Exponential pathways

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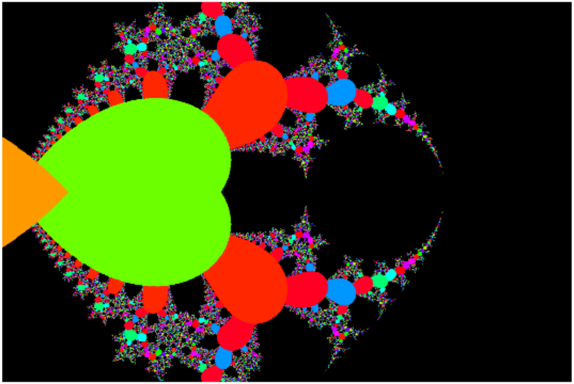


Dal lectio magistralis al Castello del Valentino, 4/12/2007
Turin to Dulwich

Contents

Three operations and three numbers

$m = \frac{1}{2} = 0.5$ satisfies $m + m = 1$

The mean m (of 0 and 1) is **rational**

$r = \sqrt{2}$ satisfies $r \times r = 2$

The root $r = 1.4142\ldots$ is **irrational** but **algebraic**

$s = \dagger 2$ satisfies $s^s = 2$

The super-root $s = 1.5596\ldots$ is **transcendental**

Roots and reciprocals

Let $s = \dagger 2$ and let $u = \dfrac{1}{s}$

$$\Rightarrow \quad u^s = \left(\frac{1}{s}\right)^s = \frac{1}{s^s} = m$$

$$\Rightarrow \quad u^{su} = (u^s)^u = m^u$$

$$\Rightarrow \quad u = m^u$$

How do we solve this equation for u?

Power towers

Consider the function $f(x) = m^x$. Start with $x = 1$ and keep applying f. For example, after 5 times, we get $m^{(m^{(m^{(m^m)})})}$

If we apply f infinitely often we get a quantity

$$u = m^{m^{m^{m^{m^{m^{m^{\nearrow}}}}}}}$$

that should satisfy $m^u = u$

Does it work?

Convergence

$$m^{\nearrow 2} = m^m = \frac{1}{r} = 0.707\ldots$$
$$m^{\nearrow 3} = m^{(m^m)} = 0.6125\ldots$$
$$m^{\nearrow 4} = m^{(m^{(m^m)})} = 0.6540\ldots$$
$$m^{\nearrow 5} = m^{(m^{(m^{(m^m)})})} = 0.63549\ldots$$
$$m^{\nearrow 6} = 0.64371864172286913\ldots$$
$$m^{\nearrow 8} = 0.64168580704299834\ldots$$
$$m^{\nearrow 10} = 0.64128450906658502\ldots$$
$$m^{\nearrow 20} = 0.64118577420003478\ldots$$

... to a limit

$$m^{\nearrow 30} = 0.6411857445139144 6\ldots$$

$$m^{\nearrow 40} = 0.64118574450498867\ldots$$

$$m^{\nearrow 50} = 0.64118574450498599\ldots$$

$$m^{\nearrow 60} = 0.64118574450498598\ldots$$

In fact,

$$m^{\nearrow \infty} = \lim_{n \to \infty} m^{\nearrow n} = 0.64118574450498598\ldots$$

exists, and

$$s = \frac{1}{u} = 1.5596104694623693\ldots$$

satisfies $s^s = 2$

Euler's theorem

The limit

$$x^{\nearrow\infty} = \lim_{n\to\infty} x^{\nearrow n}$$

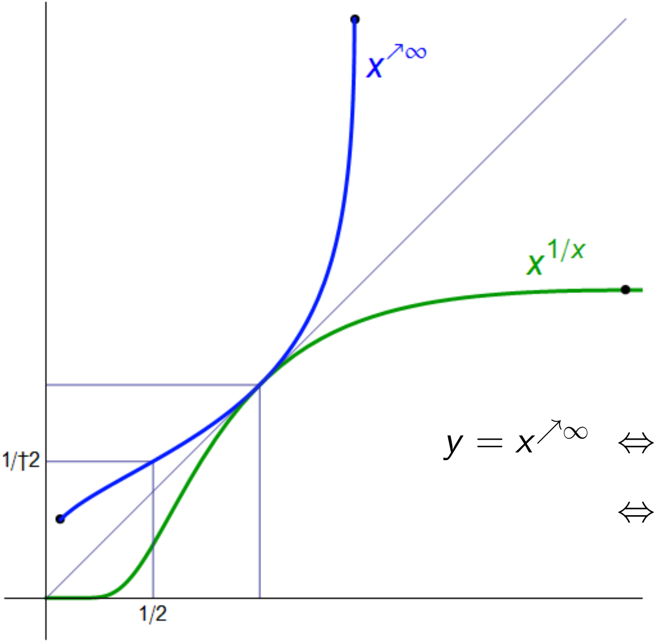
exists if and only if

$$1/e^e \leqslant x \leqslant e^{1/e}$$

which roughly means

$$0.065988 \leqslant x \leqslant 1.44466786$$

Inverse functions



$$y = x^{\nearrow\infty} \quad \Leftrightarrow \quad y = x^y$$

$$\Leftrightarrow \quad x = y^{1/y}$$

Trouble near zero

It is well known that $x^x \to 1$ as $x \to 0$

To see this, set $y = 1/x$. Then

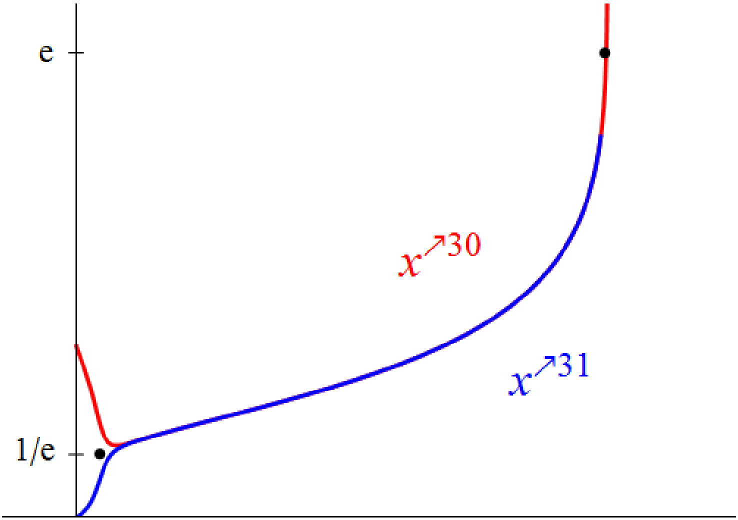
$$\log(x^x) = x \log x = \frac{-\log y}{y} \to 0 \text{ as } y \to \infty$$

It follows that $x^{(x^x)} \to 0^1 = 0$ as $x \to 0$

By induction, $x^{\nearrow n} \to 1$ as $x \to 0$ if n is even

$x^{\nearrow n} \to 0$ as $x \to 0$ if n is odd

The bifurcation

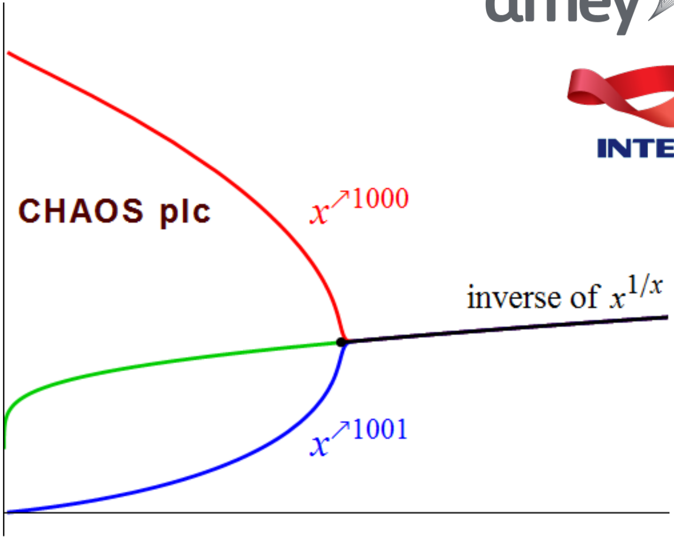


Trademarks

B5

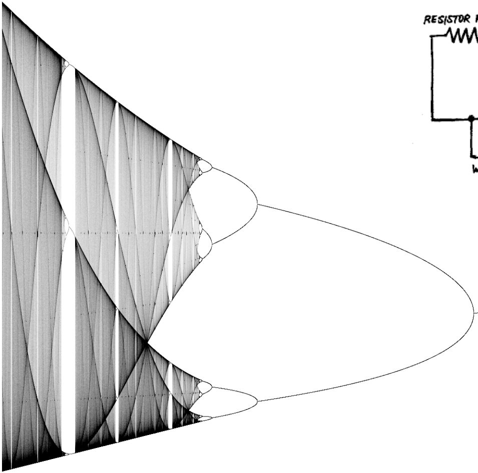
amey

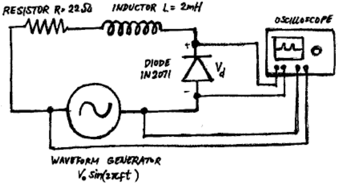
INTEGRAL



Universal onset of chaos

A picture like this is seen in many different physical situations, representing period doubling, tripling and ultimately chaos

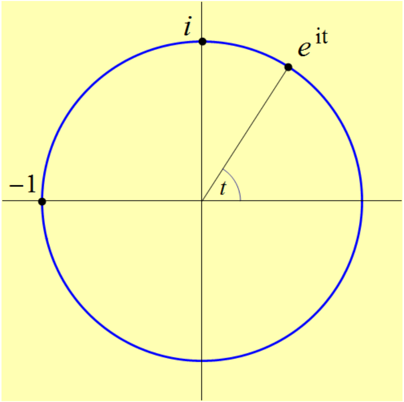




The Argand circle

To make sense of x^x for $x < 0$ one introduces $i = \sqrt{-1}$

This is an irrational number that satisfies $e^{it} = \cos t + i \sin t$

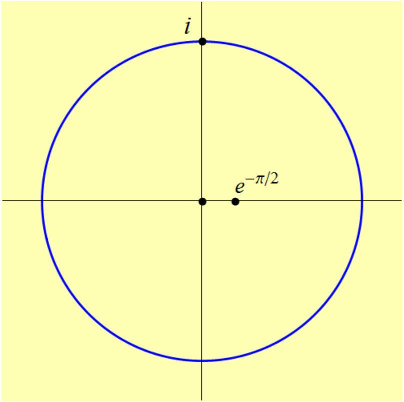


Complex exponents

One value of i^i is

$$(e^{i\pi/2})^i = e^{-\pi/2} = 0.20788\ldots$$

but there are infinitely others (all transcendental)



Exponential iteration

If $x = -t$ with $t > 0$ then

$$x^x = (t\,e^{i\pi})^{-t} = t^{-t}e^{-i\pi t} = t^{-t}(\cos \pi t - i \sin \pi t)$$

We graph $y = x^x$ by plotting in space the points

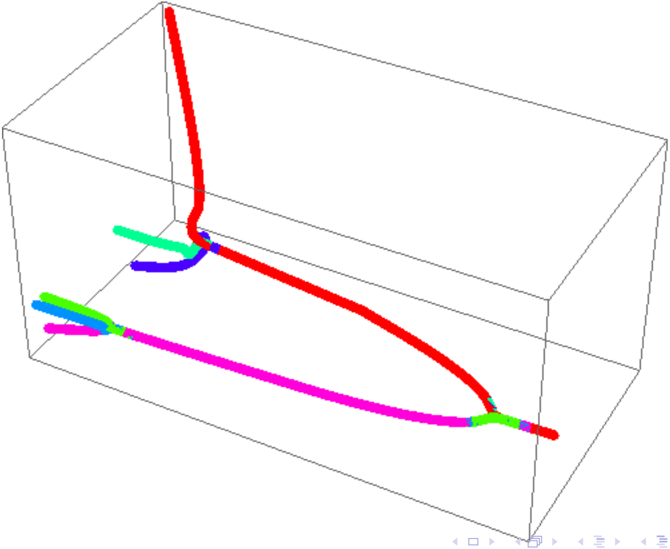
$$(-t,\ t^{-t}cos\,\pi t,\ -t^{-t}\sin \pi t)$$

One can continue by computing $x^{(x^x)}$, $x^{(x^{(x^x)})}$, ...

Two trifurcations

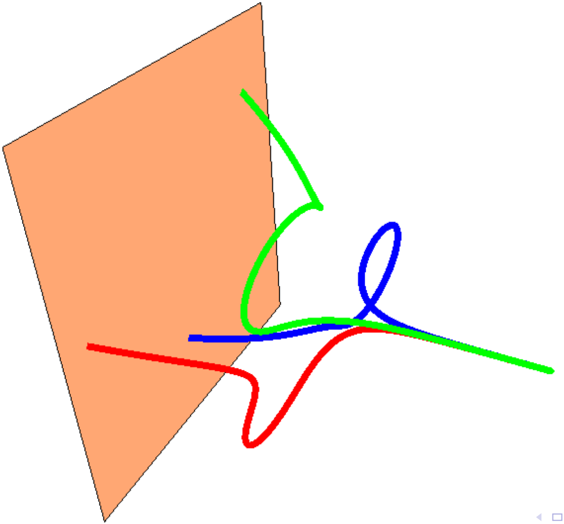
Revealed by the graph of $x^{\nearrow 400}$ for $-0.11 \leqslant x \leqslant 0.09$

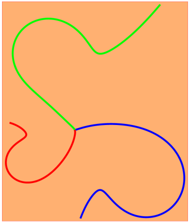
But what are the legs hiding?



Wild dynamics

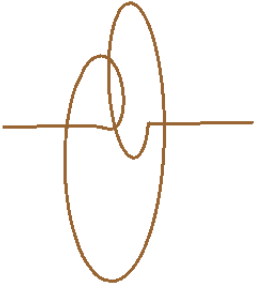
The graphs of $x^{\nearrow 480}$, $x^{\nearrow 482}$, $x^{\nearrow 484}$ between the plane $x = -0.0845$ and $x = -.075$





A slim watch spring

The graph of $x^{\nearrow 507}$ near $x = -0.08055$ forms a waveform of thickness 0.00002 but diameter 8000



A galactic spiral

The graph of $x^{\nearrow 508}$ near $x = -0.08055$ has diameter 10^{65}