Instantons defined by Lie groups

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Old references


4-dimensional origins

On an oriented Riemannian 4-manifold,

\[ \Lambda^2 T^* M = \Lambda^2_+ \oplus \Lambda^2_- \]

since \( \mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_- \), and there is an elliptic complex

\[ 0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d_-} \Omega^2_- \to 0. \]

If \( Q \) is a principal \( SU(2) \)-bundle with self-dual connection \( A \) with (so \( F = dA + A \wedge A \in \Omega^2_+ \) and \( *F = F \)) then \( H^1 \) of the complex

\[ 0 \to \Omega^0(\text{ad } Q) \xrightarrow{} \Omega^1(\text{ad } Q) \xrightarrow{} \Omega^2_- (\text{ad } Q) \to 0 \]

captures infinitesimal deformations modulo gauge equivalence. The index is \( 8k - 3 \), and a framed moduli space of dimension \( 8k \) is hyperkähler (meaning holonomy \( Sp(k) \)), given by HK quotients

\[ \text{HK}^{k(k+3)/2} / \!/ O(k) \cong \mu^{-1}_\infty(0) / G \ [D]. \]
More generally, for $G \subset SO(n)$, we can write

$$\Lambda^2 \cong sl(n) = g \oplus g^\perp.$$ 

Given a $G$-structure on $M$ (in fact, an $N(G)$-structure) this decomposition passes to 2-forms.

**Definition.** In this context, an instanton is a connection (on a bundle over $M$) whose curvature 2-forms $F^i_j$ lie in $g$.

**Example.** If the holonomy reduces to $G$ then the Levi-Civita connection is an instanton because $R_{ijkl}$ belongs to the kernel of

$$S^2(g) \subset g \otimes \Lambda^2 \subset \Lambda^2 \otimes \Lambda^2 \longrightarrow \Lambda^4.$$ 

On the other hand, the Killing form in $S^2(g)$ maps to a non-zero 4-form unless it represents curvature of a Riemannian symmetric space. So 4-forms arise in abundance!
Hitchin-Kobayashi correspondence

If \( G = SU(n) \subset SO(2n) \) so \( N(G) = U(n) \) then

\[
\Lambda^2 = \left[ [\Lambda^{2,0}] \oplus [\Lambda_{0}^{1,1}] \oplus \langle \omega \rangle \right],
\]

and \( g = su(n) \cong [\Lambda_{0}^{1,1}] \). An instanton is a connection with \((1,1)\) curvature and vanishing trace \( F \wedge \omega^{n-1} \), though this would force \( c_1 = 0 \). More generally we require that the trace be a (constant) multiple of the identity, the Hermitian-Einstein condition.

Over a complex manifold:
- A connection with \((1,1)\) curvature on a vector bundle renders it a holomorphic bundle [cf. NN].
- A holomorphic bundle admits a unique connection with \((1,1)\) curvature compatible with a given Hermitian metric on its fibres.

**Theorem** [D,UY]. On a compact Kähler manifold, an irreducible holomorphic vector bundle admits a HE connection iff it is stable.
Using a 3-form

If \( G = G_2 \subset SO(7) \) so \( N(G) = G \) then

\[
\Lambda^2 = \Lambda^2_{14} \oplus \Lambda^2_7 \cong g_2 \oplus \Lambda^1 \\
\Lambda^3 = \Lambda^3_{27} \oplus \Lambda^3_7 \oplus \langle \varphi \rangle \cong S^2_0 \oplus \Lambda^1 \oplus \mathbb{R}.
\]

*Example.* If \( \varphi = (12 - 34)5 + (13 - 42)6 + (14 - 23)7 + 567 \), we have \( 12+34, 13+24, 14+23 \in \Lambda^2_+ \subset g_2 \).

An instanton is characterized by the equivalent equations

\[
F_7 = 0, \quad F \wedge (\ast \varphi) = 0, \quad F \wedge \varphi = \ast F.
\]

Instantons are YM connections because

\[
c_2(F) \cup [\varphi] = \int F \wedge F \wedge \varphi = 4\|F_{14}\|^2 - 18\|F_7\|^2
\]

and \( \|F\|^2 \) has an absolute minimum if \( F_7 = 0 \).
Suppose that $M^7$ has a $G_2$ structure. Consider

\[ 0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{D_1} \Omega^2_7 \xrightarrow{D_2} \Omega^3_1 \to 0 \]

where $D_1 = \pi_7 \circ d$ arises from the cross product. It is complex iff

\[ D_2 \circ D_1 = 0 \iff d(\Omega^2_{14}) \subseteq \langle \phi \rangle^\perp \iff d \ast \phi = 0. \]

**Lemma [CN].** If $M^7$ is oriented and spin it has a $G_2$ structure, indeed one with $d \ast \phi = 0$.

Given an instanton on a bundle $Q$, we can extend the operators so $D \circ D = F_7 = 0$ and obtain an elliptic complex

\[ 0 \to \Omega^0(\text{ad } Q) \to \Omega^1(\text{ad } Q) \to \Omega^1(\text{ad } Q) \to \Omega^0(\text{ad } Q) \to 0 \]

whose $H^1$ parametrizes infinitesimal deformations. Close analogue with the de Rham complex $1 \to 3 \to 3 \to 1$ over a 3-manifold.
Integrability

We shall construct a differential complex for any $G \subset SO(n)$ that begins $0 \to \Omega^0 \to \Omega^1 \to \ldots$

Set $\Lambda^{-2} = \Lambda^{-1} = 0$ and $A^k = (g \wedge \Lambda^{k-2})^\perp$. Define

$$D: \mathcal{A}^k \subset \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{\pi} \mathcal{A}^{k+1}.$$

Here $\mathcal{A}^k = \Gamma(M, A^k)$, so $\mathcal{A}^k = \Omega^k$ for $k = 0, 1$.

**Proposition.** $D^2 = 0$ if only if $d: \Omega^2 \to \Omega^3$ maps sections of $g$ to those of $g \wedge \Lambda^1$.

This is obviously true if the holonomy of $M$ lies in $N(G)$; in general it is a condition on the intrinsic torsion $\tau \in \Gamma(M, g \otimes \Lambda^1)$.

In any case, we would like $0 \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \ldots \to 0$

to be elliptic. It is when $G$ equals $SU(n), G_2, Spin 7, Sp(n), \ldots$
Nearly-Kähler 6-manifolds

If $G = SU(3)$ so $N(G) = U(3)$, the complex becomes

$$0 \to \Omega^0 \to \Omega^1 \to \Gamma([\Lambda^2,0]) \oplus \langle \omega \rangle \to \Gamma([\Lambda^3,0]) \to 0.$$ 

with dimensions $1 \to 6 \to 7 \to 2$, provided most of the Nijenhuis tensor vanishes! We get a theory of instantons over nearly-Kähler 6-manifolds (meaning $(\nabla_X J)X = 0$).

*Example.* The twistor spaces $\mathbb{CP}^3 \to S^4$ and $F^3 \to \mathbb{CP}^2$ both have a $U(1)$-connection $A_1$ whose curvature $F_1$ is a Kähler form and another NK 2-form $F_2$ such that $F_2 \wedge F_2 = d\psi$ with $F_1 \wedge \psi = 0$.

$$0 = d(F_1 \wedge \psi) = dF_1 \wedge \psi + F_1 \wedge (F_2 \wedge F_2) = 2F_1 \wedge (*F_2).$$ 

Thus $A_1$ is an instanton for the NK metric.
If $G = \text{Sp}(n) \subset \text{SO}(4n)$ so $N(G) = \text{Sp}(n)\text{Sp}(1)$ and

$$(\Lambda^1)_c = E \otimes H \quad \text{(cf. } S \otimes \tilde{S})$$

$$(\Lambda^2)_c = S^2E \oplus S^2H \oplus (\Lambda^2_0 E \otimes S^2H)$$

$\cong \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \oplus \mathfrak{m}.$

Manifolds $M^{4n}$ ($n \geq 2$) like $\mathbb{HP}^n$ with holonomy in $N(G)$ are quaternion-Kähler and behave as if they were nearly hyperkähler.

Since $S^2E = \Lambda^{1,1}_i \cap \Lambda^{1,1}_j \oplus \Lambda^{1,1}_K$, instantons give rise to holomorphic bundles over the twistor space $Z^{2n+1}$, which fibres over $M$.

Using a 4-form again, one shows that the Yang-Mills functional has a critical point whenever any 2 of the 3 components vanish.

Not an abs max/min if $F \in \Gamma(M, \mathfrak{m})$, but no examples known.
Instantons via quaternions

Take $M^{4n} = \mathbb{HP}^n$. Let $q = (q_0, q_1, \ldots, q_n) \in \mathbb{H}^{n+1} \setminus 0$, $m = [q]$.

A linear form $\sum a_r q_r$ (with $a_r \in \mathbb{H}$) defines a section of the tautological line bundle $H$ (fibre $\mathbb{H} = \mathbb{C}^2$) inside $\mathbb{H}^{n+1}$, and

$$E_m = \ker \left( q^\top : \mathbb{H}^{n+1} \to H_m \right), \quad \text{so} \quad E = H^\perp.$$

Take matrices $A_0, A_1, \ldots, A_n \in \mathbb{H}^{n+k,k}$ and set $A = \sum_{r=0}^n A_r q_r$.

**Theorem** If $A_r^* A_s$ is symmetric for all $r, s$ and $A$ has rank $k$ for all $q \neq 0$ then $\ker A$ is an $\text{Sp}(n)$ instanton on $\mathbb{HP}^n$.

**Proof.** Relies on the fact that the real components of

$$dq_r \wedge d\bar{q}_s = (du_r + j dv_r) \wedge (d\bar{u}_r - d\bar{v}_r j)$$

span the subspace $\mathfrak{sp}(n)$ of $\Lambda^2$. Projection to $\ker A$ equals

$$\Pi = 1 - A(A^* A)^{-1} A^*$$

and the induced curvature is $\Pi(d\Pi \wedge d\Pi) \Pi$. 
The twistor space \( \mathbb{HP}^n \) is \( \mathbb{CP}^{2n+1} \), which is the total space of \( \mathbb{P}_c(H) \) ↓ \( \mathbb{HP}^n \).

The instantons \( F = \ker A \) pull back to holomorphic bundles \( \pi^* F \) (fibre \( \mathbb{C}^{2n} \)) with \( c(F) = (1 - h)^{-k} \), characterized by

\[
H^q(\mathbb{CP}^{2n+1}, \pi^* F \otimes \mathcal{O}(p)) = 0 \quad \begin{cases} q = 1, & p \leq -2 \\ 2 \leq q \leq n, & p \in \mathbb{Z}. \end{cases}
\]

**Example.** For \( n = k = 2 \), we can take \( A = \begin{pmatrix} q_0 & 0 \\ 0 & q_0 + q_2 \\ q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} \).

The deformation complex for \( n = 1 \) has \( h^1 = 8k - 3 \).

**Proposition.** If \( n = 2 \) the deformation complex of the instantons above has \( h^1 - h^2 = \frac{3}{2} k(17 - k) - 10 = 14, 35, 53, \ldots \).
Many interesting geometries in dim 8 are characterized by 4-forms: elements of $\Lambda^4(\mathbb{R}^8)^*$, the isotropy representation of the symmetric space $E^7/SU(8)$. The complicated orbit structure for $SL(8, \mathbb{R})$ acting on the 4-forms can be understood via roots of $E_7$ \[V\].

We focus on the inclusions

$$
\begin{align*}
\text{Sp}(2) & \hookrightarrow \text{Sp}(2)\text{Sp}(1) & \hookrightarrow \text{SO}(8).
\end{align*}
$$

$\text{Sp}(2)$ fixes a triple $\omega_1, \omega_2, \omega_3$, whilst

$\text{Sp}(2)\text{Sp}(1)$ stabilizes $\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$

$\text{Spin} 7$ stabilizes $\Phi = -\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$.

We shall investigate the topology defined by the two rank 3 groups.
Proposition. If a compact, oriented $M^8$ has a Spin 7 or a Sp(2)Sp(1) structure then $M$ is spin and $8\chi = 4p_2 - p_1^2$.

Proof.

$$\Delta_+ \cong \begin{cases} 
\Lambda^0 \oplus \Lambda_7^2 = 1 + 7 & \text{for Spin 7} \\
S^2H \oplus \Lambda_0^2E = 3 + 5 & \text{for Sp(2)Sp(1)}
\end{cases}$$

and in both cases $\Delta_- \cong \Lambda^1 \cong TM$. The Euler class $e$ satisfies

$$\text{ch}(\Delta_+ - \Delta_-) = e \hat{A}^{-1} = e(1 - \frac{1}{24}p_1 + \hat{A}_2 + \cdots)^{-1}$$

where $\hat{A}_2 = \frac{1}{5760}(7p_1^2 - 4p_2)$.

Theorem. If $\text{Hol}(M) \subseteq \text{Sp}(2)\text{Sp}(1)$ and $s > 0$ then $\hat{A}_2 = 0$. If $\text{Hol}(M) = \text{Spin 7}$ then $\hat{A}_2 = 1$.

So the QK 8-manifolds $\mathbb{H}\mathbb{P}^2$, $G_2/\text{SO}(4)$, $\text{Gr}_2(\mathbb{C}^4)$ all admit a Spin 7 structure but not the rival holonomy!
The remarkable space $G_2/\text{SO}(4)$

- parametrizes coassociative 4-planes in $\mathbb{R}^7 \subset \mathcal{O}$ [HL].
- As an application, its orbits under $\text{SO}(4)$ are relevant to the classification of coassociative submanifolds of the $G_2$ manifold $\Lambda^2_-(S^4)$ that are deformations of $T^*S^2$ [KS].
- Removing $\mathbb{CP}^2$ and quotienting out by $\mathbb{Z}_3$, we get $\mathcal{N}/\mathbb{H}^*$, where $\mathcal{N}$ is the principal nilpotent orbit in $\text{sl}(3,\mathbb{C})$. The latter is HK and the quotient QK [K,S].
- There is a construction of QK metrics from 5-manifolds with generic 2-plane distributions that are modelled asymptotically on the noncompact dual $G_2^s/\text{SO}(4)$ [L,B,D].
If $M^8$ has a $\text{Spin}_7$ structure then $(\mathcal{A}^\bullet, D)$ coincides with

$$0 \to \Omega^0 \to \Omega^1 \to \Omega_7^2 \to 0.$$

Only if it is written backwards do we need the holonomy condition!

It is easy to compute the deformation index

$$h^0 - h^1 + h^2 = \int \text{ch(ad } Q) \hat{A}$$

though $h^2$ is again unknown.

**Example.** For the Levi-Civita instanton $\Lambda^1$ on a manifold with holonomy equal to $\text{Spin}_7$, it equals $8 - \frac{1}{3} \chi$. The latter is an integer because

$$25 + b_2 + 2b_4^- = b_3 + b_4^+. $$
Intrinsic torsion

...of a $\text{Sp}(2)\text{Sp}(1)$ structure lies in $\Lambda^1 \otimes \mathfrak{m} = \begin{array}{cc} \mathcal{E} \mathcal{H} & \mathcal{K} \mathcal{H} \\ \mathcal{E} S^3 \mathcal{H} & \mathcal{K} S^3 \mathcal{H} \end{array}$

$M^8$ is quaternionic if $\tau \in \text{row 1}$; it is ideal if $\tau \in \text{col 1}$.

Surprise. $\text{SU}(3)$ possesses invariant structures of both types $[J, M]$.

If $M^8$ is quaternionic, there is an elliptic complex

$$0 \to \Gamma(S^2 \mathcal{H}) \to \Gamma(\mathcal{E} \mathcal{H}) \to \Gamma(\Lambda^2 \mathcal{E}) \to 0,$$

which corresponds to the sheaf $\mathcal{O}(-2)$ on the twistor space $Z^5$.

Tensor by $S^2 \mathcal{H}$ to obtain $(\mathcal{A}^\bullet, \mathcal{D})$. Passing to $\mathcal{O}(-3)$ gives

$$0 \to \Gamma(\mathcal{H}) \overset{\partial}{\longrightarrow} \Gamma(\mathcal{E}) \overset{\Box}{\longrightarrow} \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{H}) \to 0$$

where $\partial$ is a Fueter operator and $\Box$ is second order $[B]$. 
Associated to the Dirac operator on $\mathbb{HP}^2$ is the virtual vector bundle

$$\Lambda_0^2(E - H) = \Lambda_0^2E - E H + S^2H.$$ 

Now, $E - H$ can’t be a genuine vector bundle because:

- any monomorphism $H \to E$ would define a nowhere zero section of $E \otimes H \cong T\mathbb{HP}^2$ but $\chi = 3$;
- $E - H$ has rank 2, but a calculation shows $c_4(E - H) \neq 0$!

In fact, $c(H) = 1 - h$ so

$$c(E - H) = c(\mathbb{C}^6 - 2H) = (1 - h)^{-2} = 1 + 2h + 3h^2.$$ 

By contrast,

$$c(\Lambda_0^2E - H) = c(\Delta_+ - S^2H - H) = 1 - 3h.$$ 

We shall see that this time the difference is a vector bundle.
Theorem. There exists a rank 3 complex vector bundle $V$ over $\mathbb{HP}^2$ with $c_2 = 3h$, and an $SU(3)$-connection with $F \in \Gamma(\mathfrak{sp}(2))$.

Proof. Recall that $E = \ker(p_1: \mathbb{C}^6 \to H)$. Similarly,

$$\Lambda^2 E = \ker(\Lambda^2_0(\mathbb{C}^6) \to \mathbb{C}^6 \wedge H).$$

Fix a reduction of $Sp(3)$ to $SU(3)$, giving $\mathbb{C}^6 = \Lambda^{1,0} \oplus \Lambda^{0,1}$ and

$$p_2: \Lambda^2_0(\mathbb{C}^6) \to \mathbb{C}^6.$$

Then $p_1 \circ p_2$ has rank 2 everywhere. The instanton connection on $V = \ker(p_1 \circ p_2)$ is induced from that on $\Lambda^2_0 E$ and ultimately $E$.

The moduli space of such instantons is then the total space of

$$\frac{\text{SL}(3, \mathbb{H})}{SU(3)} \to \frac{\text{SL}(3, \mathbb{H})}{Sp(3)}, \text{ whose } T_o \cong \Lambda^2_0(\mathbb{C}^6)$$
In parallel to the theory of manifolds with reduced holonomy, there is a unified theory of instantons (which arguably preceded it in the exceptional cases).

The quaternionic version of ADHM has many unanswered questions and still open problems [O]. Unlike for $G_2$ or $Spin 7$, there is no rich theory of submanifolds.

The link between $G_2$ and the nearly-Kähler case is striking, especially since it is unknown if there are compact NK manifolds other than the four usual suspects ($S^6$, $CP^3$, $F^3$, $S^3 \times S^3$).

More applications are needed of the differential complexes, their cohomology and index theory. A big problem is to determine $h^2$.

The exceptional cases (NK, $G_2$, $Spin 7$) suffer from being non-holomorphic theories, with no twistor spaces to retire to. But this did not stop algebraic geometry being used in the construction of new compact manifolds with holonomy $G_2$ [CHNP].