Flags and twistors revisited

Real and Complex Manifolds

The mathematical heritage
of Edoardo Vesentini

Simon Salamon

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Plan

1. History: Twistor theory

Mainly for Riemannian 4-manifolds

2*. Notation: The (simplest) flag manifold $\mathbb{F}$

Projective and unitary descriptions of $\mathbb{F}$

3*. New results: Twistor projections

Dumbbells, Fubini structures, ramification

*Joint work with Amedeo Altavilla, Edoardo Ballico, and Chiara Brambilla.

1.1 Penrose's theory

is based on choosing a real form of the Klein correspondence between lines in complex projective 3-space $\mathbb{CP}^3$ and points in the complex quadric Q^4:

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{C}^4) & \\ \swarrow & & \searrow \\ \mathbb{CP}^3 & & \mathbb{G}\mathrm{r}_2(\mathbb{C}^4) = Q^4 \subset \mathbb{CP}^5 \end{array}$$

Each subgroup of $\mathrm{SL}(4,\mathbb{C})$ below distinguishes a subset of 'real' lines in $\mathbb{CP}^3$:

$$\begin{array}{llcl} \mathbb{R}^4 & \subset S^4 & \text{acted on by} & \mathrm{SO}(5,1) \simeq \mathrm{SL}(2,\mathbb{H}) \\ \mathbb{R}^{3,1} & \subset S^3 \times S^1 & \text{"} & \mathrm{SO}(4,2) \simeq \mathrm{SU}(2,2) \\ \mathbb{R}^{2,2} & \subset \mathbb{G}\mathrm{r}_2(\mathbb{R}^4) & \text{"} & \mathrm{SO}(3,3) \simeq \mathrm{SL}(4,\mathbb{R}). \end{array}$$

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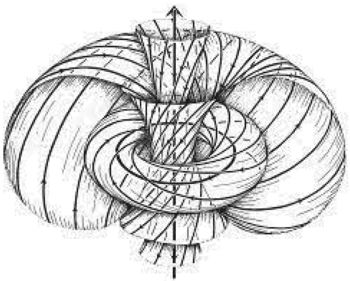

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$\mathbb{R}^{3,1}$	$\subset S^3 \times S^1$	"	$\mathrm{SO}(4,2) \simeq \mathrm{SU}(2,2)$

A point of Minkowski space $\mathbb{R}^{3,1}$ defines a $\mathbb{CP}^1$ in the real hypersurface $N = \{\mathrm{Re}(z_0\overline{z}_2 + z_1\overline{z}_3) = 0\}$ parametrizing light rays in $\mathbb{R}^{3,1}$. A point of $\mathbb{CP}^3 \setminus N$ defines a congruence of light rays in $\mathbb{R}^{3,1}$ represented by twisted* circles in $\mathbb{R}^3$.

1.2 Hopf geometry

In the first case, the fibres of

$$\begin{array}{c} \mathbb{CP}^3 \\ \pi \big\downarrow \\ S^4 = \mathbb{HP}^1 \subset \mathbb{G}\mathrm{r}_2(\mathbb{C}^4) \end{array}$$

are ‘real’ lines $\mathbb{CP}^1 \cong S^2$ relative to the antilinear involution $j\colon \mathbb{CP}^3 \to \mathbb{CP}^3$.

Because the fibres are complex, any point of $z \in \mathbb{CP}^3$ induces an almost complex structure on $T_{\pi(z)}S^4$, and a complex surface defines an orthogonal complex structure (OCS) on an open subset of S^4. This is the analogue of a more general shear-free congruence of light rays in $\mathbb{R}^{3,1}$ [Robinson-Kerr, Hughston-Mason].

Examples. A plane $\mathbb{CP}^2$ determines a conformally constant OCS on $\mathbb{R}^4 = S^4 \setminus \infty$. *A real quadric determines an OCS on $\mathbb{H} \setminus \mathbb{R}$ related to slice-regular functions on the quaternions [Gentili-Stoppato-S].*

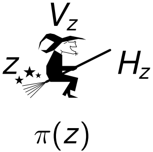
1.3 Self-duality

In place of S^4, start with any oriented Riemannian 4-manifold M. The splitting

$$\Lambda^2 T^*M = \Lambda^+ \oplus \Lambda^-$$

into self-dual and ASD 2-forms allows one to construct the 2-sphere bundle

$$\begin{array}{ccc} Z & \subset & \Lambda^- \\ \pi \big\downarrow & & \\ M & & \end{array}$$



Each point $z \in \pi^{-1}(m)$ defines an almost complex structure on its horizontal space $H_z \cong T_m M$, which can be combined with

$(+)$ the natural complex structure on $V_z \cong T_z S^2$, or

$(-)$ its negative.

This equips Z with almost complex structures $\mathbb{J}^+, \mathbb{J}^-$; the latter is never integrable.

1.4 Complex integrability

A local section $s\colon U \to Z$ determines an almost complex structure J_s on U, and

s is $\mathbb{J}^+$-holomorphic $\Leftrightarrow$ J_s is integrable

[de Bartolomeis-Nannicini].

$$\begin{array}{ccc} & & Z \subset \Lambda^- \\ & & \pi \big\downarrow \\ U & \subseteq & M \end{array} \quad .$$

Theorems.

$\mathbb{J}^+$ is integrable $\Leftrightarrow$ M is self-dual, i.e. its Weyl tensor satisfies $W_- \equiv 0$ [Atiyah-Hitchin-Singer].

$(Z, \mathbb{J}^+)$ is (compact) Kähler $\Rightarrow$ M is isometric to S^4 or $\mathbb{CP}^2$ [Besse, Hitchin].
In the latter case Z is the flag manifold $\boxed{\mathbb{F} = \mathbb{F}_{1,2}(\mathbb{C}^3) \cong \mathrm{SU}(3)/T^2}$

Given M^4 (compact, oriented), there exists n such that $M\#n\mathbb{CP}^2$ admits a self-dual metric [Poon, LeBrun, Joyce; Floer, Donaldson-Friedman, Taubes].

A local section $s\colon U \to Z$ is $\mathbb{J}^-$-holomorphic $\Leftrightarrow d\omega_s = 0$.

Let Σ be a Riemann surface. An immersion

$$\phi\colon \Sigma \longrightarrow M$$

can be lifted to $\psi\colon \Sigma \to Z$, so as to render $\phi_*(T_\sigma\Sigma)$ a complex line w.r.t. $\psi(\sigma)$.

Proposition [Eells-S]. ϕ is harmonic $\Leftrightarrow \psi$ is $\mathbb{J}^-$-holomorphic.

Examples. If ψ is also $\mathbb{J}^+$-holomorphic then ϕ is 'superminimal'. Such immersions can be constructed for any genus using Bryant's formula $\left[1,\ f - \frac{1}{2}g\frac{df}{dg},\ g,\ \frac{1}{2}\frac{df}{dg}\right]$.

Inherent in this construction is a birrational correspondence between $\mathbb{CP}^3$ and $\mathbb{F}$ that preserves their holomorphic contact structures [Lawson, Gauduchon]. Both contain an open orbit of the complex Heisenberg group [Burstall].

2.1 $\mathbb{F}$ as an algebraic variety

Let $\mathbb{P}^2 = \mathbb{CP}^2$ denote the complex projective plane, and $\mathbb{P}^{2\vee} = \mathrm{Gr}_2(\mathbb{C}^3)$ its dual.

$$\mathbb{F} = \{(p,\ell) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee} : p\ell = 0\},$$

where $p = [p_0, p_1, p_2]$ and $\ell = [\ell_0, \ell_1, \ell_2]^\top$. So there is a double fibration

$$\begin{array}{ccccc} & & \mathbb{F} & & \\ \pi_1 & \swarrow & & \searrow & \pi_2 \\ \mathbb{P}^2 & & & & \mathbb{P}^{2\vee} \end{array}$$

and all the $\mathbb{CP}^1$ fibres are tangent to the holomorphic contact 1-form

$$\theta = p \cdot d\ell = -dp \cdot \ell \ \in \ \Omega^1(\mathcal{O}(1,1)).$$

Everything is equivariant for the action $(p,\ell) \mapsto (pB^{-1}, B\ell)$ with $B \in \mathrm{SL}(3,\mathbb{C})$, and and the map $(p,\ell) \mapsto \ell p$ embeds $\mathbb{F}$ into $\mathbb{CP}^7$.

2.2 Curves and surfaces in $\mathbb{F}$

A divisor of $\mathcal{O}(a,b)$ is said to have bidegree (a,b). The Hirzebruch surfaces

$$\begin{aligned} H_m &= \{(p,\ell)\in\mathbb{F} : pm = 0\} &= \pi_1^{-1}(\ker m) \\ {}_qH &= \{(p,\ell)\in\mathbb{F} : q\ell = 0\} &= \pi_2^{-1}(\ker q) \end{aligned}$$

have bidegrees $(1,0)$ and $(0,1)$. They are blow-ups of $\mathbb{P}^{2\vee}$ (at m), and $\mathbb{P}^2$ (at q).

One can define the bidegree of an algebraic curve by intersection. So $\pi_1^{-1}(p)$ has bidegree $(0,1)$, whereas $\pi_2^{-1}(\ell)$ has bidegree $(1,0)$. Any curve of bidegree $(1,1)$ has the form

$$\mathcal{L}_{q,m} = \{(p,\ell)\in\mathbb{F} : q\ell = 0,\ pm = 0\} = {}_qH \cap H_m,$$

for fixed $q\in\mathbb{P}^2$, $m\in\mathbb{P}^{2\vee}$. It is smooth unless $qm = 0$ ($\Rightarrow \mathcal{L}_{q,m} = \pi^{-1}(q)\cup\pi^{-1}(\ell)$).

2.3 Surfaces of bidegree (1,1) in $\mathbb{F}$

Each is defined by a complex 3×3 matrix A modulo a multiple of the identity:

$$S_A = \{(p, \ell) \in \mathbb{F} : pA\ell = 0\}.$$

Up to the action of $\mathrm{SL}(3, \mathbb{C})$, we may assume that A is one of:

$$A_1 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \lambda \end{pmatrix}, \quad A_2 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad A_5 = \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$\lambda \in \mathbb{C} \setminus \{0, 1\}$

smooth | reducible | one singular point

$\lambda \sim \frac{1}{\lambda} \sim 1 - \lambda \sim \frac{1}{1-\lambda} \sim \frac{\lambda}{\lambda-1} \sim \frac{\lambda-1}{\lambda}$

2.4 The five projective types

$A \sim A_1 \;\; \Leftrightarrow S_A$ is smooth and a Del Pezzo surface of degree 6
π_1 realizes S_A as the blow up of 3 points in $\mathbb{P}^2$
π_2 realizes S_A as the blow up of 3 points in $\mathbb{P}^{2\vee}$.

$A \sim A_2$ or $A \sim A_4 \;\; \Leftrightarrow S_A$ is reducible and the union ${}_qH \cup H_m$
$qm \neq 0$ so ${}_qH \cap H_m = \mathcal{L}_{q,m}$ is smooth for A_2
$qm = 0$ so ${}_qH \cap H_m = \pi^{-1}(q) \cup \pi^{-1}(m)$ for A_4.

$A \sim A_3$ or $A \sim A_5 \;\; \Leftrightarrow S_A$ has a unique singular point.
It is weak Del Pezzo with a node for A_3
It is weak Del Pezzo with a cusp for A_5 [Dolgachev].

The classification needs to be refined for unitary automorphisms.

2.5 Real and complex manifolds

The $\mathrm{SL}(3,\mathbb{C})$ structure already defines a cross product $\Lambda^2(\mathbb{C}^3) \to \mathbb{C}^{3\vee}$. We can reduce to the special unitary group $SU(3)$, which preserves the Hermitian product

$$\langle p, q\rangle = pq^* = p_0\bar{q}_0 + p_1\bar{q}_1 + p_2\bar{q}_2,$$

where $q^* = \overline{q}^\top$ defines the line orthogonal to p. These operations allow us to add a third map π:

$$\begin{array}{ccccc} & p^* \times \ell & = p^* \cap \ell & & \\ & \uparrow \pi & & & \\ & (p,\ell) & & & \\ \swarrow & & \searrow & & \\ p & & & \ell^* & \end{array}$$

This twistor projection* π is neither holomorphic nor anti-holomorphic.

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2.6 The Weyl group of SU(3)

A point if $\mathbb{F}$ should now be regarded as an orthogonal splitting

$$\begin{aligned} \mathbb{C}^3 &= \langle p\rangle \oplus \langle p\times \ell^*\rangle \oplus \langle \ell^*\rangle \\ &= L_1 \oplus L \oplus L_2 \end{aligned}$$

rather than the flag $L_1 \subset (L_1 \oplus L) = \ell$.

Having fixed an origin, write $\mathbb{F} = \mathrm{SU}(3)/\mathbb{T}$ where $\mathbb{T} = T^2$ is a maximal torus of $\mathrm{SU}(3)$, and $W = N(\mathbb{T})/\mathbb{T} \cong S_3$. This finite group acts on $\mathbb{F}$ permuting the three fibrations. In particular, the anti-holomorphic involution

$$j(p,\ell) = (\ell^*, p^*)$$

maps (L_1, L, L_2) to (L_2, L, L_1). Let $B \in \mathrm{SL}(3,\mathbb{C})$. If $B \in \mathrm{SU}(3)$ then $B \circ j = j \circ B$.

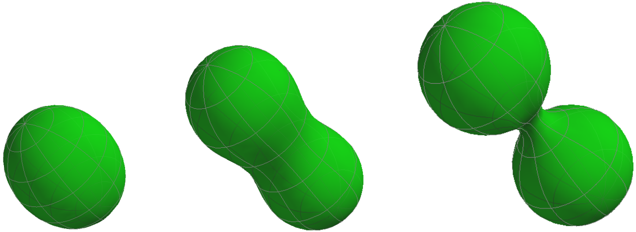
S_3 permutes the 6 invariant complex structures on $\mathbb{F}$, but preserves the almost complex structure $\mathbb{J}^-$ (up to sign). This fact allows one to generate harmonic maps of surfaces into $\mathbb{P}^2$ from holomorphic ones.

3.1 Twistor fibres and their deformations

Each fibre $\pi^{-1}(q) = \mathcal{L}_{q,q^*}$ is a $(1,1)$ curve. What is the image of a generic $(1,1)$ curve $\mathcal{L}_{q,m}$ under π? Let Φ denote the matrix $(qm)I - mq$. Then

$$\pi(\mathcal{L}_{q,m}) = \{z \in \mathbb{P}^2 : z\Phi z^* = 0\}$$

defines a 2-sphere $x_1^2 + x_2^2 + y_2^2 = r^2$ in an affine patch $\mathbb{R}^3 \subset \mathbb{C}^2$ with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Only the S^1 symmetry persists relative to the Fubini-Study metric. As r increases:



3.2 Projections of surfaces

The $(1,0)$ surface ${}_qH$ contains $\mathcal{L}_{q,q^*} = \pi^{-1}(q)$. The bijection

$$\pi : H_m \setminus \pi^{-1}(q) \longrightarrow \mathbb{P}^2 \setminus \{q\}$$

is not holomorphic, but endows $\mathbb{P}^2 \setminus \{q\}$ with a 'Fubini structure' J_q, i.e. an OCS J_q compatible with the Fubini Study metric on $\mathbb{P}^2$.

Then J_q is obtained from the standard complex structure J at $p \in \mathbb{P}^2$ by reversing sign on the tangent space to the line pq [Rawnsley]. Note that J_q, J correspond to sections of Λ^-, Λ^+ respectively.

For suppose that $x = (p, \ell) \in \pi^{-1}(q)$ corresponds to the splitting $\mathbb{C}^3 = L_1 \oplus L \oplus L_2$.

$$\begin{aligned} \Rightarrow \quad T_x^{1,0}(\mathbb{F}, \mathbb{J}^+) &\cong (L_1 \otimes \overline{L}) \oplus (L_1 \otimes \overline{L_3}) \oplus (L \otimes \overline{L_3}) \\ T_q^{1,0}(\mathbb{P}^2, J) &\cong (L \otimes \overline{L_1}) \oplus (L \otimes \overline{L_3}). \end{aligned}$$

3.3 Smooth diagonal (1,1) surfaces

A surface of bidegree $(1,1)$ in $\mathbb{F}$ will intersect a generic twistor fibre in 2 points.
Let $\lambda \in \mathbb{C} \setminus \{0,1\}$. Then

$$S_\lambda = S_{A_1} = \{(p,\ell) : p_0\ell_0 + p_1\ell_1 + p_2\ell_2 = 0 = p_1\ell_1 + \lambda p_2\ell_2\}$$

is a smooth toric 'hexagonal' Del Pezzo surface. It is j-invariant iff $\lambda \in \mathbb{R}$.

Main theorems.

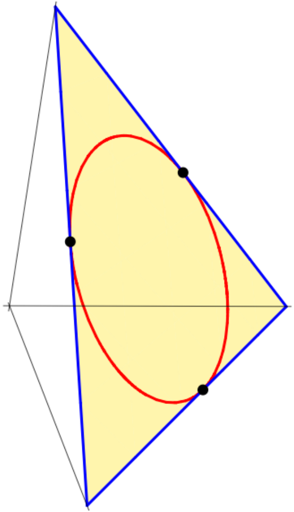
$\lambda \in \mathbb{R} \;\Rightarrow S$ contains infinitely many twistor fibres that π maps to a circle C_λ in $\mathbb{P}^2$.

$\lambda \in \mathbb{C} \setminus \mathbb{R} \;\Rightarrow S$ has no twistor fibres but is ramified over a smooth 2-torus in $\mathbb{P}^2$.

Corollary.

$\lambda \in \mathbb{R} \;\Rightarrow S \setminus \pi^{-1}(C_\lambda)$ has two components and $\mathbb{P}^2 \setminus C_\lambda$ admits a Fubini structure.

Does this produce an analogue of slice-regular functions?



The choice of a maximal torus give rise to

$$\begin{array}{rcl} \mu\colon \mathbb{P}^2 & \longrightarrow & \Delta \subset \mathbb{R}^3 \\ [q_0 : q_1 : q_2] & \longmapsto & (|q_0|^2, |q_1|^2, |q_2|^2)/\|q\|^2 \end{array}$$

The image of μ is the 2-simplex Δ.

The discriminant locus of S_λ maps to a point of Δ.

The locus $q_0 q_1 q_2 = 0$ is mapped onto $\partial\Delta$, which corresponds to $\lambda \in \mathbb{R}$.

Vertices correspond to $\lambda = 0, 1, \infty$.

Midpoints m correspond to $\lambda = -1, \frac{1}{2}, 2$, for which $C_\lambda = \mu^{-1}(m)$ is maximal.

3.5 Bounds on twistor fibres

Theorem. Suppose that S is an algebraic surface of $\mathbb{F}$ of bidegree (a, b). If $j(S) \neq S$ then S contains at most $a^2 + ab + b^2 - 1$ twistor fibres.

Corollaries.

If S has infinitely many twistor fibres then $a = b$.

If S_A has bidegree $(1, 1)$ it has $0, 1, 2$ or ∞ twistor fibres.

In the latter case, the strategy is to determine th connected set $S \cap j(S)$.

Let $A = \begin{pmatrix} \cdot & \beta & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 2 \end{pmatrix}$. Then

$$\begin{aligned} \beta = 2 \quad &\Rightarrow \quad S_A \text{ has no twistor fibre} \\ \beta = 2\sqrt{2} \quad &\Rightarrow \quad S_A \text{ has one twistor fibre} \\ \beta = 3 \quad &\Rightarrow \quad S_A \text{ has two twistor fibres.} \end{aligned}$$

3.6 $\mathbb{F}$ versus $\mathbb{CP}^3$

One expect the twistor spaces $\mathbb{F}$ and $\mathbb{CP}^3$ to have features in common, given the birational contact correspondence between them.

Quadrics and cubic surfaces in $\mathbb{CP}^3$ have been classified up to conformal equivalence (using the group $\mathrm{SL}(2,\mathbb{H})$) relative to the Penrose fibration $\mathbb{CP}^3 \to S^4$:
Any smooth cubic in $\mathbb{CP}^3$ has 27 lines with at most 6 skew. But it can have at most 5 twistor fibres over S^4 [Armstrong-S]. How many can lie inside N? [Hughston].
A smooth quartic in $\mathbb{CP}^3$ can have up to 64 lines [Segre]. At most 16 can be skew [Nikulin]. $\exists$ a j-invariant quartic in $\mathbb{CP}^3$ with 8 twistor fibres [Viaclovsky-S].

A surface of bidegree $(2,2)$ in $\mathbb{F}$ is also of K3 type. The involutions defined by π_1, π_2 do not commute and their product generates an infinite group [Wehler]. But how many twistor lines can we find?!