}(2)} \quad \in \quad \mathscr{M}(\tfrac{14}{3})$$

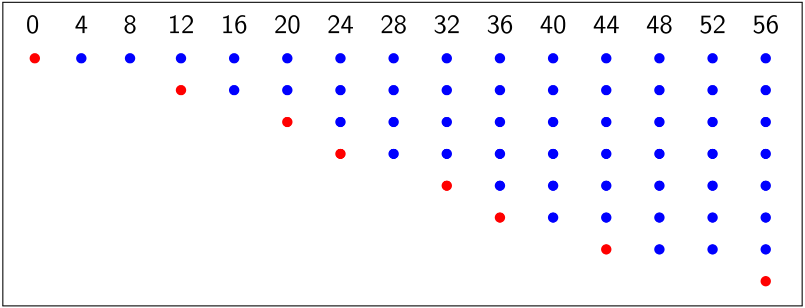
Although the definition of ϕ was motivated by holonomy, the flavour of symmetric spaces is determined by the *isometry* group!

Symmetric spaces: a strange duality

$\dfrac{E_8}{E_7 \times \mathrm{SU}(2)}$ has real dimension 112, and eight primitive cohomology classes $\sigma_k \in H^{4k}(M, \mathbb{R})$;

$$H^{56}(M, \mathbb{R}) = \left\langle \sigma_k \cup u^{14-k} : k = 0, 3, 5, 6, 8, 9, 11, 14 \right\rangle$$

The Betti numbers display symmetry about degree 28:



Higher dimensions: more rigidity

When an isometry group S^1 or G acts on M, the indices become virtual G modules. Which operators (like $D \otimes \Delta_+$) have indices that are sums of trivial modules? Define a sequence R'_i by

$$R'(q) = \sum_{i=0}^{\infty} q^k R'_k = \bigotimes_{i=1}^{\infty} \Lambda(q^{2i-1})/\Lambda(-q^{2i}).$$

$R'_0 = \mathbb{C}$ corresponds to the Dirac operator [AH], spin $\frac{1}{2}$
$R'_1 = TM = \Lambda^1$ arises from the Rarita-Schwinger operator, $\frac{3}{2}$
$R'_2 = \Lambda^2 + \Lambda^1$,
$R'_3 = \Lambda^3 + \Lambda^2 + \mathsf{S}^2 + \Lambda^1$,
$R'_4 = 2\Lambda^1 + \Lambda^2 + \Lambda^3 + \Lambda^4 + 2\mathsf{S}^2 + V_{\mathrm{sw}}$.

Theorem [W, BT]. If M^{2n} is a compact spin manifold then the index $\mathrm{ind}(R'_k)$ of $D \otimes R'_k$ is rigid for each k.

What are the best specializations for $G = E_*$ or $H = K\mathrm{SU}(2)$?

References

[BT] Bott and Taubes
[BH] Bryant and Harvey
[CCV] Čadek, Crabb and Vanžura
[CN] Crowley and Nordström
[FS] Fino and Salamon
[GG] Gray and Green
[G] Guan
[HS] Herrera and Salamon
[J] Joyce
[K] Kovalev
[MS] Moroianu and Semmelman
[S] Salamon
[W] Witten