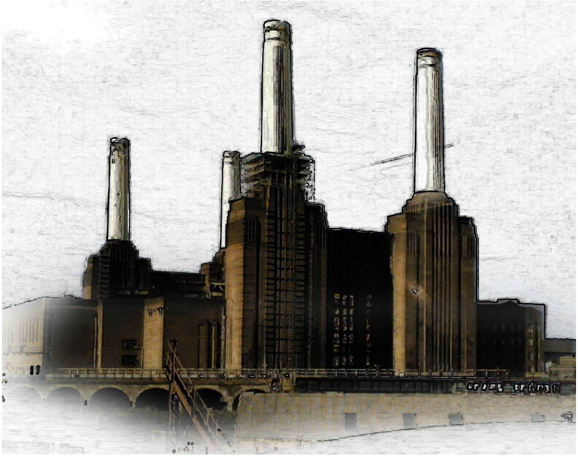
The Mathematical Skyline



Gresham College, London, 18 January 2017

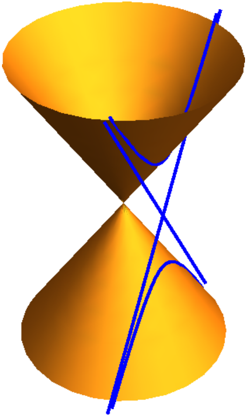
Bad timing

Dismantling the chimneys

Points and conics

4 points marked by the chimneys

© Matthew Train 2013



A hyperbola and its asymptotes

David Singmaster's question

"It appears to me that there will be some point where the chimneys will appear regularly spaced along the skyline ... where does one have to be to see this effect?" (M500–258.1)

Problem 258.1 – Battersea Power Station

David Singmaster

For those who don't know London, Battersea Power Station is a London landmark beside Chelsea Bridge and the Thames. This was (is?) the world's (or Europe's?) biggest brick building, using 61M bricks. Giles Gilbert Scott was asked to improve the architecture in 1931. Battersea A was started in 1929 and the first part, of 138MW, started work in 1933, with an additional 105MW in Sep 1935. Construction of Battersea B started in 1937. The first part, of 100MW, was in service in 1941. After the war, another 60MW was added and a final 100MW was added in 1953. The final result is a massive rectangular building with four massive chimneys, at the four corners of the rectangular building, which are visible from much of London. The chimneys are 337ft high. Battersea A closed in 1975. Battersea B closed on 31 Oct 1983. It was planned to be converted into a theme park by 1990, but the idea fell victim to a recession and the building remains half open to the elements. A friend recently described it as looking like a dead table.

The chimneys are basically at the corners of a rectangle, whose long sides run approximately north–south. The dimensions are approximately $50\text{m} \times 160\text{m}$. The problem arises because one sees the chimneys on the skyline as one drives into London from the west and one notices that the relative positions of the chimneys shift as one drives along the north side of the Thames. It appears to me that there will be some point where the chimneys will appear regularly spaced along the skyline. Is this true? If so, where does one have to be to see this effect? For consistency, let us label the four chimneys as A, B, C, D, going clockwise from A at the SW corner which we take as the origin of a coordinate system. So A is at $(0,0)$, B is at $(0,160)$, C at $(50,160)$, D at $(50,0)$. John Sharp has found a site which does architectural views and has sent the view below, but there are no dimensions given. I want to be able to go to a correct viewpoint and take a photo.

The question is well posed!

Its solution will involve surprisingly rich mathematics, including:

- results concerning *permutations* (re-orderings) of four objects;
- the so-called *cross-ratio*, a projective invariant;
- theorems on *conics*, curves defined by equations of degree 2;
- *asymptotes* of hyperbolae;
- examples of *cubic curves*, defined by equations of degree 3.

Thanks to John Armstrong and John Silvester for helping to uncover this.

Viñoly's masterplan

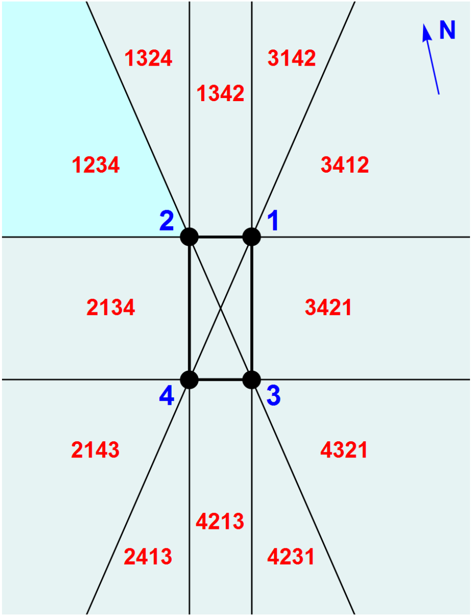
The boiler house is encircled by curved geometrical forms.

Designs by Gehry, Foster and partners

The equal-spacing problem involves:
photography and perspective,
lengths, not angles,
4 chimneys, not 3!

Permutations

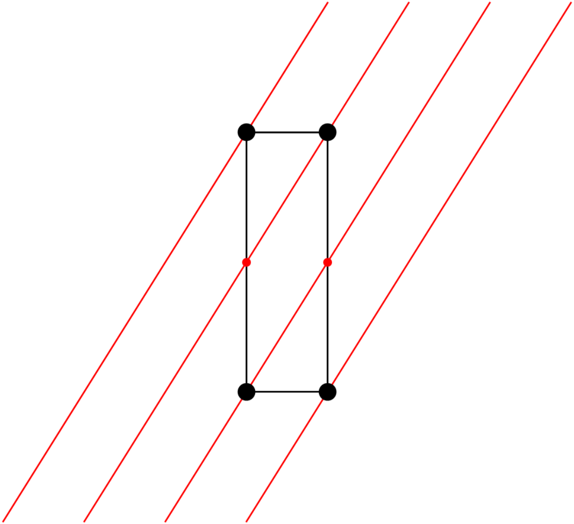
One can see the chimneys in different orders in various regions, but only 12 of the $4! = 24$ permutations are possible:



Parallel lines

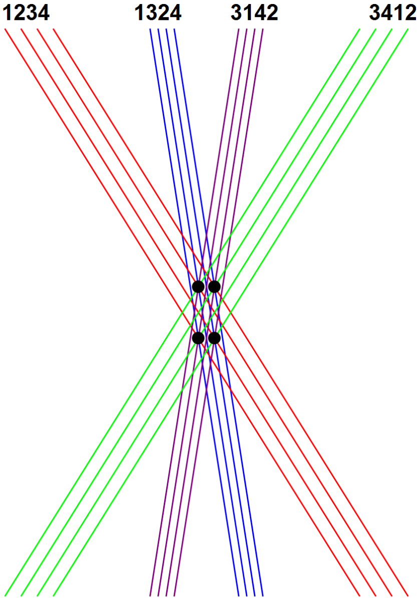
can be spaced equally between the chimneys using midpoints.

The rectangle measures 160m$\times$50m, and these lines have slope 8/5:



Parallel lines

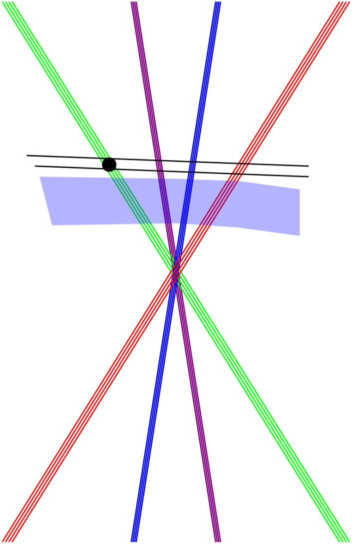
Four sets with different slopes can be drawn to realize 8 of the chimney permutations:



Solution at infinity

Ebury Bridge

Grosvenor Road



The cross-ratio

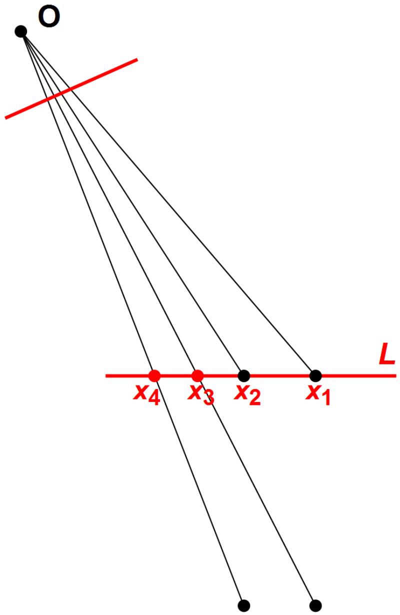
Draw a line from a point O to each chimney.

Take note of distances x_1, x_2, x_3, x_4 on any line L that cuts the other four.

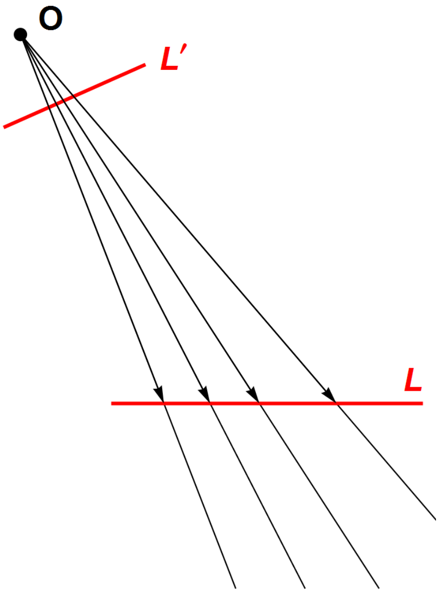
The quantity

$$r = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}$$

does not depend on the position of the 5th line L.



A perspectivity



Projection from O defines a mapping

$$f: \ L' \to L$$

that preserves the cross-ratio of 4 points.

Projective transformations of the plane

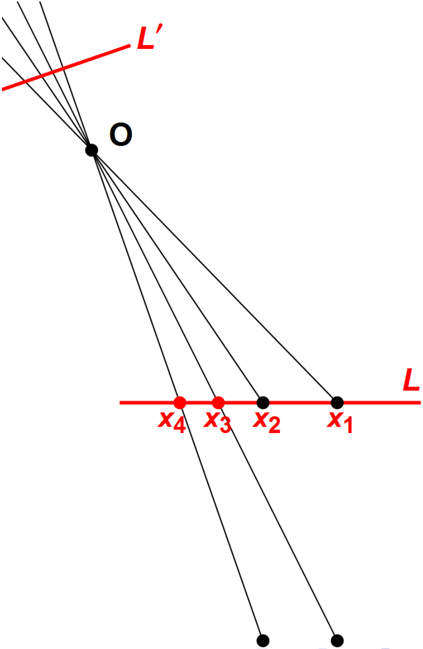
Equal spacing

If the 4 lines through O are equally spaced on some film L' then

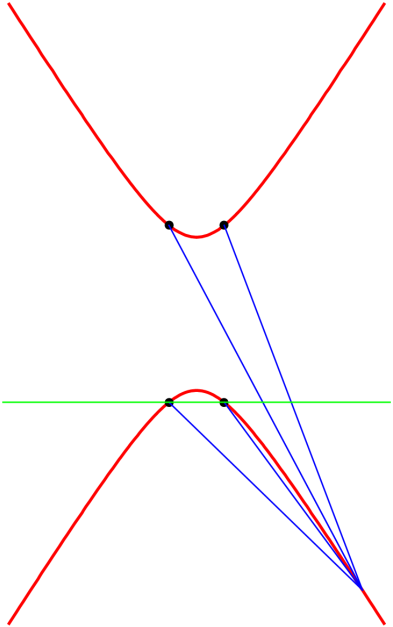
$$r = \frac{2 \times 2}{3 \times 1} = \frac{4}{3}.$$

The problem becomes:

Find the locus of those points O for which $r = \frac{4}{3}$.



This locus is a hyperbola $\mathscr{H}$



The equation of $\mathscr{H}$ is

$$25y^2 - 64x^2 = 120000$$

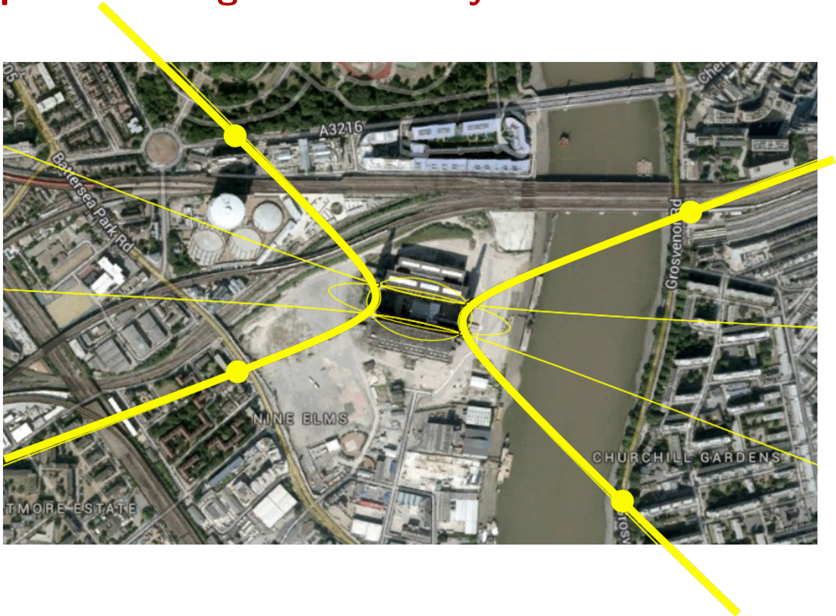
or

$$\left(\frac{y}{40\sqrt{3}}\right)^2 - \left(\frac{x}{25\sqrt{3}}\right)^2 = 1.$$

Its asymptotes are

$$5y \pm 8x = 0.$$

$\mathscr{H}$ passes through the chimneys



Points and conics

- *2 points lie on a unique line.*

- *3 points (if not collinear) lie on a unique circle.*

- *5 points (no 3 of which are collinear) lie on a unique conic, which is either an ellipse, parabola or hyperbola.*

Now let O be a "correct" viewpoint for the chimney order 1234. Then our "viewing hyperbola" $\mathscr{H}$ is the unique conic through

$$C_1,\ C_2,\ C_3,\ C_4,\ O.$$

The point O could be the one beyond Ebury Bridge at infinity.

Chasles' Theorem

Given 5 points C_1, C_2, C_3, C_4, O on a conic curve, the cross-ratio r of the lines

$$OC_1,\ OC_2,\ OC_3,\ OC_4$$

does not depend on O.

This explains why *any* point on $\mathscr{H}$ has cross-ratio $\frac{4}{3}$ relative to

$$C_1, C_2, C_3, C_4.$$

Michel Chasles 1793–1880
Hon. member LMS 1867

$$\vec{AC} = \vec{AB} + \vec{BC}$$

TRAITÉ

DES

SECTIONS CONIQUES,

FAISANT SUITE AU

TRAITÉ DE GÉOMÉTRIE SUPÉRIEURE,

PAR M. CHASLES,

Membre de l'Institut; Professeur de Géométrie supérieure à la Faculté des Sciences de Paris; Membre de la Société Royale de Londres; Associé étranger des Académies Royales des Sciences de Bruxelles et de Naples; Correspondant de l'Académie Impériale des Sciences de Saint-Pétersbourg; des Académies Royales des Sciences de Berlin, de Madrid et de Turin; de l'Académie Pontificale des *Nuovi Lincei* de Rome; de l'Académie des Sciences de l'Institut de Bologne.

PREMIÈRE PARTIE.

RBM

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE IMPÉRIALE POLYTECHNIQUE,

SUCCESSEUR DE MALLET-BACHELIER,

Quai des Augustins, 55.

1865.

(L'Auteur et l'Éditeur de cet Ouvrage se réservent le droit de traduction.)

NAPOLI

Traité des Sections Coniques

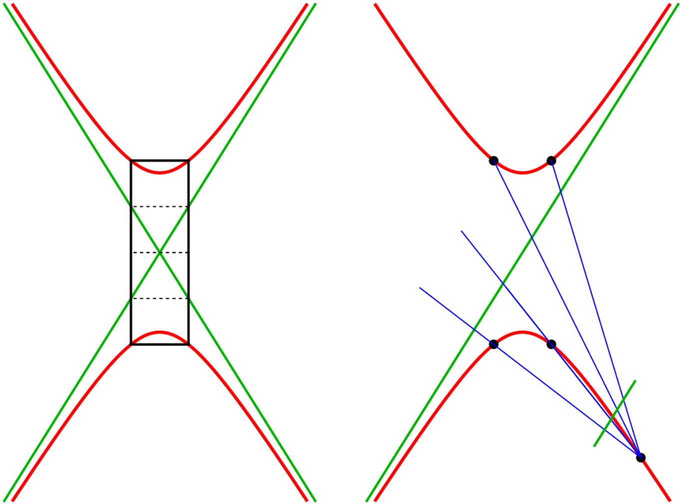
Extract from page 3:

4. PROPRIÉTÉ FONDAMENTALE RELATIVE AUX POINTS D'UNE CONIQUE. — *Si de quatre points d'une conique on mène des droites à un cinquième point de la courbe : le rapport anharmonique de ces droites a une valeur constante, quel que soit le cinquième point.*

En effet, les droites menées de quatre points *a*, *b*, *c*, *d* à un cinquième point P ont leur rapport anharmonique égal, d'après le théorème précédent, à celui des quatre points dans lesquels les quatre tangentes en *a*, *b*, *c*, *d* rencontrent une cinquième tangente. Donc ce rapport anharmonique est constant, quel que soit le cinquième point P.

"rapport anharmonique" = "cross-ratio"

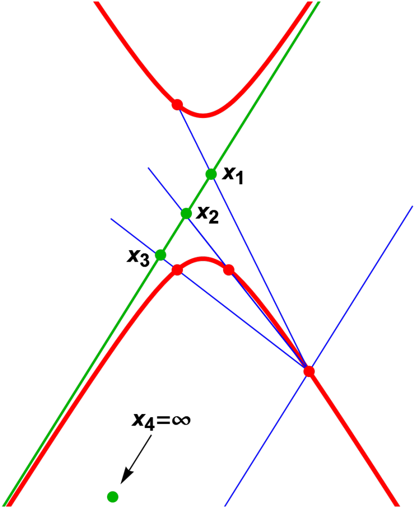
Where to position the camera?



Parallel to the other asymptote!

Relative to our hyperbola $\mathscr{H}$:

- the 4 four chimneys define a cross-ratio of $\frac{4}{3}$.
- any 3 chimneys and a point at ∞ define a cross-ratio of 2.



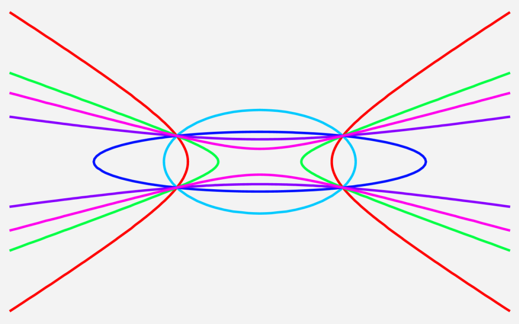
Chasles' theorem then implies that $x_1 - x_2 = x_2 - x_3$.

Permutating cross-ratios

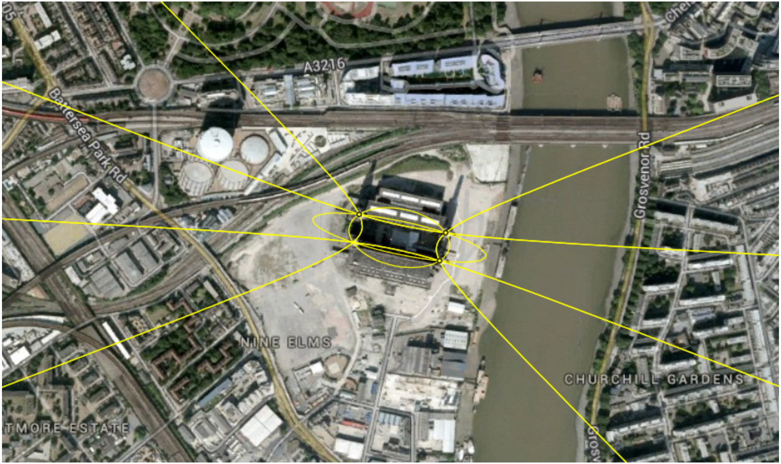
A re-ordering may or may not affect the cross-ratio r. Swapping two chimneys (like C_2 and C_4) does:

$$\frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)} = \frac{4}{3} \quad \Rightarrow \quad \frac{(x_1-x_3)(x_4-x_2)}{(x_1-x_2)(x_4-x_3)} = 4.$$

One achieves 6 different values: $r = \frac{4}{3},\ 4,\ \frac{3}{4},\ -3,\ \frac{1}{4},\ -\frac{1}{3}$.



Four meaningful conics



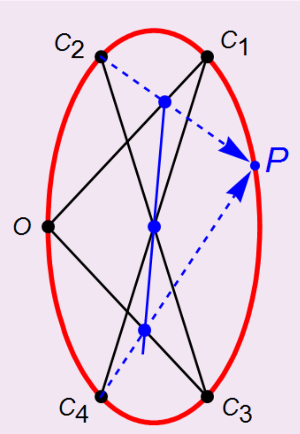
The hyperbolas correspond to $r = \frac{4}{3}, 4$, the ellipses to $r = \frac{3}{4}, -3$.

Pascal's Theorem

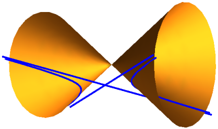
If a "hexagon" is inscribed in a conic, the points of intersection of opposite sides are collinear.

(Blaise Pascal 1623–1662)

This can be used to construct a generic point P on the conic through C_1, C_2, C_3, C_4, O.

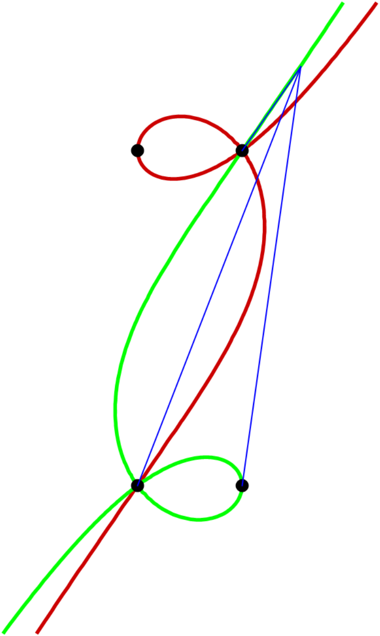


An ellipse over Battersea



Let $\mathscr{H}$ be the outer hyperbola passing through the chimneys. Consider all the double circular cones in space that contain $\mathscr{H}$. Their vertices trace out an ellipse that lies above the power station like Wembley's arch, but half the span and truly vertical!

Nodal cubic curves



Using gradients and the formula

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B},$$

the angle subtended by chimneys 1,4 and the angle subtended by chimneys 4,3 are equal if

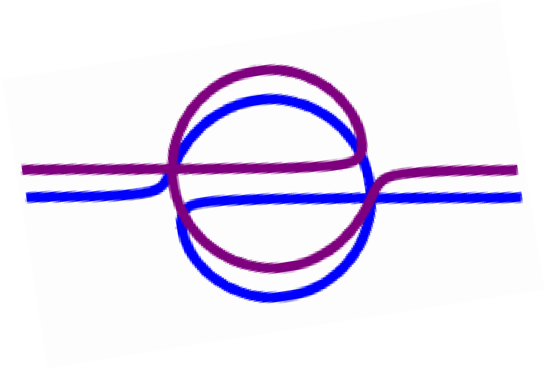
$$\begin{gathered}8x^3 + 200x^2 + 46200x + 3125y \\ -1280xy - 5x^2y + 600y^2 \\ +8xy^2 - 5y^3 = 1405000.\end{gathered}$$

Two designer cubics

represent equal pairs of angles

$$\angle 24 = \angle 41 \quad \text{and} \quad \angle 41 = \angle 13$$

between the chimney triples 241 and 413:



Pink Floyd's vision

Sir Elton John demonstrates Chasles' Theorem to a celebrity crowd at Battersea Power Station

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