Special metrics on 8-manifolds

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4-forms and symmetry

Let M be an n-dimensional oriented Riemannian manifold.

Throughout, we shall suppose that its structure group is reduced to some proper subgroup $G \subset SO(n)$, giving

$$\mathfrak{g} \subset \mathfrak{so}(n) \;\cong\; \Lambda^2 = \Lambda^2 T_x^* M.$$

Any invariant quadratic form on $\mathfrak{g} \cong \mathfrak{g}^*$ defines a 4-form Ω on M by wedging:

$$B \;\in\; S^2(\mathfrak{g}) \;\subset\; S^2(\Lambda^2) \;\stackrel{\mathfrak{S}}{\longrightarrow}\; \Lambda^4$$

If $\Omega = 0$ then B defines a curvature tensor, making $\mathfrak{g} \oplus \Lambda^1$ into a symmetric Lie algebra. Then there exists a symmetric space with holonomy group G.

Mostly n will be 8.

Linear equivalence of 4-forms

Can one classify 4-forms algebraically? Yes and no: this problem is tractable in dimensions up to 8.

Example for $n=7$. Consider $*\varphi$, where $\varphi\in\Lambda^3$ has stabilizer G_2. Both forms are 'stable': they have open orbits under $GL(7,\mathbb{R})$ $(49-14=35)$. The 3-form φ is determined by the 4-form and an orientation.

Any hypersurface of $\mathbb{R}^8$ admits such a 4-form which is closed, induced from the constant $Spin(7)$-invariant 4-form

$$\Omega_{-1}=*\varphi+\nu\wedge\varphi.$$

Moreover, any spin manifold M^7 has a coclosed G_2 structure [Crowley-Nordström]. Local existence of metrics on $M^7\times(0,\epsilon)$ with holonomy $Spin(7)$ can then be established by evolution.

4-forms in 8 dimensions

There are stable 3-forms in 8 dimensions. On $\mathfrak{su}(3) = \mathbb{R}^8$,

$$\gamma(x, y, z) = \langle\, [x, y], z\, \rangle$$

has an open orbit in Λ^3 $(64 - 8 = 56)$. This leads to the study of $PSU(3)$ structures for which $d\gamma = 0 = d *_\gamma \gamma$ [Hitchin, Witt].

There are no stable 4-forms. And the classification of 4-forms on $\mathbb{R}^8$ under $GL(8, \mathbb{R})$ is highly non-trivial. It can be done using roots of E_7 and the fact that the symmetric space

$$M^{70} = \frac{E_7}{SU(8)}$$

has holonomy (isotropy) representation $\Lambda^4(\mathbb{R}^8)$ [Antonyan].

Groups containing Sp(2)

On $\mathbb{R}^8 = \mathbb{H}^2 \ni (p, q)$, we can define a 'hyperkähler triple'

$$\tfrac{1}{2}(dp \wedge dp + dq \wedge dq) = \omega_1 i + \omega_2 j + \omega_3 k$$

of 2-forms

$$\begin{cases} \omega_1 = e^{12} + e^{34} + e^{56} + e^{78} \\ \omega_2 = e^{13} + e^{42} + e^{57} + e^{86} \\ \omega_3 = e^{14} + e^{23} + e^{58} + e^{67}. \end{cases}$$

Consider

$$\Omega_\lambda = \tfrac{1}{2}(\lambda\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3).$$

Generically the stabilizer of Ω_λ is $Sp(2)U(1) \subset SU(4)$, but:

- $\mathrm{stab}(\Omega_1) = Sp(2)Sp(1)$.
- $\mathrm{stab}(\Omega_{-1}) = Spin(7)$.

A conical metric with holonomy Spin(7)

There exist 'compatible' isotropy irreducible spaces

$$M^7 = \frac{SO(5)}{SO(3)} = \frac{Sp(2)}{SU(2)}, \qquad M^{11} = \frac{G_2}{SO(3)},$$

meaning that the isotropy representation of the first threads through the second:

$$SU(2) \to SO(3) \subset G_2 \longrightarrow \mathrm{Aut}(S^6(\mathbb{C}^2)).$$

Incredibly, there is only 'one' $SO(3)$-invariant 3-form

$$\Lambda^3(S^6(\mathbb{C}^2)) \;\cong\; \Sigma^{12} \oplus \Sigma^8 \oplus \Sigma^6 \oplus \Sigma^4 \oplus \Sigma^0.$$

It follows that the G_2 structure on M^7 satisfies $d\varphi = *\varphi$, and $r *\varphi + dr \wedge \varphi$ is a closed 4-form, defining a Ricci-flat metric with holonomy group $Spin(7)$ on $\mathbb{R}^+ \times M^7$.

Closed versus parallel

A holonomy reduction occurs when $\nabla\Omega = 0$.

For the Levi Civita connection, obviously $\nabla\Omega = 0 \;\Rightarrow\; d\Omega = 0$.

- If $\Omega = \Omega_{-1}$ has stabilizer $Spin(7)$ then $d\Omega = 0 \Rightarrow \nabla\Omega = 0$.
  For if $G = \mathrm{stab}(\Omega)$ then $\mathfrak{g}$ 'kills' Ω and only $\mathfrak{so}(n)/\mathfrak{g} \cong \mathfrak{g}^\perp$ acts effectively. For $G = Spin(7)$,

  $$\nabla\Omega \;\in\; \Lambda^1 \otimes \mathfrak{g}^\perp \cong \Lambda^3 \cong \Lambda^5 \quad (8 \times 7 = 56).$$

- If $\Omega = \Omega_1$ has stabilizer $Sp(2)Sp(1)$ then $d\Omega$ does not determine $\nabla\Omega$ [Swann]. If $\nabla\Omega = 0$ then M^8 is called 'quaternion-Kähler' and is Einstein. It is natural to generalize this class to allow Ω is closed (so harmonic) but not parallel.

Instrinic torsion for Sp(2)Sp(1)

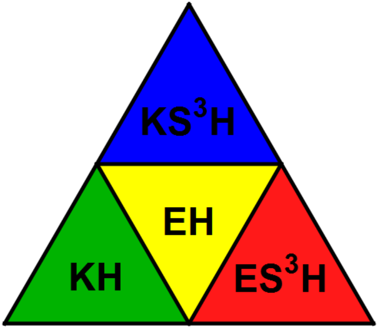
The space $\Lambda^1 \otimes (\mathfrak{sp}(2)+\mathfrak{sp}(1))^\perp$ has 4 components:

If $\nabla\Omega$ lies in…

blue then $\mathbf{d\Omega = 0}$

red then 'ideal':
$d\omega_i = \sum \alpha_i^j \wedge \omega_j$

green then quaternionic,
i.e. integrable twistor space



Corollary. (Ideal or quaternionic) and $d\Omega = 0 \;\Rightarrow\; \nabla\Omega = 0$

Quest for harmonic Sp(2)Sp(1) reductions

A first example was found on $M^8 = M^6 \times T^2$ where $M^6 = \Gamma\backslash N$ is a symplectic nilmanifold with a pair of simple closed 3-forms, giving a type of 'tri-Lagrangian geometry'. The structure group of M^6 reduces to a (diagonal) $SO(3)$.

There are many more examples of the form $M^7 \times S^1$ obtained by setting $\Omega_1 = \beta + e^8 \wedge \alpha$ and using the fact that

$$Sp(2)Sp(1) \cap SO(7) = G^*_{2\alpha} \cap G^*_{2\beta} = SO(4).$$

[Conti-Madsen classify 11 nilmanifolds and find solvmanifolds]. But are there simply-connected examples?

Theorem [CMS]. The parallel QK 4-form on $G_2/SO(4)$ can be 'freely' deformed to a 4-closed form with stabilizer $Sp(2)Sp(1)$ invariant by the cohomogenous-one action by $SU(3)$.

Symmetric spaces in 8 dimensions

Apart from $\mathbb{CP}^4$ (whose holonomy $U(4)$ is not so special), there are 4 compact models. They all admit a cohomogenous-one action, with principal orbits $SU(3)/U(1)_{1,-1}$ and two ends chosen from

$$S^5, \qquad \mathbb{CP}^2, \qquad \mathbb{L} = SU(3)/SO(3)$$

[Gambioli], and $\mathbb{L}$ parametrizes special Lagrangians in $\mathbb{R}^6$. The first 3 models are QK: their holonomy is contained in $Sp(2)Sp(1)$.

$\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$	$SU(4)/U(2)Sp(1)$	$\mathbb{CP}^2$, $\mathbb{CP}^2$
$\mathbb{HP}^2$	$Sp(3)/Sp(2)Sp(1)$	$\mathbb{CP}^2$, S^5
$G_2/SO(4)$	$G_2/SU(2)Sp(1)$	$\mathbb{CP}^2$, $\mathbb{L}$
$SU(3)$	$SU(3)^2/\Delta$	S^5, $\mathbb{L}$

Topological remarks

The spaces $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$, $\mathbb{HP}^2$, $G_2/SO(4)$ satisfy $\widehat{A}_2 = 0$ and

$$8\,\chi = 4p_2 - p_1^2.$$

The latter is also valid for any 8-manifold whose structure group reduces to $Spin(7)$, and the Wolf spaces all have such structures, but not holonomy equal to $Spin(7)$ as this would require $\widehat{A}_2 = 1$.

Quaternion-Kähler manifolds with $s > 0$ are 'nearly hyperkähler', and analogous to nearly-Kähler 6-manifolds. But in 8 dimensions they must be symmetric; in any case

$$\dim G = 5 + c^2,$$

precluding $SU(3)$ as an isometry group if $b_4 = 1$.

Nonetheless the approach of [Foscolo-Haskins] motivates the search for closed 4-forms, which look like being abundant.

Relevance of SU(3)

There are 2 non-conjugate TDA's in $\mathfrak{su}(3)$: block $\mathfrak{su}(2)$ and real $\mathfrak{so}(3)$. By general theory [Kronheimer, Swann], each gives rise to HK and QK metrics. For $\mathfrak{su}(2)$ one merely gets $\mathbb{CP}^2$. For $\mathfrak{so}(3)$, $\mathbb{L}$ is a critical manifold for a Nahm flow, resulting (mod $\mathbb{Z}_3$) in

$$\begin{array}{ccc} \dfrac{G_2}{SO(4)} \setminus \mathbb{CP}^2 & \cong & \mathbb{V} \\ \downarrow & & \downarrow \text{ obvious rank 3 VB with fibre } \Sigma^2 \\ SU(3)/SO(3) & = & \mathbb{L} \end{array}$$

Passing to an S^2 bundle over the 8-manifold, this twistor space is the projectivization $\mathscr{N}/\mathbb{C}^*$ of the principal nilpotent orbit

$$\mathscr{N} = \{A \in \mathfrak{sl}(3,\mathbb{C}) : A^3 = 0,\ A^2 \neq 0\}.$$

Quest for the 4-form on $\mathbb{V}$

The construction of exceptional metrics on vector bundles over 3- and 4-manifolds made use of ‘dictionaries’ of tautological differential forms. It was natural to use similar techniques to identity the parallel 4-forms over $\mathbb{V}$, but this took a few years:

Proposition [CM]. The parallel QK 4-form Ω can be expressed $SU(3)$-equivariantly on $\mathbb{V}$ as

$$\frac{3\sin^2(r)\cos^2(r)}{r^2}\mathbf{bb}\boldsymbol{\beta}+\frac{\sqrt{3}\sin(2r)}{r}\mathbf{b}\tilde{\boldsymbol{\beta}}+\frac{\sin^2(r)\cos^2(r)}{r^2}\mathbf{a}\tilde{\boldsymbol{\beta}}\boldsymbol{\epsilon}-\frac{-5\sin(2r)+\sin(6r)+4r\cos(2r)}{128\sqrt{3}r^3}\boldsymbol{\gamma\epsilon\epsilon}$$

$$+\frac{\sin^4(r)(\cos(2r)+\cos(4r)+1)}{2\sqrt{3}r^4}\mathbf{bbb}\,\mathbf{a}\boldsymbol{\epsilon}+\frac{\sqrt{3}(2r\cos(2r)-\sin(2r))}{8r^3}\mathbf{b}\boldsymbol{\beta}\,\mathbf{a}\boldsymbol{\epsilon}$$

$$+\frac{3(2r\sin(4r)+\cos(4r)-1)}{4r^4}\mathbf{ab}\,\mathbf{ab}\boldsymbol{\beta}+\frac{\sin^2(r)(5r-6\sin(2r)-3\sin(4r)+r(13\cos(2r)+5\cos(4r)+\cos(6r)))}{96\sqrt{3}r^5}\mathbf{ab}\,\boldsymbol{\epsilon\epsilon\epsilon}$$

$$+\frac{\sin^3(2r)(\sin(2r)-2r\cos(2r))}{32r^6}\mathbf{abb}\,\mathbf{a}\boldsymbol{\epsilon\epsilon}-\frac{\sin^3(2r)\cos(2r)}{8r^3}\mathbf{a}\boldsymbol{\gamma}\,\mathbf{a}\boldsymbol{\gamma}$$

and tends to $3\,\mathbf{bb}\boldsymbol{\beta}+2\sqrt{3}\,\mathbf{b}\tilde{\boldsymbol{\beta}}$ as $r\to 0$.

Letters, syllables and words

The $SU(3)$-invariant differential forms on $\mathbb{V}$ arise from forms defined on the fibre with values in the exterior algebra of the base, everything invariant by $SO(3)$. Syllables arise by contracting letters using the inner product or the volume form on $\mathbb{R}^3$. Examples:

- the syllable **aa** equals $r = \sum (a_i)^2$;
- $\Lambda^2(T_x^*\mathbb{L}) \cong \mathfrak{so}(5) \cong \Sigma^2 \oplus \Sigma^6$, and the value of the syllable $\mathbf{a}\boldsymbol{\beta}$ is the pullback of the 2-form in Σ^2 it represents:

$$a_1(-e^{12}+2e^{34}) + a_2(e^{13}-e^{24}-\sqrt{3}e^{25}) + a_3(e^{14}+\sqrt{3}e^{15}-e^{56});$$

- differentiating the a_i gives $b_i = da_i +$ connection forms, then $\mathbf{bbb} = b_1 \wedge b_2 \wedge b_3$ and $\mathbf{bb}\boldsymbol{\beta} = \mathfrak{S}\ b_i \wedge b_j \wedge \beta_k$;
- words like $\mathbf{bb}\boldsymbol{\beta}$ and $\mathbf{bbb}\,\mathbf{a}\boldsymbol{\epsilon}$ of degree 4 can be formed by wedging 1 or 2 syllables together.

More invariant 4-forms

A generic $SU(3)$-invariant 4-form on $\mathbb{V}$ is

$$k_1\mathbf{bb}\beta + k_2\mathbf{b}\tilde{\beta} + k_3\mathbf{ab}\tilde{\beta} + k_4\mathbf{b}\gamma\epsilon + k_5\mathbf{a}\tilde{\beta}\epsilon + k_6\gamma\epsilon\epsilon$$
$$+k_7\mathbf{bbb}\,\mathbf{a}\epsilon + k_8\mathbf{b}\beta\,\mathbf{a}\epsilon + k_9\mathbf{ab}\,\mathbf{ab}\beta + k_{10}\mathbf{ab}\,\mathbf{a}\gamma\epsilon$$
$$+k_{11}\mathbf{ab}\,\epsilon\epsilon\epsilon + k_{12}\mathbf{abb}\,\mathbf{a}\epsilon\epsilon + k_{13}\mathbf{ab}\beta\,\mathbf{a}\epsilon + k_{14}\mathbf{a}\gamma\,\mathbf{a}\gamma.$$

It extends smoothly across $\mathbb{L}$ iff k_i are smooth even functions of r.
It is closed if and only if

$$\begin{cases} k_1' = rk_9, \quad k_2' = 8rk_8, \\ k_3 = k_{13} = 0, \quad k_5 = \frac{1}{3}k_1, \\ 2rk_{12}' + 12k_{12} - \frac{1}{r}k_{14}' = 0, \\ 24\frac{1}{r}k_6' + \frac{1}{r}k_8' - 72k_{11} - 6k_7 = 0 \end{cases}$$

Group parameters

In order to express the QK 4-form Ω relative to the standard basis (e_i, b_j), we need to express the k_i in terms of parameters

$$e^{i\lambda_1}, e^{i\lambda_2}, e^{i\lambda_{13}}, e^{i\lambda_{14}}, \begin{pmatrix} \lambda_8 & \lambda_9 \\ \lambda_{10} & \lambda_{11} \end{pmatrix}, e^{i\lambda_3}, \begin{pmatrix} \lambda_4 & \lambda_5 \\ \lambda_6 & \lambda_7 \end{pmatrix}, \lambda_{12}$$

for the group $U(1)^4 \rtimes GL(2,\mathbb{R}) \times U(1) \times GL(2,\mathbb{R}) \times \mathbb{R}^*$ that commutes with the $U(1)$ stabilizer of each $SU(3)$ orbit. Closure imposes ODE's on the λ_i (but not λ_7) and we find a solution

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_9 = \lambda_{10} = 0,\ \lambda_{12} = -1,$$
$$\lambda_5 = -\cos(2r),\ \lambda_6 = \sqrt{3},\ \lambda_8 = \tfrac{1}{2}(3 - 2\cos^2 r)\cos r,$$
$$\lambda_{11} = \tfrac{\cos(2r)\sqrt{3}\sin r}{2r},\ \lambda_{13} = \tfrac{(1+2\cos^2 r)\sin r}{2r},$$
$$\lambda_{14} = \tfrac{\sqrt{3}}{2}(-1 + 2\cos^2 r)\cos r.$$

Linear deformation

Problem. To preserve the stabilizer $Sp(2)Sp(1)$ by solving

$$\Omega + t\phi = g(t)\Omega, \qquad g(t) \in GL(8,\mathbb{R}).$$

NB. If $A \in \mathfrak{gl}(8,\mathbb{R})$ satisfies $A \cdot (A \cdot \Omega) = 0$ then $\phi = A \cdot \Omega$ works.

Surprisingly, this can be applied in the $SU(3)$-equivariant case with $A = e_{56}$ to obtain a new triple with $\widetilde{\omega}_2 = \omega_2 - \lambda e^{58}$, $\widetilde{\omega}_3 = \omega_3 + \lambda e^{57}$. It is the interpretation of what happens if $\lambda_7 \neq 0$.

Proposition. The closed 4-form

$$\widetilde{\Omega} = \Omega + f(r)(\mathbf{aa}\,\mathbf{b}\gamma\epsilon + 3\mathbf{ab}\,\mathbf{a}\gamma\epsilon)$$

defines a metric on $G_2/SO(4)$ with an $Sp(2)Sp(1)$-structure that is not QK, for any smooth function $f\colon [0,\pi/4] \to \mathbb{R}$ vanishing on neighbourhoods of the endpoints.

Other structures on the 8-manifolds

Theorem [Gauduchon-Moroianu-Semmelmann]. Apart from the Grassmannians $\mathbb{G}\mathrm{r}_2(\mathbb{C}^n)$, the Wolf spaces (including $E_8/E_7Sp(1)$) do not admit almost complex structures even stably.

NB. $\mathbb{V}\setminus\mathbb{L}$ does admit an $SU(3)$-invariant almost Hermitian structure of generic type defined by ω_1, but this can't therefore extend to the ends.

Proposition. There does not exist an $SU(3)$-invariant $Spin(7)$ structure on $\mathbb{V}$: only $Sp(2)Sp(1)$ reductions extend to the zero section $\mathbb{L}$.

Work is in progress to:

- establish the existence or otherwise of harmonic $Sp(2)Sp(1)$ structures on $\mathbb{HP}^2$ and $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$;
- solve the ODE's on the λ_i parameters to find other harmonic structures on the 4 symmetric spaces, including $SU(3)$.

Consimilarity in SU(3)

It is known that the compact Lie group $SU(3)$ admits

- a left-invariant hypercomplex structure [Joyce];
- an invariant $Sp(2)Sp(1)$ metric that is 'ideal' [Macía];
- a $PSU(3)$ structure with harmonic 3-form [obvious].

The cohomogeneity-one action is twisted conjugation:

$$X \mapsto P X \overline{P}^{-1} = PXP^{\top}, \qquad X,\, P \in SU(3).$$

The stabilizer of the identity is $SO(3)$ and its orbit ($\cong \mathbb{L}$) is

$$\{PP^{\top} : P \in SU(3)\} \stackrel{\text{ex}}{=} \{X \in SU(3) : X\overline{X} = I\}.$$

In fact, $\Gamma : X \mapsto X\overline{X}$ maps $SU(3)$ equivariantly* onto the hypersurface $\mathscr{H} = \{P \in SU(3) : \operatorname{tr} P \in \mathbb{R}\}$, which can be identified with the Thom space of the vector bundle $\Lambda^2_- T^*\mathbb{CP}^2$.

Application to G_2 holonomy

The action of $SU(3)$ on $\mathbb{HP}^2$ commutues with S^1, so there's a residual quotient to be performed:

$$\begin{array}{ccccccc} & & \mathbb{L} & \xrightarrow{\color{red}{\Gamma}} & \mathrm{pt} \\ \mathbb{HP}^2 \setminus \mathbb{CP}^2 & \cong & SU(3) \setminus \mathbb{L} & \to & \Lambda^2_- T^*\mathbb{CP}^2 \\ \downarrow & & \downarrow & & \downarrow \\ S^5 & = & S^5 & \to & \mathbb{CP}^2 \end{array}$$

S^5 is the zero set of a QK moment map, and $\mathbb{CP}^2 = \mathbb{HP}^2 /\!/\!/ S^1$. The 7-dimensional quotent admits a complete metric with holonomy group equal to G_2 [Bryant-S], and can be identified with $S^7 \setminus \mathbb{CP}^2$ [Atiyah-Witten, Miyaoka] as well as $\mathscr{H} \setminus \mathrm{pt} \subset SU(3)$.