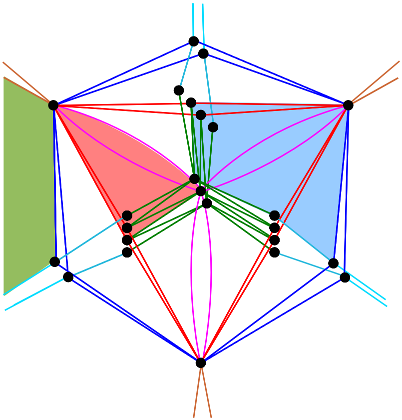
Orthogonal twistor theory

Simon Salamon



INdAM, 13 September 2016

Orthogonal complex structures

Let $\Omega \subseteq \mathbb{R}^{2n}$ be a connected open subset of Euclidean space, endowed with its flat Riemannian metric g_0.

An *orthogonal complex structure* (OCS) on Ω is an integrable almost-complex structure J on Ω satisfying

$$g_0(JX, JY) = g_0(X, Y).$$

One can replace g_0 by the conformally equivalent metric on S^{2n}. Goal is to find and classify such objects. Integrability means that

$$[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$$

for all vector fields X, Y on Ω.

Quiz

Which of the following admit *non-constant* orthogonal complex structures:

$$\mathbb{R}^2, \qquad \mathbb{R}^4, \qquad \mathbb{R}^6, \qquad T^6 \quad ?$$

Consider the conformal compactification $S^{2n} = \mathbb{R}^{2n} \sqcup \infty$.

Which of the following admit orthogonal complex structures:

$$S^2, \qquad S^4, \qquad S^6 \quad ?$$

Spaces of pure spinors

Suppose first that J is constant, and preserves orientation. It then belongs to the Hermitian symmetric space

$$\begin{aligned} Z_n &= SO(2n)/U(n) \\ &= \left\{ g \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} g^{-1} : g \in SO(2n) \right\} \\ &= \left\{ J \in \mathfrak{so}(2n) : J^2 = -I,\ \mathrm{Pf}\, J = 1 \right\} \\ &= \left\{ [\phi] : \phi \in \Delta^+,\ \phi \otimes \phi \in \Lambda^n_+ \mathbb{R}^{2n} \right\} \subseteq \mathbb{CP}^{N-1}, \end{aligned}$$

where Δ^+ is the spin representation of dimension $N = 2^{n-1}$. Then

$$\begin{aligned} Z_1 &= \mathrm{pt} \\ Z_2 &= \mathbb{CP}^1 \cong S^2 \\ Z_3 &= \mathbb{CP}^3 \\ Z_4 &= Q^6 \subset \mathbb{CP}^7. \end{aligned}$$

More generally, J takes values in an obvious *bundle* with fibre Z_n. Remarkably, the total space over the sphere belongs to the series:

$$\begin{array}{c} Z_{n+1} \\ {\scriptstyle Z_n}\Big\downarrow \\ S^{2n}. \end{array}$$

A section $S^{2n} \supset \Omega \stackrel{s}{\to} Z_{n+1}$ defines an almost-complex structure J_s on Ω. There are no global sections unless $n = 1, 3$.

Key proposition. The section s determines an OCS if and only if it is "self-holomorphic":

$$s_* \circ J_s = J \circ s_*,$$

where J is the Kähler complex structure of Z_{n+1}.

n=2: the Penrose fibration

$$\begin{array}{ccc} Z_3 & = & \mathbb{CP}^3 \\ & & \Big\downarrow \pi \\ & & S^4 = \mathbb{HP}^1 \end{array}$$

Let $\mathbb{H}^*$ act on $\mathbb{H}^2$ on the right. On $\mathbb{C}^3 \subset \mathbb{CP}^3$,

$$\pi[1, u, W_1, W_2] = [1 + ju,\ W_1 + jW_2] = [1,\ q],$$

where

$$W_1 + jW_2 = q(1 + ju) = (z_1 + jz_2)(1 + ju),$$

giving

$$\begin{cases} W_1 = z_1 - u\bar{z}_2 \\ W_2 = z_2 + u\bar{z}_1, \end{cases}$$

a deformation of the “standard” complex structure on $\mathbb{R}^4 = \mathbb{C}^2$.

Constant OCS's

Setting u constant distinguishes a plane $\mathbb{CP}^2$ that contains the fibre $\pi^{-1}[0,1]$ over $\infty \in S^4$ Moreover, W_1, W_2 are holomorphic functions for some complex structure J_u on $\mathbb{R}^4$ with $(1,0)$-forms

$$\begin{cases} dW_1 = dz_1 - u\,d\bar{z}_2 \\ dW_2 = dz_2 + u\,d\bar{z}_1. \end{cases}$$

$\Lambda^{1,0}$ is isotropic w.r.t. the Euclidean metric g_0, and $(\mathbb{R}^4, g_0, J_u)$ is Kähler. The associated 2-form

$$\tfrac{1-|u|^2}{1+|u|^2}\omega_1 + \tfrac{2\,\mathrm{Im}\,u}{1+|u|^2}\omega_2 + \tfrac{2\,\mathrm{Re}\,u}{1+|u|^2}\omega_3 \ \in\ \Lambda^2_+\mathbb{R}^4.$$

Folklore theorem. Any OCS on $(\mathbb{R}^4, g_0)$ is J_u for some $u \in \mathbb{CP}^1$.

The conformal group

The action of

$$SO_{\rm o}(5,1) \cong \frac{SL(2,\mathbb{H})}{\mathbb{Z}_2}$$

by conformal transformations of S^4, or equivalently Möbius transformations

$$q \mapsto (aq+b)(cq+d)^{-1}$$

of $\mathbb{HP}^1$ lifts to $\mathbb{CP}^3$, and acts transitively on $\mathbb{CP}^3$ and the set $(\mathbb{CP}^3)^*$ of planes defining constant OCS's.

Theorem [S+Viaclovsky]. Let A be a closed subset of $\mathbb{R}^4$, and J an OCS defined on $\mathbb{R}^4 \setminus A$. If $\mathcal{H}^1(A)=0$ then J is conformally equivalent to a constant OCS (and is defined on $\mathbb{R}^4 \setminus \{p\}$).

There exist other examples with A a curve.

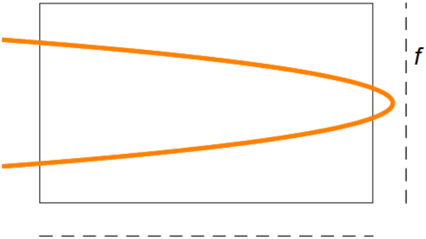
Graphs

We have defined

$$\begin{array}{ccc} \mathbb{R}^4 \times \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^3 \\ ((z_1, z_2), u) & \mapsto & [1,\ u,\ W_1,\ W_2]. \end{array}$$

This incidentally shows that $\mathbb{R}^6$ admits a non-constant OCS!

If $u = f(z_1, z_2)$ is now a function, $f(\Omega)$ will be the graph of an almost complex structure J_f over an open subset $\Omega \subseteq \mathbb{R}^4$:



Self-holomorphic functions

Recall that the almost-complex structure J_f is integrable iff

$$f: \ (\Omega, J_f) \longrightarrow \mathbb{CP}^3$$

is holomorphic. Such OCS's can therefore be constructed from algebraic subvarieties of $\mathbb{CP}^3$. Analytically, the condition is

$$\frac{\partial f}{\partial \bar{z}_1} - f\frac{\partial f}{\partial z_2} = 0, \quad \frac{\partial f}{\partial \bar{z}_2} + f\frac{\partial f}{\partial z_1} = 0.$$

Such an f satisfies

$$\sum_{i=1,2} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} = 0,$$

and is a *harmonic morphism* $\Omega \to \mathbb{CP}^1$.

A real quadric

Consider the smooth quadric

$$Q = \{[1, u, W_1, W_2] \in \mathbb{CP}^3 : W_2 = uW_1\}.$$

so $u = W_2/W_1$. This gives $\bar{z}_2 u^2 - 2iy_1 u + z_2$, and

$$u = f(z_1, z_2) = i\frac{y_1 \pm \sqrt{y_1^2 + x_2^2 + y_2^2}}{\bar{z}_2},$$

with roots u and $-1/\bar{u}$, corresponding to J_u and $-J_u$ ("reality").

The *discriminant locus*

$$D = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : y_1 = x_2 = y_2 = 0\}$$

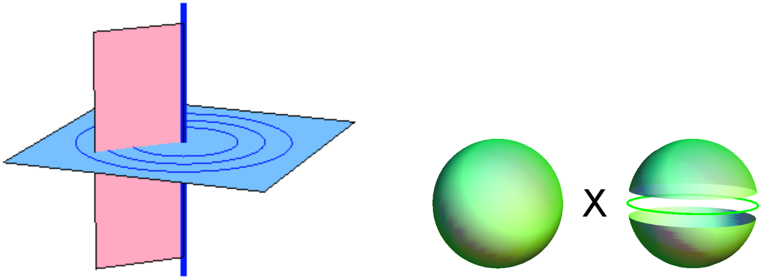
is a line in $\mathbb{R}^4$ or a circle $c \subset S^4$.

$\mathbb{R}^4$ minus a line

It follows that $\pi^{-1}(c) \cong S^1 \times S^2$, and $Q \setminus \pi^{-1}(c)$ double covers

$$\Omega \;=\; S^4 \setminus c \;\cong\; \mathbb{CP}^1 \times \mathbb{C}^+.$$

The associated OCS $(\pm J_c, g_0)$ is conformal to a Kähler product [Pontecorvo]. It is defined by a half-plane $\mathbb{C}_j^+$ $(j \in S^2)$ rotating around the real axis, cf. slice-regular $\mathbb{H}$-valued functions:



Classification of quadrics

under the conformal group:

Let $\mathscr{Q}$ be a smooth quadric in $\mathbb{CP}^3$, and set

$$D = \{p \in S^4 : \#\pi^{-1}(p) \neq 2\}.$$

Theorem [SV]. There are four possibilities: D is

0. a smooth 2-torus
1. a 2-torus pinched at one point p_1
2. a 2-torus pinched at two points p_1, p_2

∞. a round circle, and $\mathscr{Q}$ is equivalent to Q.

In the last case, $\pi^{-1}(D) \subset \mathscr{Q}$. In cases 1 and 2, $\pi^{-1}(p_i) \subset \mathscr{Q}$.

Quaternion power series

Take $W_2 = uW_1$ again, but with $W_1 = v \in \mathbb{C}^+$. Then

$$\begin{array}{ccc} [1, u, v, uv] & \in & Q \\ \pi \downarrow & & \\ q & & = (1 + ju)v\,(1 + ju)^{-1} = UvU^{-1}, \end{array}$$

and "u rotates v into position": $\mathbb{R}^4 \setminus \mathbb{R} = \mathbb{CP}^1 \times \mathbb{C}^+$.

Since $q^n = Uv^nU^{-1}$, we can convert a convergent power series

$$F(q) = \sum a_n q^n, \quad a_n \in \mathbb{H},$$

defining a smooth map $S^4 \to S^4$, into (?stem) data:

$$[1,\ F(q)] = [U,\ \textstyle\sum a_n(1 + ju)v^n] = \pi[\,1,\ u,\ \phi(u, v),\ \psi(u, v)\,],$$

with ϕ, ψ are holomorphic functions, linear in u.

Twistor lifts

Corollary [S+Gentili+Stoppato]. On a domain in which it is injective, a slice-regular function F maps the OCS J_Q defined by Q onto another *orthogonal* complex structure.

There is a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\tilde{F}} & \mathbb{CP}^3 \\ \big\downarrow & & \big\downarrow \\ S^4 & \xrightarrow{F} & S^4. \end{array}$$

The class of maps with this property incorporates J_Q-holomorphic ones (these correspond to power series with $a_k \in \mathbb{R}$) and elements of the conformal group $SO_{\rm o}(5,1)$.

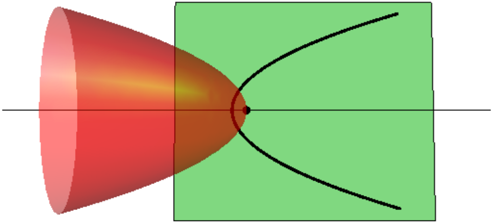
The example $F(q) = q^2 + iq$

Lemma. F commutes with the S^1 action $q \mapsto e^{i\theta} q e^{-i\theta}$. It maps

- $\mathbb{R}$ onto the parabola $\gamma = \{x^2 + xi : x \in \mathbb{R}\}$
- $\mathbb{C}$ onto $\mathbb{C}$ and $-\frac{1}{2}i$ to the focus $\frac{1}{4}$ of γ
- the plane $\Pi = -\frac{1}{2}i + j\mathbb{C}$ bijectively onto the paraboloid

$$\Gamma = \{\tfrac{1}{4} - y^2 - z^2 + yj + zk : y, z \in \mathbb{R}\}$$

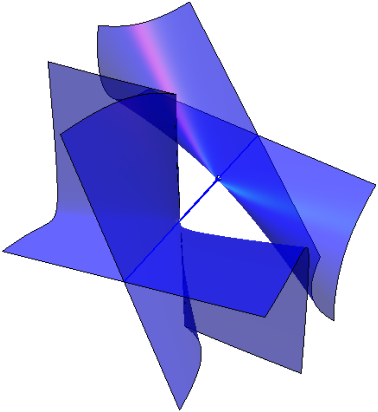
- $\mathbb{H} \setminus \Pi$ onto $\mathbb{H} \setminus \Gamma$ as a double cover.



The parabola's scroll

Proposition. $\tilde{F}(u,v) = [1, u, v^2 + iv, u(v^2 - iv)]$, and $\tilde{F}$ maps Q birationally onto the quartic surface

$$\mathscr{K} = \left\{[W_i] \in \mathbb{CP}^3 : (W_0 W_3 - W_2 W_2)^2 = 2W_0 W_1 (W_0 W_3 + W_2 W_2)\right\}.$$



NB. Twistor lifts with cubic images have been studied by Altavilla.

Geometry of $\mathscr{K}$

It is a quartic of type IV(B) In Edge's list, and has 3 singular lines. It is a ruled surface associated to a complex curve $\mathbb{CP}^1 \to \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$.

The discriminant locus of $\mathscr{K}$ is

$$\gamma \ \sqcup \ (\mathbb{C} \setminus \gamma) \ \sqcup \ (\Gamma \setminus \tfrac{1}{4})$$

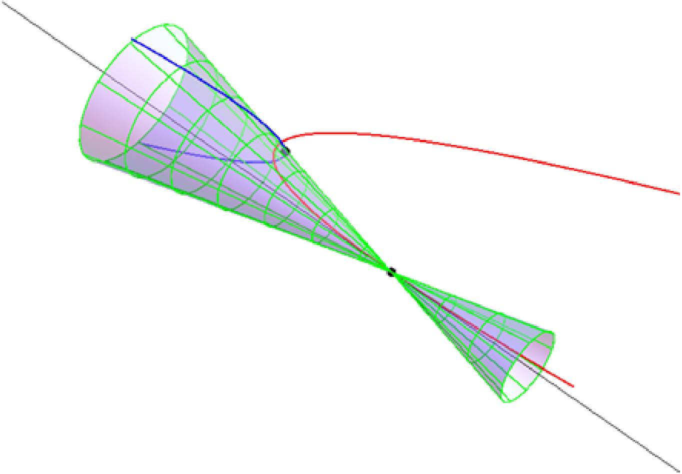
where

- $\mathscr{K}$ contains $\pi^{-1}(\gamma)$
- if $p \in \mathbb{C} \setminus \gamma$, then $\pi^{-1}(p)$ has two singular points of $\mathscr{K}$
- if $p \in \Gamma$, then $\pi^{-1}(p)$ is tangent to $\mathscr{K}$ at two points.

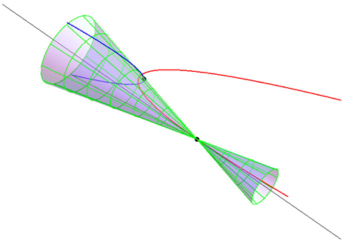
No OCS determined by an open set of $\mathscr{K}$ extends across Γ. So it is a "focal set" for an OCS on $\mathbb{R}^4 \setminus \gamma$. In what other sense does the parabola γ determine the paraboloid Γ?

Dual parabolas and cones

Factoring out by rotation in the last two coordinates of $\mathbb{R}^4$,
Γ becomes a (red) parabola in a plane orthogonal to (blue) γ:



Cones and quadrics



Theorem [GSS]. In four dimensions, Γ is the locus in $\mathbb{R}^4$ of vertices v of circular 2-dimensional cones that contain (blue) γ. Moreover,

- each such cone C_v is the discriminant locus of a quadric Q_v in $\mathbb{CP}^3$. In $\mathbb{R}^4$, it is a 2-torus pinched at v and ∞.
- as a real surface, C_v lifts to a cylinder/2-torus in Q_v.
- Q_v contains the fibres $\pi^{-1}(\infty)$ and $\pi^{-1}(v)$, and is tangent to $\mathscr{K}$ along a twisted cubic.

$\mathscr{K}$ (blue) and Q_v (pink)