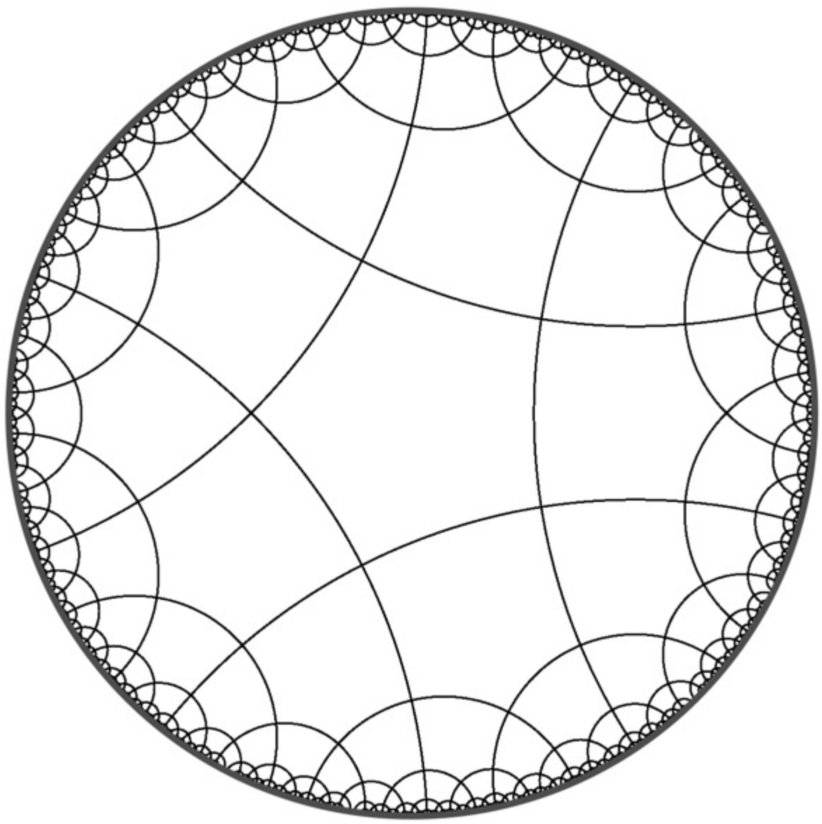
GEOMETRY I

LECTURE NOTES FOR 2012-13



1 Preliminaries

1.1. Course summary

A mixture of elementary and abstract ideas…

First part: Euclidean plane geometry

Postulates for distances, lines, angles and similar triangles.

Sums of angles, Pythagoras' theorem, regular polygons.

Perpendicular bisectors, parallel lines, transversals.

Circles. Tangents, inscribed angles.

Second part: "Higher geometry"

Classification of isometries of the plane.

A bit of analytic geometry in 2 and 3 dimensions.

The sphere. Spherical triangles.

Hyperbolic geometry (which is like that on a sphere of radius $\sqrt{-1}$)

1.2. Historical perspective

600BC: Thales' theorems. Ratios of intercepting line segments, angles subtended inside a circle.

550BC Pythagoras' theorem. Gives a simple construction of irrational lengths. But how can one list triples of integers a, b, c so that $a^2 + b^2 = c^2$?

300BC: The Elements. Euclid's masterpiece (13 books, 6 on plane geometry) includes the postulates:

E1. A straight line segment can be drawn joining any two points.

E2. Any straight line segment can be extended indefinitely to a straight line.

E3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

E4. All right angles are congruent.

E5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines must intersect each other on that same side if extended far enough.

The fifth requires one to imagine infinity. Repeated attempts were made to prove that it either followed from the other postulates or is not "inevitable". It is equivalent to

P5 (Playfair's 1795 version: the Parallel Postulate). Given a line ℓ and a point P not on ℓ there is one and only one line through P that does not meet ℓ.

We shall investigate *hyperbolic geometry* in which there infinitely many lines through P that do not meet ℓ, and yet Euclid's other postulates hold. It was discovered through work of

- **Gauss (1777–1855)**
- Lobachevsky (1792–1856)
- Bolyai (1802–1860)
- Beltrami (1835–1900)

and others. Far-reaching generalizations were developed by

- Riemann (1826–1866)

who founded what is now called Riemannian geometry that was studied by

- Clifford (1845–1879, he was at King's as a teenager), and
- **Einstein (1879–1955)**

to formulate the theory of General Relativity. Non-Euclidean geometry is nowadays an essential tool in physical theories that attempt to unite gravitation with other fundamental forces. Returning to E1, one is reminded that to construct a straight line in practice one uses a DIY laser, and photons travel along *geodesics* in space-time.

1.3. Logical arguments

A cornerstone of *The Elements* is the rigorous use of laws of deduction, starting from small number of postulates. There is a sense in which the whole of mathematics has developed as the study of such axiomatic systems. Here is a simple example, based on the following axiom (that we shall adopt):

Given two distinct points, there is one and only one line passing through them.

From this, we can deduce the

Proposition. *Any two distinct lines ℓ, m have at most one point in common.*

Proof. Suppose (just for a moment) that P, Q are distinct points that lie on both ℓ and m. Then both ℓ and m pass through P and Q, contradicting the axiom. So our supposition cannot hold, and we have proved the Proposition. $\square$

This method is called "proof by contradiction" or "reductio ad absurdam" (RAA). Here is a famous algebraic example:

Proposition. $\sqrt{2}$ is irrational: there are no integers p, q such that $\sqrt{2} = p/q$.

Proof. Suppose (just for the moment) that $\sqrt{2} = p/q$ with p, q integers. We can also suppose that p, q are not both even, by first cancelling any factors of 2 top and bottom. Then

$$p^2/q^2 = 2, \qquad \text{so} \qquad p^2 = 2q^2,$$

and p^2 is even. It follows that p itself must also be even (why?). So $p = 2m$ for some integer m, and

$$4m^2 = 2q^2, \qquad \text{so} \qquad 2m^2 = q^2,$$

and q^2 and q are also even. But now *both* p and q are even, which contradicts our supposition. Therefore we cannot write $\sqrt{2}$ as a rational number p/q. □

1.4. Maps and their inverses

In mathematics, any object a can always be thought of as belonging to some *set* $\mathscr{A} = \{a, \ldots\}$, chosen to best reflect the characteristics of the element a we are interested in.

A *mapping*, *map*, or *function* $f\colon \mathscr{A} \to \mathscr{B}$ between two sets $\mathscr{A}, \mathscr{B}$ is an assignment of a point $f(a)$ of $\mathscr{B}$ to every point a of $\mathscr{A}$:

$$a \in \mathscr{A} \quad \Rightarrow \quad f(a) \in \mathscr{B}.$$

(Geometers say "point" instead of "element" because they regard any set as a "space" of some sort!) We also write

$$a \mapsto f(a).$$

The set of elements of $\mathscr{B}$ that "come from" $\mathscr{A}$ is called the *image* of f:

$$\operatorname{Im} f = \{b \in \mathscr{B} : b = f(a) \text{ for some } a \in \mathscr{A}\} = \{f(a) : a \in A\},$$

more usefully denoted $f(\mathscr{A})$.

The mapping f is called

- *injective* or *one-to-one* if $f(a) = f(a') \;\Rightarrow\; a = a'$,f for all a, a' in $\mathscr{A}$;
- *sujective* or *onto* if $f(\mathscr{A}) = \mathscr{B}$;
- *bijective* if both the above are true.

Let us discuss these concepts in more detail.

A map $f\colon \mathscr{A} \to \mathscr{B}$ is injective if no two elements of $\mathscr{A}$ map to the same element of $\mathscr{B}$.

A map $f\colon \mathscr{A} \to \mathscr{B}$ surjective if every element of $\mathscr{B}$ is "f of something in $\mathscr{A}$". If this is not true, all we have to do is replace $\mathscr{B}$ by the *image* $f(\mathscr{A})$, since f always defines a surjective mapping $\mathscr{A} \to f(\mathscr{A})$.

If $\mathscr{A}$ and $\mathscr{B}$ are finite sets (which in this course they won't usually be) of size $|\mathscr{A}|$ and $|\mathscr{B}|$, it is easy to see that

- f is injective $\Rightarrow$ $|\mathscr{A}| \leqslant |\mathscr{B}|$;

- f is surjective $\Rightarrow$ $|\mathscr{A}| \geqslant |\mathscr{B}|$;
- f is bijective $\Rightarrow$ $|\mathscr{A}| = |\mathscr{B}|$.

Moreover, if $|\mathscr{A}| = |\mathscr{B}|$ then f is injective iff it is surjective, and clearly there exists a bijective map $f\colon \mathscr{A} \to \mathscr{B}$. More generally, two (possibly infinite) sets are said to have the same *cardinality* if and only if there exists a bijection from either one to the other.

A bijective mapping (also called *bijection*) sets up a correspondence in which each element of $\mathscr{A}$ can be paired with exactly one element of $\mathscr{B}$ and vice versa. In this case, given $b \in \mathscr{B}$, there exists a unique element in $a \in \mathscr{A}$ such that $f(a) = b$. We denote this element a by $f^{-1}(b)$, so

$$f(f^{-1}(b)) = b.$$

In this way, we have defined a mapping $\mathscr{B} \to \mathscr{A}$ called the *inverse* of f, and any element $a \in \mathscr{A}$ satisfies

$$f^{-1}(f(a)) = a.$$

If f is injective then it defines a bijection from $\mathscr{A}$ to $f(\mathscr{A})$, and f^{-1} is defined as a mapping $f(\mathscr{A}) \to \mathscr{A}$. But even if f is not bijective, one uses the notation

$$f^{-1}(b) = \{a \in \mathscr{A} : f(a) = b\}$$

to indicate the *subset* of all elements of $\mathscr{A}$ mapping to b (it is called the *inverse image* of b and it may be that $f^{-1}(b) = \varnothing$). In this context, the symbol "f^{-1}" has no meaning on its own.

Examples. 1. Let $\mathscr{A} = \{1, 2, 3\}$. There are 3^3 different mappings f from $\mathscr{A}$ to $\mathscr{A}$ because we can freely choose the images $f(1), f(2), f(3)$. One possibility is

$$f\colon \quad 1 \mapsto 1,\;\; 2 \mapsto 1,\;\; 3 \mapsto 1;$$

this is an example of a *constant mapping* – the image is a singleton set. A bijection from $\mathscr{A}$ to itself is called a *permutation*, and there are only 6 of these because once we know $f(1)$, there are only two choices for $f(2)$ and (then) none for $f(3)$. One example is

$$f\colon \quad 1 \mapsto 1,\;\; 2 \mapsto 2,\;\; 3 \mapsto 3;$$

this is called the *identity mapping* and can always be considered when $\mathscr{A} = \mathscr{B}$. More generally, a set of size n has a total $n!$ permutations, including the identity one.

2. If $\mathscr{A} = \mathscr{B} = \mathbb{Z}$ is the set of integers then the map d (for "doubling") that sends n to $2n$ is injective. If E is the set of even integers then d defines a *bijective mapping* from $\mathbb{Z}$ to E even though E is a proper subset of $\mathbb{Z}$. Paradoxically, it is even possible to define a bijection from $\mathbb{Z}$ (or indeed $\mathbb{N}$) to the set $\mathbb{Q}$ of all rational numbers. Consequently the infinite sets $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ all have the same cardinality.

1.5. Real-valued functions

The previous theory is most familar when $\mathscr{A} = \mathscr{B} = \mathbb{R}$, in which case a mapping is an ordinary function. In this section, we present some examples of functions to illustrate a geometrical way of thinking.

Examples. 1. If $f(x) = x^2$ then the image of f is

$$f(\mathbb{R}) = \{x \in \mathbb{R} : x \geqslant 0\},$$

the set of non-negative numbers, also denoted $[0, \infty)$. Since this is not the whole of $\mathbb{R}$. we see that f is not onto; for example there is no real number x such that $f(x) = -1$. Nor is f injective; for example $f(2) = f(-2) = 4$ and we can write $f^{-1}(4) = \{-2, 2\}$ (even though f^{-1} does not exist).

2. The function $f(x) = x^3$ is, by contrast, bijective – any negative number has a negative cube root. There is a notational ambiguity, since both the following are correct:

$$f^{-1}(8) = 2 \quad \text{or} \quad f^{-1}(8) = \{2\}$$

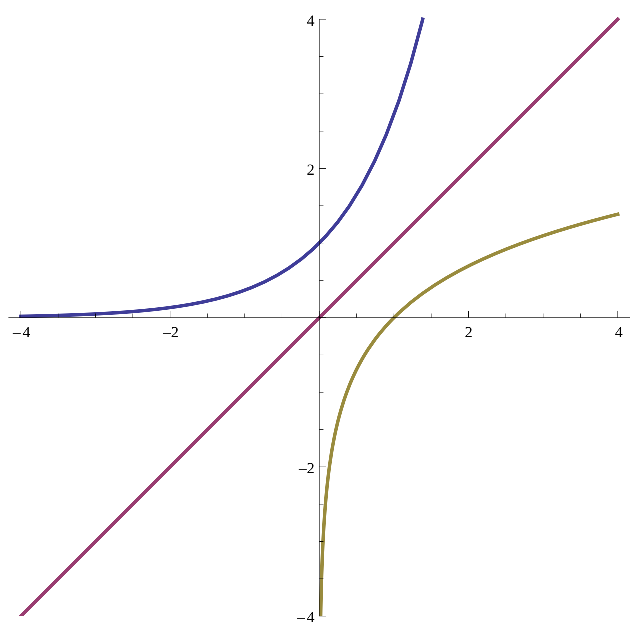
(One of the few occasions in which curly brackets don't matter!)

3. If $f(x) = e^x$ then the image

$$f(\mathbb{R}) = \{x \in \mathbb{R} : x > 0\} = (0, \infty)$$

is the set of positive numbers. We can therefore regard f as a bijective mapping from $\mathbb{R}$ to $(0, \infty)$. The inverse of this mapping is (by definition) the natural logarithm:

$$f^{-1}(c) = \log_e c = \ln c, \qquad c > 0.$$



It is clear from the (upper or blue) graph of f that it is bijective, but it is a bit harder to *prove* that f is bijective and therefore that the logarithm function really does exist. (In fact, f is injective because $x < y \Rightarrow e^x < e^y$; surjectivity requires a more advanced result about *continuous* functions.)

In general, if the inverse of a function f exists, the graph of f^{-1} is the reflection of that of f in the diagonal line $x = y$. This is because, if $y = f(x)$ then the point

$$(y, f^{-1}(y)) = (f(x), x)$$

is obtained by swapping the coordinates of (and so reflecting) the point $(x, f(x))$ on the graph of f. Overleaf the graph of the inverse function $x \mapsto \ln x$ is the lower one.

This is a powerful geometric aid: for example, it is now clear (from seeing the graph of e^x) that there is no *real* number such that $e^x = x$. For if there were, the point (x, e^x) would lie on both the graph of f and the line $x = y$.

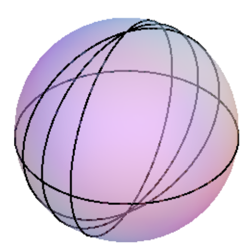
2 Principles of plane Euclidean geometry

We shall adopt an informal set of axioms developed by G. Birkhoff in the 1930's, consistent with Euclid's, to describe geometry in two dimensions. Athough these axioms are satisfied for the usual system in which points can be represented by Cartesian coordinates (x, y), we should not at this point assume that lines, distances and angles have their usual meaning.

In addition to the set $\mathbb{R}$ of real numbers and its various properties such as order, we shall suppose that we are given a set of points called the *plane* and distinguished subsets called *lines*. We shall also suppose that we can associate to any two points a number that represents their *distance* apart, and to any two lines two numbers (differing by π) that represent the *angle* at which they meet (see diagram on page 9).

These concepts are clarified by means of five postulates or axioms, labelled B1–B5, "B" for Birkhoff. They are reproduced below with the lecturer's wording. Since lines are subsets of the plane and points belong to lines, we can use the language of sets and write $A \in \ell$ to assert that a point A lies on a line ℓ. This may seem self-evident, but it was not a feature of earlier axiomatic approaches.

These postulates are all valid in the Cartesian setting, but in order to appreciate their significance, it is instructive to ask whether each one holds for



- the Fano plane, described in Sheet 1, containing just 7 points and 7 lines;
- the surface of a sphere in which lines are great circles (each one is the intersection of the sphere with a plane passing though its centre, like the ones shown).

2.1. Lines and distance

The first postulate is

B1. Given any line ℓ, there is a bijection $f\colon \ell \to \mathbb{R}$ that measures distances.

The existence of a such a map f allows us to label each point of a line unambiguously with a real number. We can choose to put 0 where we want, but distances along the line must coincide with distances in $\mathbb{R}$. Therefore the distance between two points A and B must equal $|f(A)-f(B)|$; we shall denote this distance $|AB|$ and pronounce it the "length AB". Once we have understood the consequences of having such a bijective correspondence, we need never refer to f again.

B1 fails for the Fano plane because there cannot be a bijection between a finite set (like $\ell_1=\{1,2,3\}$) and an infinite set (like $\mathbb{R}$). It fails for the sphere (with the usual notion of distance measured along the surface) because any two points on a great circle are never more than a distance πr apart (r is the radius of the sphere), whereas points in $\mathbb{R}$ can be as far apart as we please.

Postulate B1 yields the notion of "betweenness" for points on a line, something that is missing in Euclid's treatment. Given $A, B \in \ell$, we can speak of

- the *line segment* AB that consists of all points $C \in \ell$ for which

$$f(A) \leqslant f(C) \leqslant f(B) \quad \text{or} \quad f(B) \leqslant f(C) \leqslant f(A).$$

- the half-line or *ray* $\overrightarrow{AB}$ that consists of all points on the line ℓ that contains AB and all the points on the other side of B with respect to A.

It follows that

$$\ell = \overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}, \qquad AB = \overrightarrow{AB} \cap \overrightarrow{BA}.$$

Our second postulate strengthens Euclid's first:

B2. There is exactly one line passing through two given points.

It follows (see page 2) that *two distinct lines ℓ, m have at most one point P in common.* In set theory notation, we can write $\ell \cap m = \{P\}$.

B2 is true for the Fano plane. It fails for the sphere because antipodal points (meaning opposite points on a diameter) are contained in infinitely many great circles. However it *is* true if we restrict attention to (say) the upper hemisphere excluding the equator.

2.2. Angles

These are formed by two separate rays $\overrightarrow{OA}, \overrightarrow{OB}$ with the same starting point O.

B3. If $\mathscr{R}$ is the set of rays with a given startpoint, there is a bijection $g\colon \mathscr{R} \to [0, 2\pi)$ that measures angles in radians.

In the Cartesian set-up, we can place the origin at O and take $\overleftrightarrow{OA}$ to be the x-axis. Rays are then represented by points of a circle of radius 1 centre O, and we can take g^{-1} to be the mapping $\theta \mapsto (\cos\theta, \sin\theta)$ (or, $\theta \mapsto (\cos\theta, -\sin\theta)$ if we prefer to move clockwise.) Radians are chosen to make this formula simple, but recall that

$$2\pi \text{ radians } = 360^\circ = \text{one full turn}$$

2π (or any integer multiple of 2π) is excluded because it defines the same angle as 0. In general, B3 tells us that the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, denoted by $\angle AOB$, equals the number

$$\theta = \left| g(\overrightarrow{OA}) - g(\overrightarrow{OB}) \right|.$$

With a different choice of the ray that g maps to 0, this angle could become $2\pi - \theta$.

𝔑𝔬𝔱𝔢. Birkhoff's third postulate actually has a second part, involving continuity, which guarantees that rays intersect the opposite side of any triangle in a continuous way.

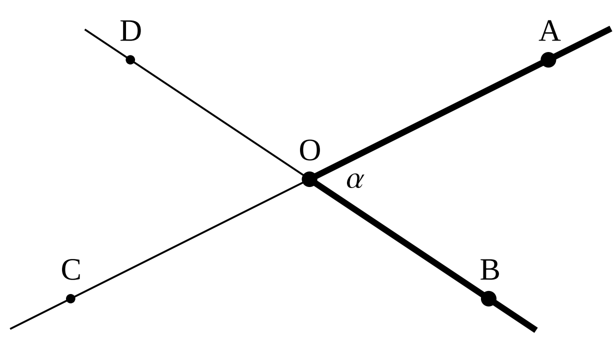
By a *straight angle*, we shall mean the case in which the two rays form different parts of the *same* line.

B4. All straight angles correspond to π radians.

B3 fails for the Fano plane because there are only three rays emanating from each point. Both B3 and B4 are however valid for the sphere – we can measure angles at a point on the surface by considering the tangent vectors to great circles (think what happens at the north pole, which is equivalent to any other point on the sphere).

Let us consider a consequence of B3 and B4. Since the two rays $\overrightarrow{OA}, \overrightarrow{OB}$ can both be extended unambiguously to lines (by B2) that "look like" $\mathbb{R}$ (by B1), we can choose points C, D on the "other sides" of O so as to specify four angles, namely

$$\angle AOB, \quad \angle BOC, \quad \angle COD, \quad \angle DOA.$$



In the picture, angle $\alpha = \angle AOB$ is the smaller of the two angles formed by our two rays, and we can also denote it by $\angle BOA$ (in this notation there is no sign and the exact positions of A, B are irrelevant).

Postulate B4 reinforces our understanding of angle measurement. Since $\alpha = \angle AOB$, and $\angle AOC$, $\angle BOD$ are straight, it tells us that

$$\angle BOC = \pi - \alpha = \angle DOA = \angle AOD,$$

and it follows that $\angle COD = \alpha$. This is the well-known statement that *opposite angles are equal*.

A special case occurs if $\alpha = \pi/2$, for then all four angles are equal. A line is said to be *perpendicular* to another line if both intersect at a point at which all four angles are equal. We can deduce from B3 and B4 the

Proposition. *If P lies on ℓ there is exactly one line through P perpendicular to ℓ.*

𝔚𝔞𝔯𝔫𝔦𝔫𝔤. We have not yet established the analogous statement in which P is a point *not* on ℓ. This is true, but does not follow so quickly from the postulates.

2.3. Similar triangles

Three points A, B, C that are not collinear (meaning that they do not lie on some line) determine a triangle whose sides are the line segments BC, CA, AB. By B2, the 3 sides cannot intersect in points other than A, B, C, which we call the *vertices* of the triangle. We denote the lengths of the sides by

$$a = |BC|, \quad b = |CA|, \quad c = |AB|.$$

The corresponding angles are

$$\alpha = \angle A = \angle BAC, \quad \beta = \angle B = \angle CBA, \quad \gamma = \angle C = \angle ACB,$$

and we shall assume that these angles are all less than π. (There is a subtle point here, related to the note on page 9, which we shall gloss over.)

Definition. Two triangles are similar if the 3 angles of one equal the 3 angles of the other, and the corresponding sides are proportional.

Accordingly, to say that two triangles $\triangle ABC$, $\triangle A'B'C'$ are similar means that their respective angles and lengths are related by *all* of the following equations:

$$\boxed{\begin{array}{l} \alpha' = \alpha, \quad \beta' = \beta, \quad \gamma' = \gamma, \\ a' = ka, \quad b' = kb, \quad c' = kc, \end{array}}$$

for some real number $k > 0$. We write $\triangle ABC \sim \triangle A'B'C'$, or $\triangle ABC \overset{k}{\sim} \triangle A'B'C'$ if it is important to mention k. If the factor k of proportionality equals 1 then the triangles are said to be *congruent*.

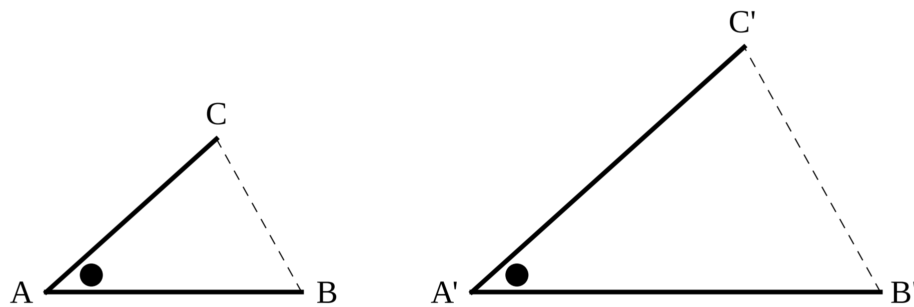
Our last postulate asserts that we can draw this conclusion by assuming only half the equations:

B5 (the "SAS" rule). Two triangles are similar if an angle of one equals an angle of the other and the sides forming these angles are proportional in length.

If the triangles are again $\triangle ABC$ and $\triangle A'B'C'$, this means that

$$\begin{aligned} &\alpha' = \alpha, \quad b' = kb, \quad c' = kc \\ \Rightarrow \quad &\beta' = \beta, \quad \gamma' = \gamma, \quad a' = ka. \end{aligned}$$

The assumption involving k can be replaced by $b'/b = c'/c$, and then (by way of conclusion) this number will also equal a'/a. The drawing is based on straight lines we are familiar with:



B5 is a postulate: in this course, never try to prove it!

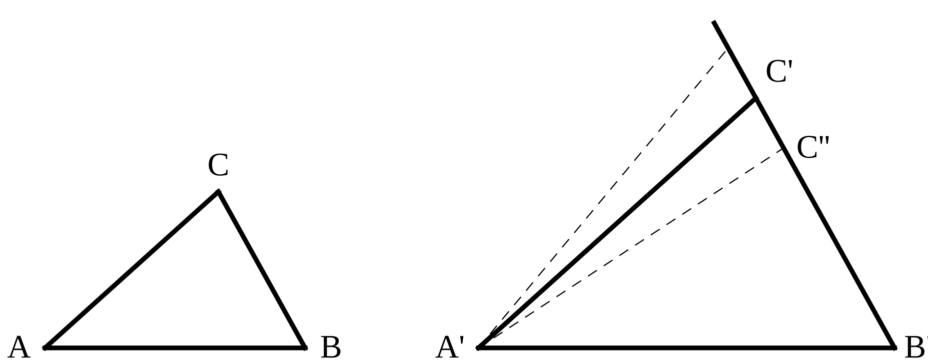
𝔚𝔞𝔯𝔫𝔦𝔫𝔤. B5 is false if the two sides are not those adjacent to the angle (Sheet 1).

3 Similarity theorems

3.1. The AA rule

Theorem (AA). *Two triangles are similar if two angles of one equal two angles of the other.*

Proof. We can label the triangles $\triangle ABC, \triangle A'B'C'$ so that $\angle A = \angle A'$ and $\angle B = \angle B'$. Suppose that $c' = |A'B'| = k|AB| = kc$ with $k > 0$. If we knew that $|B'C'|$ were equal to $k|BC|$ then the two triangles would be similar by B5. In any case, by B1, we know that there exists a point C'' on $\overrightarrow{B'C'}$ for which $|B'C''| = k|BC|$:



Triangles $\triangle ABC$, $\triangle A'B'C''$ now share proportional lengths adjacent to a common angle $\angle B' = \angle B$. They are therefore similar by B5 and as a consequence,

$$\angle B'A'C'' = \angle A.$$

By hypothesis, this angle also equals $\angle A'$. The rays $\overrightarrow{A'C'}$, $\overrightarrow{AC''}$ must now coincide since (by B3) they are determined by their angles with $\overrightarrow{A'B'}$. Therefore

$$C' = C''$$

($\overrightarrow{AC'}$ can't meet $\overrightarrow{B'C'}$ in more than one point), and $\triangle ABC$ *is* similar to $\triangle A'B'C'$. $\square$

3.2. Isosceles triangles

Recall that two triangles $\triangle ABC$, $\triangle A'B'C'$ are said to be similar if corresponding angles are equal, and corresponding sides are proportional. The notion "corresponding" involves the bijection

$$A \mapsto A', \quad B \mapsto B', \quad C \mapsto C',$$

assuming the vertices are ordered consistently.

Theorem. *If two sides of a triangle are equal then the angles opposite these sides are equal.*

Proof. Suppose that $b = a$. The idea is to re-label the vertices according to the scheme

$$A' = B, \quad B' = A, \quad C' = C,$$

so that $\alpha' = \beta$ and $\beta' = \alpha$. We can now legitimately apply B5 with $k = 1$ to $\triangle ABC$ on the one hand, and $\triangle A'B'C' = \triangle BAC$, on the other, considered as two different triangles. They have a common angle $\angle C = \angle C'$, whose adjacent sides are proportional with $k = 1$ since $a = b = a'$ and $b = a = b'$. The SAS rule allows us to conclude that $\triangle ABC$ and $\triangle BCA$ are similar (indeed, congruent), and in particular $\angle A = \angle A'$ and $\angle B = \angle B'$. Thus, $\angle A = \angle B$ or $\alpha = \beta$. $\square$

A triangle satisfying the hypothesis of Theorem 1 is called *isosceles*.

Theorem. *If two angles of a triangle are equal then the opposite sides are equal in length.*

Proof. This result is the converse of Theorem 1, and is proved by applying AA in place of SAS.

3.3. The SSS rule

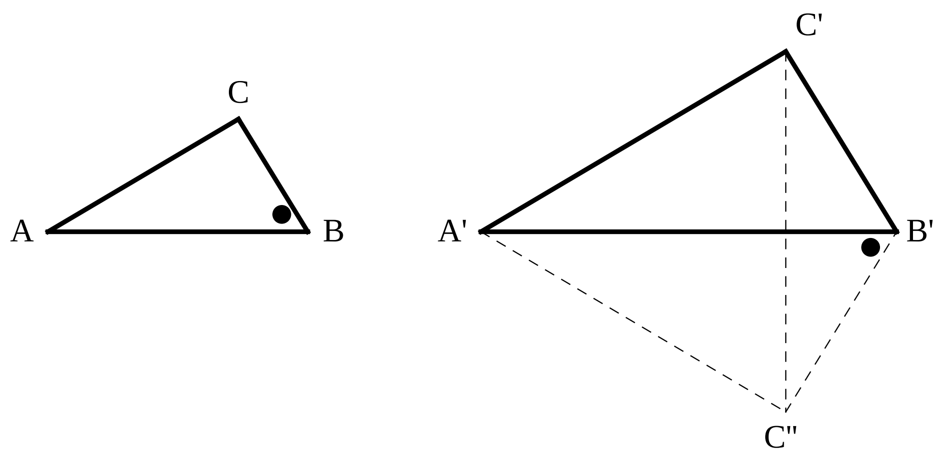
This section is devoted to establishing another known criterion for similarity. The proof is longer and is based on isosceles triangles.

Theorem (SSS). *Two triangles $\triangle ABC$, $\triangle A'B'C'$ are similar if their respective sides are proportional.*

In symbols, this means

$$a' = ka, \quad b' = kb, \quad c' = kc \quad \text{for some } k > 0$$
$$\Rightarrow \quad \alpha = \alpha', \quad \beta = \beta', \quad \gamma = \gamma'.$$

Proof. We begin by constructing a triangle $\triangle A'B'C''$ such that $\angle A'B'C'' = \angle ABC$ and $|B'C''| = |B'C'| = k|BC|$. The diagram assumes that $\beta = \angle ABC < \pi/2$, but the following proof works without this assumption.



By construction, $\triangle ABC$ and $\triangle A'B'C''$ share a common angle, and the sides adjacent to this angle are proportional. If follows, by the SAS rule, that they similar, and this tells us that

$$\angle A'C''B' = \angle ACB \quad \text{and} \quad |A'C''| = k|AC|.$$

Since $|A'C'| = k|AC|$, Theorem 1 tells us that $\triangle A'C'C''$ is isosceles and its angles at C', C'' are equal. The same argument applies to $\triangle B'C'C''$, so its angles at C', C'' are also equal. Now $\angle A'C'B'$ is determined by $\angle A'C'C''$ and $\angle C''C'B'$ (in the diagram, it is their sum). Similarly for $\angle A'C''B'$, so

$$\angle A'C'B' = \angle A'C''B'.$$

It follows that from the SAS rule (with this common angle) that $\triangle AB'C' \sim \triangle A'B'C''$. But $\triangle A'B'C'' \sim \triangle ABC$, so we can conclude that $\triangle A'B'C' \sim \triangle ABC$, which is what we want. □

The last step of the proof makes us of the fact that if $\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3$ are three triangles then

- $\mathscr{T}_1 \sim \mathscr{T}_2, \quad \mathscr{T}_2 \sim \mathscr{T}_3 \quad \Rightarrow \quad \mathscr{T}_1 \sim \mathscr{T}_3.$

This follows easily from the definition of similarity, as do the following properties:

- $\mathscr{T}_1 \sim \mathscr{T}_1,$
- $\mathscr{T}_1 \sim \mathscr{T}_2 \quad \Rightarrow \quad \mathscr{T}_2 \sim \mathscr{T}_1,$

valid for all triangles $\mathscr{T}_1, \mathscr{T}_2$. A relationship (between objects in a set) like $\sim$ that satisfies all three properties is called an *equivalence relation*. We shall meet more examples.

3.4. Angles sum to 180^o!

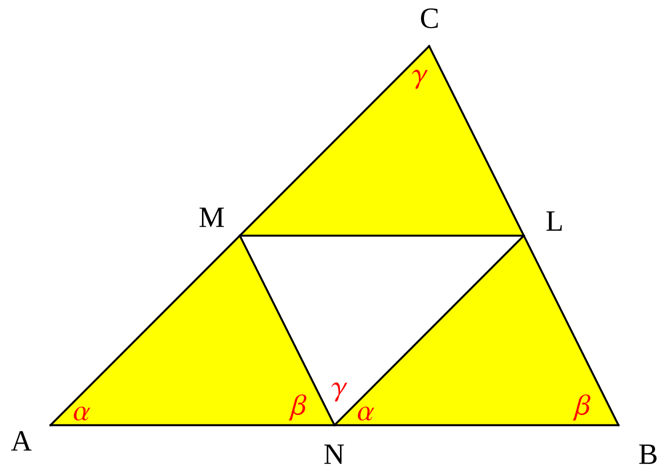
Recall that the three angles of a triangle are by assumption all less than π. We are finally in a position to prove

Theorem. *The angles of a triangle add up to π radians.*

Proof. Given $\triangle ABC$, let L, M, N be the respective midpoints of the sides opposite A, B, C. For example, $|BL| = |LC|$, and the existence of L is guaranteed B1. We have already seen (on Sheet 2, using B5) that the coloured triangles (with their vertices in the obvious order) are similar to $\triangle ABC$. In particular

$$\angle A = \angle BNL, \qquad \angle B = \angle ANM,$$

as indicated.



The middle triangle has all its lengths half as big as those of $\triangle ABC$. So applying SSS (with $k = \frac{1}{2}$) we deduce that $\triangle LMN \sim \triangle ABC$. The vertices are listed in the correct order, so it is $\angle LNM$ that corresponds to $\angle C$.

We know (by B3) that the straight angle $\angle ANB$ is the sum $\alpha+\beta+\gamma$ of the three angles shown, which (by B4) must equal π radians. □

If a triangle has three equal sides it is called *equilateral*. By SSS, all its angles are equal.

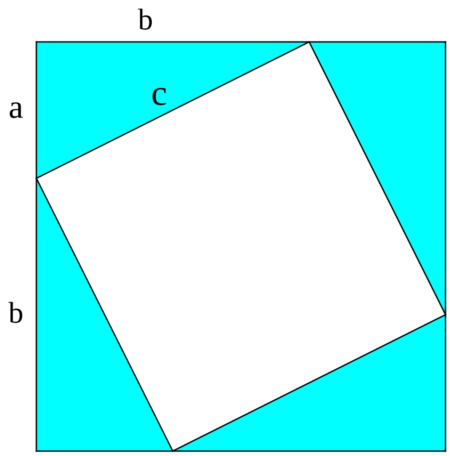
Corollary. *An equilateral triangle has all its angles equal to $\pi/3$, and any two are similar.*

4 Pythagoras' theorem

In a triangle ABC with a right angle at C, the (length c of the) side opposite the right angle is called the *hypotenuse*. The following fundamental theorem lies at the heart of algebra, geometry and calculus.

Theorem (Pythagoras). *If* $\angle BCA = \pi/2$ *then* $a^2 + b^2 = c^2$.

Here is a "binomial proof" that relies on the concept of area (which we shall not however develop in this course). Four congruent right-angled triangles are arranged cyclically around the inside of a square of length $a + b$:

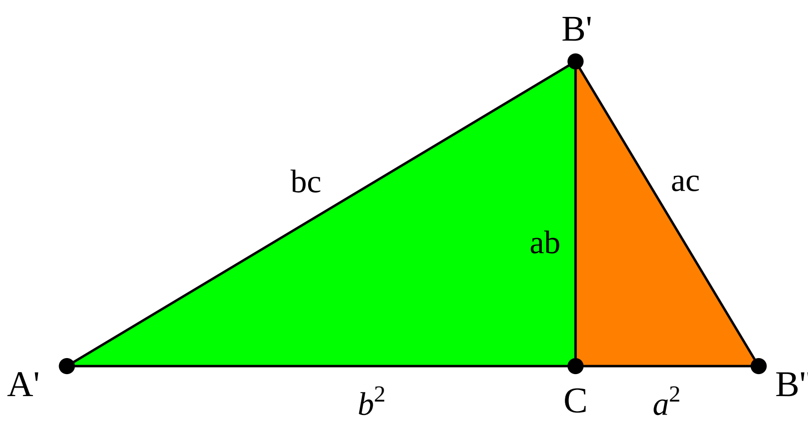


$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^2 = c^2 + 4 \cdot \tfrac{1}{2}ab$$

4.1. Two more proofs

Proof using B5. Suppose that $\triangle ABC$ has lengths $a, b, c,$ and a right angle at C. Extend (or diminish) the segments CA, CB to CA', CB' by a factor b so as to form the green triangle. Then scale by a factor a so as to construct the orange triangle $\triangle CB''B'$ with the ray $\overrightarrow{CB'}$ bisecting the straight angle $\angle A'CB''$. By B5 (SAS), the green and orange triangles have hypotenuses of length bc and ca respectively.



The big triangle now has angles $\angle A'$, $\angle B''$ in common with each of the smaller ones, and is therefore similar to $\triangle ABC$ by the AA rule. From the sides adjacent to B', the factor of proportionality must be $k = c,$ and it follows that

$$b^2 + a^2 = c \cdot c,$$

by the definition of similarity. □

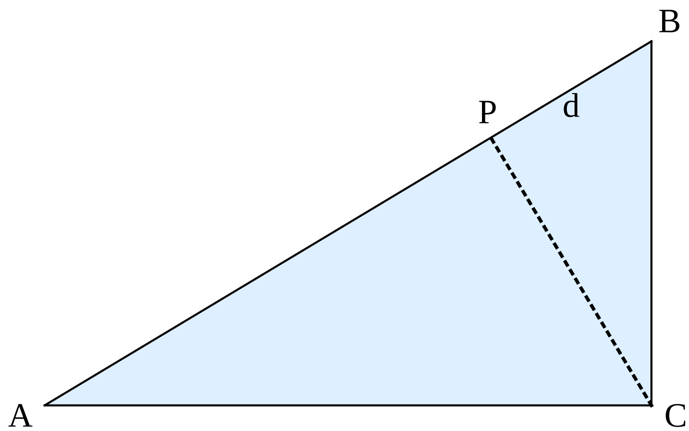
Here is a variant of this argument in which we construct a similar diagram but working *within* the given triangle ABC. It involves dropping a perpendicular from C to AB so that the two angles at P are $\pi/2$. It follows from the AA rule that the two smaller triangles are both similar to $\triangle ABC$, and so

$$\frac{d}{a} = \frac{a}{c}, \qquad \frac{c-d}{b} = \frac{b}{c},$$

where $d = |PB|$. Then

$$cd = a^2, \qquad c(c-d) = b^2,$$

again giving $c^2 = a^2 + b^2$.



The problem with this proof is that as yet we have no direct way of constructing the perpendicular CP. We could fix P by requiring that $|PB| = a^2/c$ (as was done to get the computer to draw the picture!). In this case $\angle CPB$ *will* be $\pi/2$, but then the proof is very similar to the one we have already given above.

4.2. Perpendicular bisectors

Let AB be a line segment with midpoint M, so that $|AM| = |MB|$.

Definition. The line m passing through M and making angles of $\pi/2$ with $\overleftrightarrow{AB}$ is called the *perpendicular bisector* of AB.

We know that M exists by B1, and m is unique since rays emanating from a point are completely determined by the angle they make (B3).

Lemma (on perpendicular bisectors). The perpendicular bisector m of AB consists of exactly those points equidistant from A and B, so in symbols:

$$m = \{P : |AP| = |BP|\}.$$

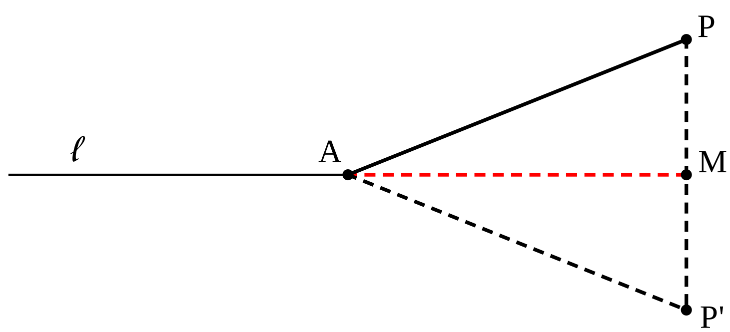
Proof. To prove that two sets are equal, we need to show that an element of one belongs to the other *and* vice versa.

If $P \in m$ then $\triangle AMP$ and $\triangle BMP$ are congruent by SAS (with equal sides adjacent to right angles). It follows that $|AP| = |BP|$. (One could also use Pythagoras' theorem here, although the first argument is more direct.)

Conversely, if $|AP| = |BP|$ then $\triangle AMP$ and $\triangle BMP$ are similar by SSS so all their angles are equal. In particular, $\angle AMP = \angle BMP$, so both are $\pi/2$. Therefore, $\overleftrightarrow{MP}$ is the perpendicular bisector of AB. □

Theorem (on the existence of perpendiculars). *Given any line ℓ and a point P, there exists exactly one line m containing P and meeting ℓ at an angle of $\pi/2$.*

Proof. If $P \in \ell$ this is true by B3. So assume that $P \notin \ell$. Choose any point $A \in \ell$ and consider the segment PA. By B3, we can construct a ray AP' so that ℓ bisects $\angle PAP'$. We can also choose P' so that $|AP| = |AP'|$:



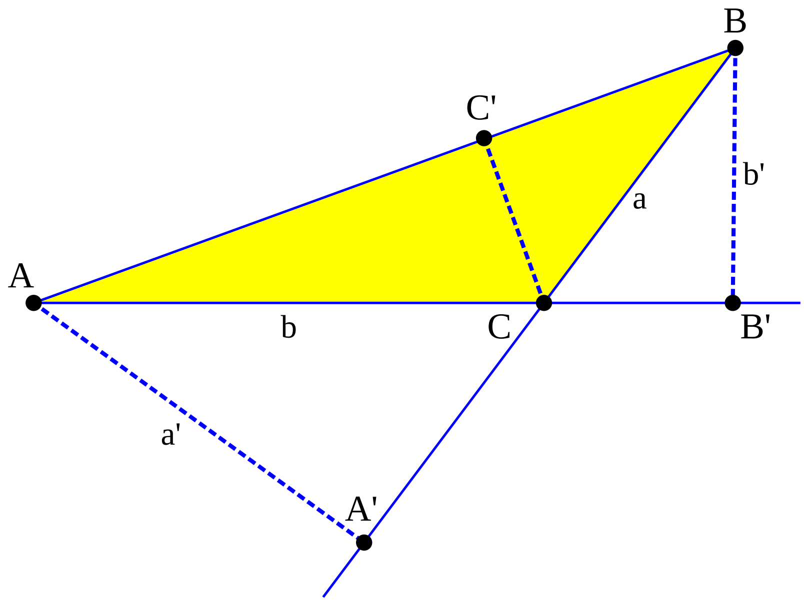
Consider the midpoint M of PP'; we do not yet know that AM is part of ℓ. But, by SSS, $\triangle AMP$ and $\triangle AMP'$ are similar and so $\angle PAM = \angle P'AM$. It follows $\overleftrightarrow{AM}$ *does* bisect $\angle PAP'$ and so equals ℓ. It is also the perpendicular bisector of PP' (by the lemma or because $\angle AMP = \angle AMP'$). Therefore ℓ and PP' meet at $\pi/2$ and $\overleftrightarrow{PP'}$ is the line m we are seeking.

The perpendicular $m = \overleftrightarrow{PM}$ is unique because if there were two such lines $\overleftrightarrow{PM}$, $\overleftrightarrow{PM'}$ then we obtain a triangle PMM' with a non-zero angle at P and $\angle M = \angle M' = \pi/2$, which is impossible. □

4.3. Altitudes of a triangle

Definition. The line from a vertex of a triangle perpendicular to the opposite side is called an *altitude*.

Suppose that the altitudes from A, B, C meet $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$ in the points A', B', C', and consider the respective lengths $a' = |AA'|$, $b' = |BB'|$, $c' = |CC'|$.



We must have $aa' = bb' = cc'$, since any one of these products equals twice the area of $\triangle ABC$. However, we shall not discuss area further, and to prove (for example) the equality $aa' = bb'$ one can merely apply the AA rule to $\triangle AA'C$ and $\triangle BB'C$.

Later we shall show that the three altidues of a triangle ABC are always *concurrent*, that is they meet in a common point H, called the *orthocentre* of the triangle. If $\triangle ABC$ is *obtuse* (meaning one of its angles is greater than $\pi/2$, as in the diagram) then H will lie outside of the triangle.

Here is another use of altitudes. It is natural to define the *distance* between two points A, B of the plane as the length $|AB|$ of the segment AB. We shall also express this number as $d(A, B)$, to emphasize that it is a *function* of the two points. This function satisfies three rules:

- $d(A, B) = 0$ if and only if $A = B$;
- $d(A, B) = d(B, A)$;
- $d(A, B) \leqslant d(A, C) + d(C, B)$ for any points A, B, C.

The first two are inherent in Postulate B1. The third is easy to check if A, B, C lie on the same line, since $|AB| = |AC| + |CB|$ if $C \in AB$ whereas $|AB|$ is less than either $|AC'|$ or $|C'B|$ if $C \notin AB$. In general, the third rule is called *triangle inequality*, because it is equivalent to the

Corollary (of Pythagoras' Theorem). The sum of any two sides of a triangle (with non-zero angles) is greater than the third side.

Proof. If we construct the altitude from C meeting $\overleftrightarrow{AB}$ in C', then the segments AC and CB are the hypotenuses of right-angled triangles, and so $|AC'| < |AC|$ and $|C'B| < |CB|$. But A, C', B lie (not necessarily in that order) on the same line, so from above

$$|AB| \leqslant |AC'| + |C'B|.$$

Therefore, $|AB| < |AC| + |CB|$. □

5 Pythagorean algebra

In this section we analyse the equation

$$a^2 + b^2 = c^2$$

algebraically. To factorize the left-hand side we need complex numbers.

5.1. Complex numbers

If $z = a + ib$ with $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$ then its *complex conjugate* is defined to be $\overline{z} = a - ib$. This enables us to write

$$z\overline{z} = (a + ib)(a - ib) = a^2 + b^2.$$

Right-angled triangles come into play when we represent complex numbers in the plane as if they are vectors. The hypotenuse c represents the *modulus* or length of z, denoted $|z|$. With these definitions, Pythagoras' theorem becomes the identity

$$|z|^2 = z\overline{z}.$$

If we scale down by $1/|z|$, we obtain a complex number of modulus one that can be used to define the trigonometic functions:

$$\frac{1}{|z|}z = \frac{1}{c}(a + ib) = \cos\theta + i\sin\theta,$$

and remind us that $\cos^2\theta + \sin^2\theta = 1$. The right-hand side can also be written in exponential form $e^{i\theta}$, leading to the very useful formula of de Moivre:

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

Since

$$zw = (a + ib)(c + id) = (ac - bd) + i(ad + bc),$$

if we change the sign of i we discover that

$$\overline{z}\,\overline{w} = \overline{zw}, \qquad z, w \in \mathbb{C}.$$

This now implies that

$$|zw|^2 = zw \cdot \overline{zw} = (z\,\overline{z})(w\,\overline{w}) = |z|^2|w|^2,$$

confirming that *the modulus of a product is the product of the moduli*:

$$|zw| = |z||w|.$$

5.2. Algebraic numbers

Like $\sqrt{2}$, the number i is *algebraic* since it is a root of a polynomial equation (in this case, $z^2 + 1 = 0$) with integer coefficients.

Definition. A complex number z is said to be *algebraic* if it satisfies a polynomial with integer coefficients:

$$a_n z^n + \cdots + a_1 z + a_0 = 0,$$

with $a_i \in \mathbb{Z}$. (It would be equivalent to say "rational coefficients" since we could multiply out all the denominators to get a polynomial like the one on the left above.)

It is known that both π and e are *transcendental*, meaning "not algebraic". But of course $e^{i\pi}$ *is* algebraic since it equals -1! Other examples of transcendental numbers include $2^{\sqrt{2}}$, e^{π}, and the solution x to the equation $x^x = 2$ (proved to be irrational on Sheet 1). But there are many numbers for which it is not known whether they are algebraic or not.

Proposition. The real numbers $\cos\theta$ and $\sin\theta$ are algebraic whenever the angle θ is a rational multiple of π.

Proof. The assumption is that $\theta = m\pi/n$ with $m, n \in \mathbb{Z}$ (and $n \neq 0$). Let

$$z = \cos\theta + i\sin\theta = e^{im\pi/n}.$$

Then

$$z^{2n} = e^{2m\pi i} = 1,$$

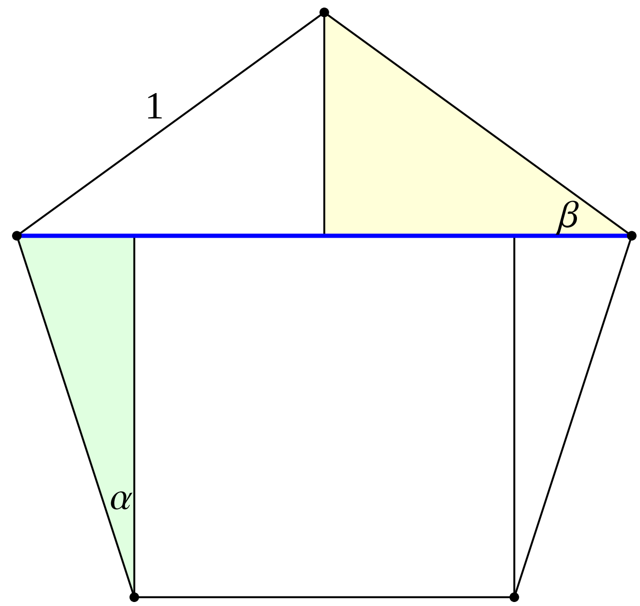
so certainly z is algebraic: it satisfies $z^{2n} - 1 = 0$.

Now it turns out that if one adds adds (or substracts, or multiplies or divides) rational multiples of algebraic numbers, the result remains algebraic. It follows that $a = \cos\theta$ and $b = \sin\theta$ are both algebraic. But one can also prove this directly by expanding

$$z^{2n} = (a + ib)^{2n}$$

by the binomial theorem, and then equating real and imaginary parts and using the equation $a^2 + b^2 = 1$. This will give polynomial equations for a and b (try it for $n = 2$ when one knows that a and b are each one of $0, 1, -1$). □

Example. $\cos\dfrac{\pi}{5} = \sigma/2$, where $\sigma = (1 + \sqrt{5})/2 = 1.618\ldots$ is the *golden ratio* that satisfies the equation $1 + \dfrac{1}{\sigma} = \sigma$.



To see this, consider a regular pentagon with all sides 1 and all interior angles equal (to $3\pi/5$). It follows easily that $\alpha = \pi/10$ and $\beta = \pi/5 = 2\alpha$ in the picture. Measurement of the blue diagonal segment yields

$$2\cos(2\alpha) = 2\sin\alpha + 1.$$

But $\cos(2\alpha) = 1 - 2\sin^2\alpha$, so $s = \sin\alpha$ satisfies $2(1 - 2s^2) = 2s + 1$, or

$$4s^2 + 2s - 1 = 0.$$

We know that $s > 0$, so

$$s = \frac{-2 + \sqrt{20}}{8} = \frac{-1 + \sqrt{5}}{4},$$

and $2\cos(2\alpha)2s + 1 = (1 + \sqrt{5})/2$, as claimed. $\square$

$\mathfrak{Warning}$. Not all algebraic numbers can be built up from integers using combinations of nth-root symbols $\sqrt[n]{}$.

5.3. Pythagorean triples

Definition. A *Pythagorean triple* is a set (a, b, c) of three *integers* such that (in order) $a^2 + b^2 = c^2$.

We may as well suppose that all of a, b, c are non-zero, and positive. Examples are

$$(3, 4, 5), \quad (5, 12, 13), \quad (7, 24, 25), \quad (20, 21, 29).$$

Given a Pythagorean triple (a, b, c), we can manufacture another by multiplying all three numbers by a common factor. For example, $(3, 4, 5)$ gives $(6, 8, 10)$. From now on, we shall only be interested in triples that are *coprime*, meaning that a, b, c have no common factor (other than ± 1).

In particular, a and b cannot both be even. On the other hand, if a and b are both odd then

$$a^2 + b^2 = (2m+1)^2 + (2n+1)^2 = 4(m^2 + n^2 + m + n) + 2$$

is divisible by 2 and not 4 and cannot equal c^2 (which being even must be divisible by 4). So we can suppose that b *is even and* a, c *are odd.*

Theorem (for constructing triples). Any coprime Pythagorean triple with b even has the form

$$a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2$$

where p, q are coprime positive integers, not both odd, with $p > q$. Any such choice of p, q will give a coprime Pythagorean triple.

Proof. We first prove the last sentence. Given p, q coprime,

$$a^2 + b^2 = (p^2 - q^2)^2 + 4p^2q^2 = (p^2 + q^2)^2 = c^2,$$

so (a, b, c) is a valid triple. Moreover, if r is a prime number that divides a, c then r divides $2p^2$ and $2q^2$ which is impossible because p, q have no common factor and we know that $r \neq 2$ (because a and c are odd).

Next, re-write the equation $a^2 + b^2 = c^2$ as

$$\left(\frac{b}{2}\right)^2 = \frac{c^2 - a^2}{4} = \left(\frac{c+a}{2}\right)\left(\frac{c-a}{2}\right).$$

We are assuming that b is even and a, c are odd, so all the fractions are in fact integers. If $r \geqslant 2$ is a prime number that divides $b/2$ then r^2 must divide $(a+c)/2$ or $(a-c)/2$. (Otherwise r will have to divide both factors on the right, and so their sum a and difference c; but then r divides a, b, c and the triple is not coprime.) Factorizing $b/2$ into prime factors, it now follows that there exist positive integers p, q such that

$$\frac{c+a}{2} = p^2, \qquad \frac{c-a}{2} = q^2.$$

Obviously $p > q$, and the argument above shows that p, q have no prime factor in common. □

Here is a table of triples with $b = 2pq$, in which each row corresponds to a fixed value of q. If q is odd, we can take p even and vice versa. The choice $(p, q) = (6, 3)$ (in red) gives us 3^2 times $(3, 4, 5)$ so can be ignored. We have stopped with $c = 97$.

q	p	(a, b, c) with b even
1	$2, 4, 6, 8$	$(3, 4, 5)$, $(15, 8, 17)$, $(35, 12, 37)$, $(63, 16, 65)$
2	$3, 5, 7, 9$	$(5, 12, 13)$, $(21, 20, 29)$, $(45, 28, 53)$, $(77, 36, 85)$
3	$4, 6, 8$	$(7, 24, 25)$, $(27, 36, 45)$, $(55, 48, 73)$
4	$5, 7, 9$	$(9, 40, 41)$, $(33, 56, 65)$, $(65, 72, 97)$

If we restrict to $q = 1$, we already get infinitely many coprime triples $(p^2 - 1, 2p, p^2 + 1)$.

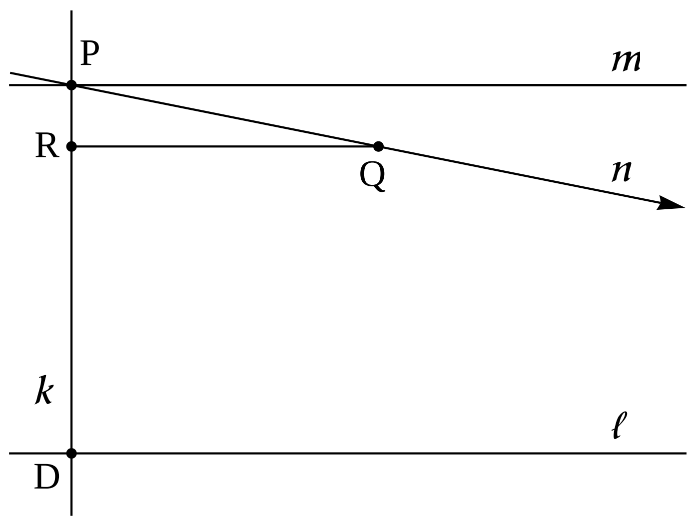
6 Parallel lines

6.1. The "Parallel Postulate"

Definition. Two lines in the plane that do not meet are said to be *parallel*. A line is also said to be parallel to itself.

Thus, ℓ and m are parallel if and only if $\ell \cap m = \varnothing$ or $\ell = m$.

Theorem. *Given a line ℓ and a point P not on ℓ there is a unique line parallel to ℓ passing though P.*



Proof. Given a line ℓ and a point P not on ℓ we know from §4.2 that there is a line k perpendicular to ℓ passsing through P. Moreover, there is a line m passing through P perpendicular to k (B3). Then ℓ and m are both perpendicular to k so we know they cannot intersect. (If $Q \in \ell \cap m$ then ℓ and m are both perpendiculars from Q to k and must coincide by the theorem on perpendiculars in §4.2.)

We now have to prove that m is the *only* line parallel to ℓ through P. We need to prove that if n is another line through P then ℓ and n must have a point in common. So suppose n makes an angle of less than $\pi/2$ at P with k (one side or the other). Choose a point $Q \in n$ different from P and drop a perpendicular from Q to k, with foot $R \in k$. If $D \in \ell$ is the foot of the perpendicular from P, there exists $S \in \ell$ such that

$$\frac{|PD|}{|DS|} = \frac{|PR|}{|RQ|}, \quad \text{or equivalently} \quad \frac{|PR|}{|PD|} = \frac{|RQ|}{|DS|}.$$

We not know at this stage that S lies on the line $\overleftrightarrow{PQ} = n$. But $\triangle PDS$ and $\triangle PRQ$ have a right angle and adjacent sides proportional, so are similar by B5. It follows that $\angle DPS = \angle RPQ$ and $\overleftrightarrow{PS} = n$ by B3. Therefore S *does* lie on n, and n cannot be parallel to ℓ. $\square$

Remarks. The theorem is also true without the hypothesis on P because the only line parallel to ℓ containing $P \in \ell$ is (by definition) ℓ itself.

The uniqueness statement in the theorem above was formulated as an *axiom* by the Scottish mathematician Playfair in 1795, and is often called the "Parallel Postulate".

6.2. Equivalence relations

Let us write $\ell \parallel m$ as an abbreviation for "ℓ is parallel to m". Then

- $\ell \parallel \ell$ (by definition),
- $\ell \parallel m \;\Rightarrow\; m \parallel \ell$ (almost by definition),
- $\ell \parallel m,\ m \parallel n \;\Rightarrow\; \ell \parallel n.$

To justify the last assertion, suppose that ℓ is parallel to m but not to n, and that m is parallel to n. Then there exists $P \in \ell \cap n$ and both ℓ and n are lines parallel to m passing through P. This contradicts the "only one" statement in the parallel postulate.

These three rules characterize what is called an *equivalence relation*. Therefore "being parallel" (in symbols, $\parallel$) is an equivalence relation on the set of lines in the plane.

Another example (which we have already seen) is similarity for triangles. Let us write

$$\triangle ABC \;\sim\; \triangle A'B'C'$$

to express the fact that the two triangles are similar with the vertices in the stated order. By Theorem 2 (the AA rule) and Theorem 5, we know that this is true if and only if $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$. It follows that

- $\triangle ABC \sim \triangle ABC$,
- $\triangle ABC \sim \triangle A'B'C' \;\Rightarrow\; \triangle A'B'C' \sim \triangle ABC$,
- $\triangle ABC \sim \triangle A'B'C',\;\; \triangle A'B'C' \sim \triangle A''B''C'' \;\Rightarrow\; \triangle ABC \sim \triangle A''B''C''$.

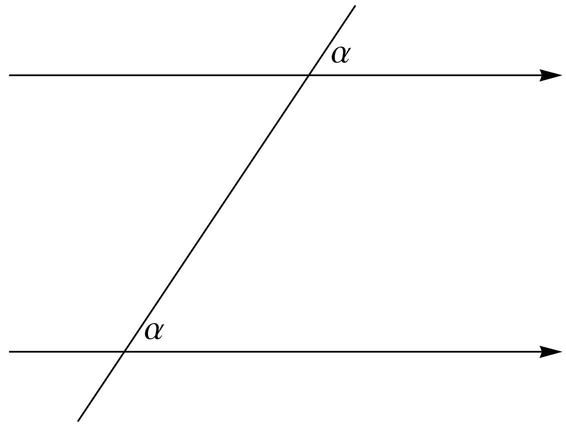
Taken in order, these rules reflect the fact that $\sim$ is *reflexive*, *symmetric*, *transitive*. Therefore similarity is an equivalence relation on the set of all triangles.

6.3. Transversals

Given one straight line ℓ, we may now consider *all* those lines parallel to ℓ : there will be one for each point of the line k in §6.1.

Lemma (on transversals). *If a line meets two lines at the same angle then those lines are parallel.*

Proof. We already know this to be true if the angle is $\pi/2$ radians. In general, the hypothesis "at the same angle" is to be interpreted by the diagram.



If the lines are not parallel they will meet one side or the other of the line giving rise to a triangles whose angles add up to more than π radians. □

This proves shows that the Parallel Postulate is closely connected with the property that angles in a triangle sum to 180^o.

Proposition (on transversals). *Suppose that ℓ and m are parallel lines, and that k meets ℓ at an angle $\alpha > 0$. Then k meets m at the same angle.*

Proof. Given the hypotheses of the theorem, k must meet m at P say, otherwise k and ℓ would both be parallel to m. But k is not parallel to ℓ. Now construct a line m' so that k meets m' at P with angle α. We want to show that $m' = m$. By the lemma, $\ell \parallel m'$. But we already know that $m \parallel \ell$, so $m \parallel m'$ and (since m and m' meet at P) $m = m'$. □

Therefore, if k meets one in a family of parallel lines at a single point, k must meet every one of them, and at the same angle. Such a line k is called a *transversal*

Theorem (on transveral ratios). *Suppose that ℓ, m, n are parallel, and that k, k' are transversals meeting the three at points L, M, N and L', M', N' respectively. Then*

$$\frac{|LM|}{|L'M'|} = \frac{|MN|}{|M'N'|},$$

and the parallel lines are cut "in proportional segments".

Proof. If k and k' intersect in a point P, we can use similar triangles. All the triangles have two equal angles by the proposition, so they are similar by the AA rule:

$$\triangle PLL' \sim \triangle PMM' \sim \triangle PNN'.$$

This tells us that

$$\frac{|PL|}{|PM|} = \frac{|PL'|}{|PM'|}, \qquad \frac{|PL|}{|PN|} = \frac{|PL'|}{|PN'|},$$

(equal ratios between corresponding sides of different triangles), or equivalently

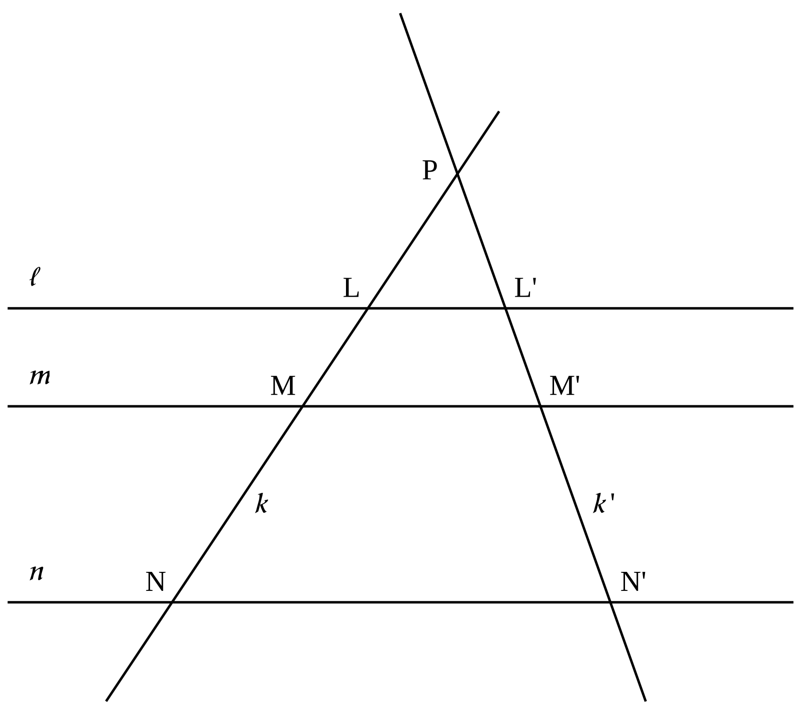
$$\frac{|PL|}{|PL'|} = \frac{|PM|}{|PM'|} = \frac{|PN|}{|PN'|}.$$

(equal ratios between the different sides of the same triangle). But one can "subtract" ratios top and bottom, so that the last three ratios are also equal to

$$\frac{|PM| - |PL|}{|PM'| - |PL'|} = \frac{|PN| - |PM|}{|PN'| - |PM'|},$$

which is the equality stated in the theorem. (To understand the last step, observe that if $a/b = c/d = \lambda$ then $(c-a)/(d-b) = (\lambda d - \lambda c)/(d-c) = \lambda$.)

If k and k' are parallel, we first need to introduce the third line $k'' = \overleftrightarrow{LN'}$, which will intersect m in a point M''. See Sheet 5. □



7 Cartesian coordinates

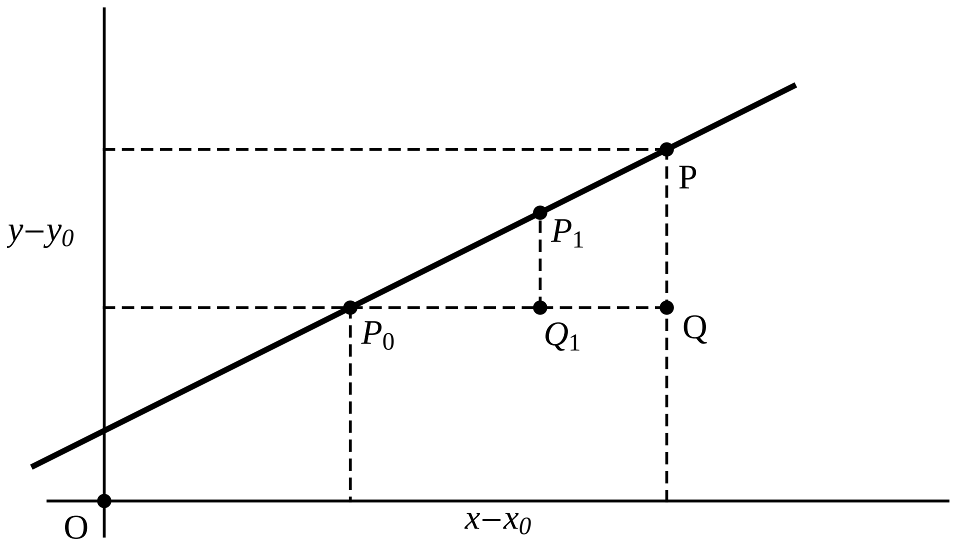
7.1. Parallel networks

If we fix a line ℓ and another line k, meeting ℓ in O and perpendicular to it, we can construct a family of perpendiculars to ℓ, each one parametrized by the distance $x \in \mathbb{R}$ (positive or negative) from O to the foot of the perpendicular. We know that each such "vertical" line k_x is parallel to $k = k_0$.

Similarly, for each $y \in \mathbb{R}$ we have the line ℓ_y perpendicular to k constructed from a point of k a distance y from O. Each "horizontal" line ℓ_y is parallel to $\ell = \ell_0$. In this way, we obtain a network of two families of parallel lines.

Theorem. *For fixed real numbers x, y, the lines k_x, ℓ_y intersect in a unique point P, denoted also $P(x, y)$, and evry point in the plane arises in this way.*

Proof. The fact that k_x, l_y intersect follows from the lemma in §6.3 (k_x is a transveral of the parallel lines ℓ, ℓ_y). They can only meet in one point (this follows from B2). Conversely, given P, we can drop perpendiculars from P to both ℓ and to k, and these perpendiculars will be k_x, ℓ_y for some $x, y \in \mathbb{R}$. $\square$



The parameters x and y are called the *Cartesian coordinates* of P, and one also writes $P = (x, y)$. The lines $\ell = \ell_0$ and $k = k_0$ (defined by the equations $x = 0$ and $y = 0$ respectively) are called the *axes* of the system. The point O is called the *origin* of the system. We can re-phrase the theorem by means of the

Corollary. *There is a* bijection *between the plane and the set* $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ *consisting of* ordered pairs (x, y).

Remark. Given any two sets $\mathscr{A}, \mathscr{B}$, one defines their *Cartesian product* as

$$\mathscr{A} \times \mathscr{B} = \{(a, b) : a \in \mathscr{A},\ b \in \mathscr{B}\}.$$

Each element of this set is an *ordered* pair (a, b) (hence round rather than curly brackets) and (a, b) is distinct from (b, a) unless $a = b$ (which can only happen if a belongs to $\mathscr{A} \cap \mathscr{B}$). For example if $\mathscr{A} = \{0, 1\}$ (size 2) and $\mathscr{B} = \{p, q, r\}$ (size 3) then

$$\mathscr{A} \times \mathscr{B} = \{(0, p), (1, p), (0, q), (1, q), (1, r), (0, r)\}.$$

Here the 6 elements form a set, so *their* order does not matter!

The corollary is usually taken for granted as a starting point for courses on analytic and vector geometry. The work so far in our course may be regarded as a "prequel" (like inserting clone wars before Star Wars). We have established the existence of a system of Cartesian coordinates starting from four basic postulates B1–B4 plus the powerful SAS postulate B5 to detect triangles that are similar (not just congruent).

This led us to *prove* other criteria for similarity (AA and SSS), the fact that the angles of a triangle sum to 180^o, and Pythagoras' theorem. It also led to the assertion (in §6.1) of a *unique* line parallel to a given one and passing through a given point. We shall now show that the lines of our theory are (like we have always drawn them) the ordinary straight lines of Cartesian geometry.

7.2. The equation of a straight line

A single equation relating the coordinates x, y will typically define a *curve* in the plane. We shall see a few examples later in this course, though a proper study of curves is a more advanced topic.

The simplest equation is a *linear* one:

$$ax + by + c = 0.$$

where $a, b \in \mathbb{R}$ are constants. We shall now show that this describes a *line* of our theory based on postulates B1–B5. Consider such a line L that makes an angle θ with one (and therefore, by §6.3, all) of the horizontal lines ℓ_y.

If θ is a right angle then L equals k_c for some $c \in \mathbb{R}$, and L has equation $x = c$ with c constant. If $\theta = 0$ then $L = \ell_c$ for some $c \in \mathbb{R}$ and has equation $y = c$. Otherwise the situation resembles that shown on the previous page. Fix two points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ on L, and consider a third point $P = (x, y)$ allowed to move along the line. Because the opposite sides of rectangles have equal lengths,

$$|P_0Q| = |x - x_0| \quad \text{and} \quad |QP| = |y - y_0|.$$

The AA rule, applied to right-angled triangles $\triangle P_0Q_1P_1$ and $\triangle P_0QP$, tells us that

$$\frac{y - y_0}{y_1 - y_0} = \frac{x - x_0}{x_1 - y_0},$$

or equivalently

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}.$$

The right-hand side is a constant m, so $y - y_0 = m(x - x_0)$, or

$$y = mx + c,$$

where $c = y_0 - mx_0$. This equation has the form above with $a = m$ and $b = -1$.

In conclusion, L is indeed an ordinary straight line of the type we are familiar with. The quantity m is called the *slope* or *gradient* of L. It follows that *given P and $m \in \mathbb{R}$, there is a unique straight line passing through P with slope m.* It is well known that two lines with slopes m_1 and m_2 are perpendicular iff $m_1 m_2 = -1$.

The hypotenuse of the right-angled $\triangle P_0QP$ has length

$$d(P_0, P) = |P_0P| = \sqrt{|x - x_0|^2 + |y - y_0|^2}.$$

This is the formula for the distance between two points of the Cartesian plane.

7.3. Circles and conics

The set $\mathscr{C}$ of points at a fixed distance r from a given point is a *circle*. Of course, r is the *radius* and the point is the *centre*. If we take the centre to be the origin of the coordinates then

$$\mathscr{C} = \{(x, y) : x^2 + y^2 = r^2\},$$

by Pythagoras' theorem. If the centre is (x_0, y_0), the equation becomes

$$(x - x_0)^2 + (y - y_0)^2 = r^2,$$

or

$$x^2 + y^2 + 2dx + 2ey + f = 0,$$

where $d = -x_0$, $e = -y_0$ and $f = x_0^2 + y_0^2 - r^2$ are constants. Conversely, this equation will define a circle provided $d^2 + e^2 - f > 0$.

The general equation for a circle can be generalized by allowing x^2 and y^2 to have unequal coefficients, and introducing a multiple of xy. Since all terms (including xy) have total degree at most 2, what results can reasonably be called a *quadratic equation* in the two variables x, y :

Definition. The set (or "locus") of points (x, y) in the plane satisfying an equation of the form

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

(where a, b, c, d, e, f are constants with a, b, c not all zero) is called a *conic*.

The numbers defining this conic can be conveniently encoded into tabular form by defining two matrices:

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad B = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}.$$

These matrices are square and symmetric about a diagonal. Any square matrix has a number associated to it, called its *determinant* (whose vanishing means an inverse matrix does not exist). The determinants of A and B are the quantities

$$\det A = ac - b^2, \qquad \det B = acf + 2bde - ae^2 - cd^2 - fb^2.$$

To publicize the usefulness of Cartesian coordinates, we quote without proof the

Theorem. *Suppose that $\mathscr{C}$ contains at least two points and* $\det B \neq 0$. *Then*

- $\mathscr{C}$ *is an ellipse if* $\det A > 0$;
- $\mathscr{C}$ *is a parabola if* $\det A = 0$;
- $\mathscr{C}$ *is a hyperbola if* $\det A < 0$.

If $\det B = 0$ *then* $\mathscr{C}$ *consists of two lines that may intersect, be parallel or even coincident.*

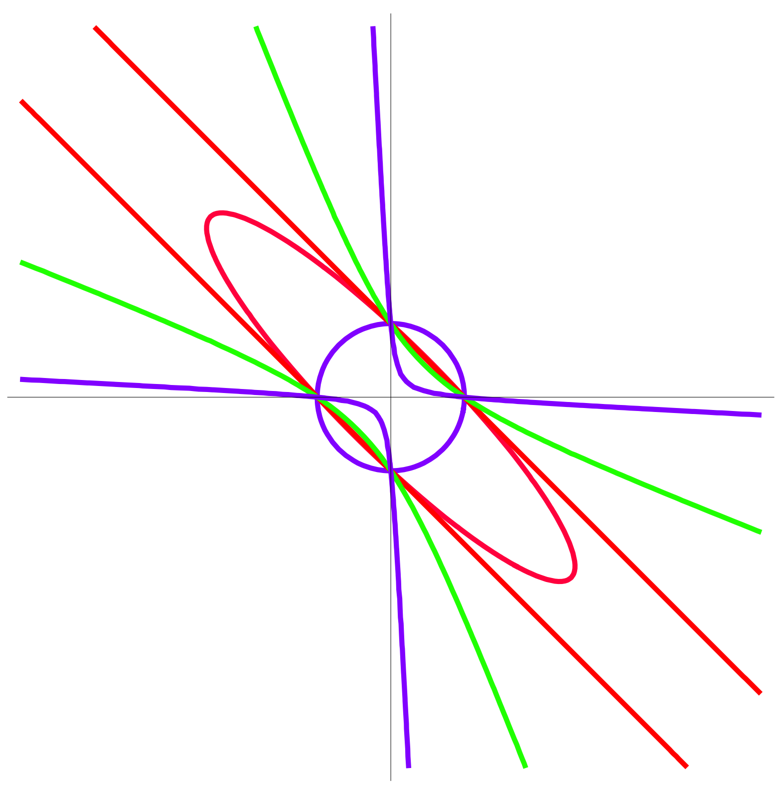
Examples. The word "conic" reflects the fact that the three main types can be realized as plane sections of a circular double cone in space (and therefore by light reaching a wall from a hand-held table light with a cylindrical lamp-shade).

• Any circle has $a = c$ and $b = 0$, so $\det A > 0$ and $\det B = f - d^2 - e^2 < 0$. A circle is a special type of ellipse. The "standard" ellipse $x^2/A^2 + y^2/B^2 = 1$ lies entirely inside a rectangle centred at the origin, width $2A$ and height $2B$.

• The graph of the function $y = x^2$ is a parabola (here we can take $a = 1,\ e = -\frac{1}{2}$ with the other coefficients zero, so $\det A = 0$ and $\det B \neq 0$.)

• The graph of $y = 1/x$ is a hyperbola with equation $xy = 1$. Other hyperbolae with the same shape are $x^2 - y^2 = 2$ and $x^2 - y^2 = -2$.

• The conics defined by $x^2 + 2bxy + y^2 = 1$ for $b = 0, \frac{11}{12}, 1, \frac{3}{2}, 10$ are plotted in the same plane. But which is which?



8 Circles and triangles

8.1. Tangents and secants

We shall begin by considering the intersection of a line and a circle. This can be solved analytically by substituting one of the equations

$$y = mx + c, \qquad x = c$$

into the general equation

$$x^2 + y^2 + 2dx + 2ey + f = 0$$

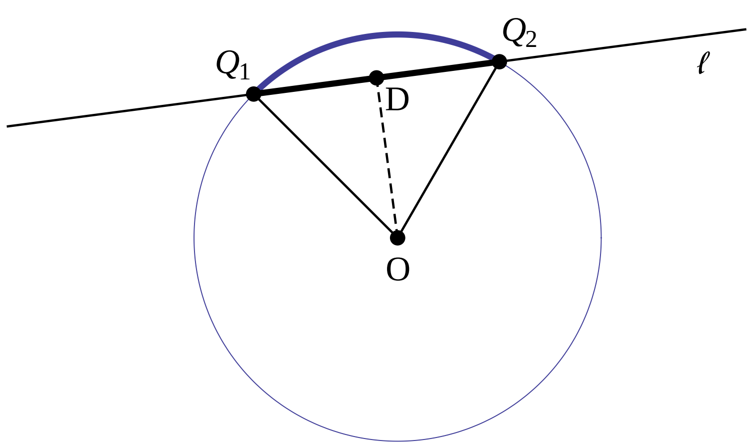
of a circle (with centre $(-d, -e)$). In the first case, we obtain

$$x^2 + (mx + c)^2 + 2dx + 2e(mx + c) + f = 0,$$

which is a quadratic equation in x. It will have real roots, one real root (more precisely, two coincident real roots) or no real roots (more precisely, a complex root z and its conjugate $\overline{z}$). Once we have a real value for x, we can determine y from the equation of the line. The second case is similar (indeed, simpler) and gives us 0,1, or 2 values for y from which we can find x.

It follows that a line must intersectq a circle in 0, 1 or 2 points. There is an obvious interpretation of these possibilities, which are best analysed geometrically. Given a circle $\mathscr{C}$ with radius r and centre O and a line ℓ, drop a perpendicular from O to ℓ with foot D. There are three cases:

- $|OD| > r$. By Pythagoras, $|OQ| = \sqrt{|OD|^2 + |DQ|^2} > r$ for all other points $Q \in \ell$, so the line does not meet the circle.
- $|OD| = r$, so $|OQ| > r$ for all other points $Q \in \ell$, and ℓ meets the circle in the one point D.
- $|OD| < r$. In this case, there exists $x \in \mathbb{R}$ such that $|OD|^2 + x^2 = r^2$ and we obtain two points $Q_1, Q_2 \in \ell$ with $|OQ_1| = r = |OQ_2|$, corresponding to x and $-x$. So the line intersects the circle in two points, as in the diagram:



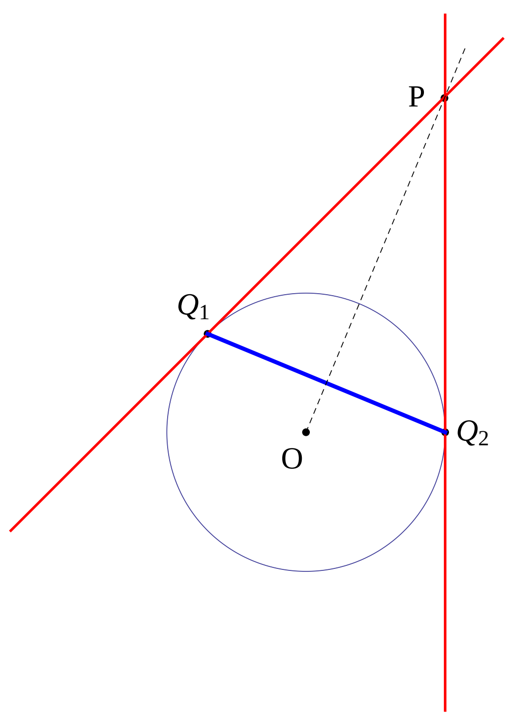
Definition. In the second case, we say that ℓ is a *tangent* of $\mathscr{C}$. In the third case, we say that ℓ is a *secant* of $\mathscr{C}$, and the segment Q_1Q_2 is called a *chord*. If $D = O$ then the chord is a *diameter* and $|Q_1Q_2| = 2r$. From the lemma in §4.2 we deduce the

Proposition. *If Q_1Q_2 is a chord of a circle with centre O then its perpendicular bisector passes through O.*

Proposition (on tangents). *A line ℓ is tangent to a circle at P if and only if ℓ passes through P and is perpendicular to $\overleftrightarrow{OP}$.*

Proof. If ℓ is perpendicular to OP at P then (as explained above), the line cannot meet the circle at points other than P, so it is (by definition) a tangent. If ℓ is tangent to $\mathscr{C}$ at P but does not make a right angle with OP then the foot of the perpendicular from O will be a point $Q \in \ell$ for which $|OQ| < |OP| = r$. But then ℓ will meet $\mathscr{C}$ is a second point, contrary to hypothesis. □

We have seen that the perpendicular bisector of any chord passes through the centre. It also passes through the point of intersection of the associated tangents:



Theorem (on two tangents). *Let ℓ, m be two lines that are tangent to the same circle $\mathscr{C}$ (centre O) at distinct points Q_1, Q_2. Suppose that ℓ and m meet in P. Then $|PQ_1| = |PQ_2|$, and $\overleftrightarrow{PO}$ bisects $\angle Q_1PQ_2$ and the chord Q_1Q_2 (at right angles)*

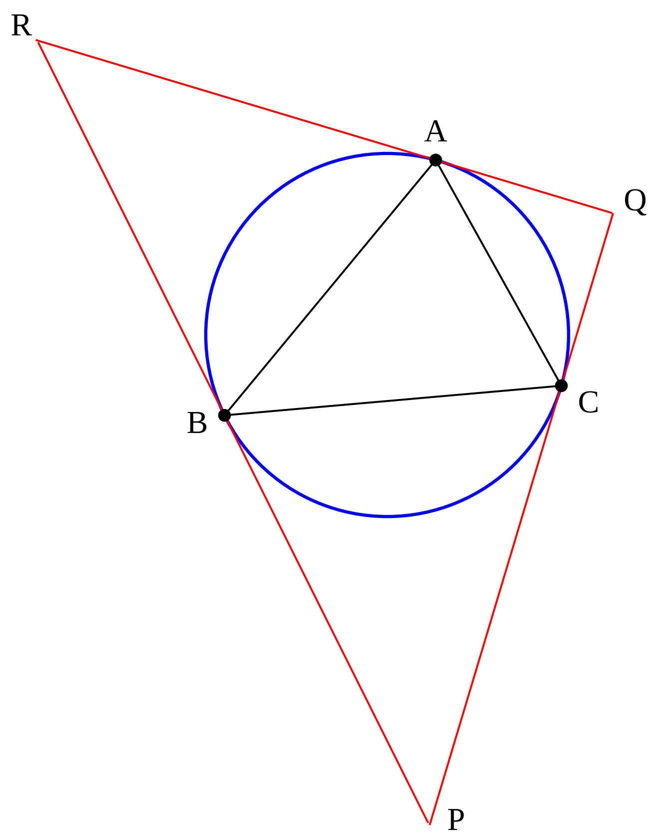
Proof. The right-angled triangles $\triangle OPQ_1$ and $\triangle OPQ_2$ share a hypotenuse and have two equal radii. It follows from Pythagoras that $|PQ_1| = |PQ_2|$, and the two triangles are similar (by SSS or SAS). It also follows that $\angle OPQ_1 = \angle OPQ_2$.

We now know that O and P are equidistant from Q_1 and Q_2. It follows (lemma in §4.2) that OP is the perpendicular bisector of Q_1Q_2. □

8.2. Altitudes revisited

Choose three distinct points A, B, C on a circle $\mathscr{C}$. We can now construct not only the (black) triangle $\triangle ABC$, but also the (red) triangle PQR formed by the three tangents to $\mathscr{C}$ at A, B, C. Results from §8.1 tell us that the centre O of the circle $\mathscr{C}$ can be recovered from either triangle:

- it is the intersection of the *perpendicular bisectors* of the sides of $\triangle ABC$;
- it is the intersection of the *angle bisectors* of $\triangle PQR$.



Next, we shall show that it was not necessary to begin with the circle above, in the sense that we may construct $\mathscr{C}$ from A, B, C :

Theorem (circle through 3 points). *There exists a unique circle passing through three given points A, B, C that are not collinear.*

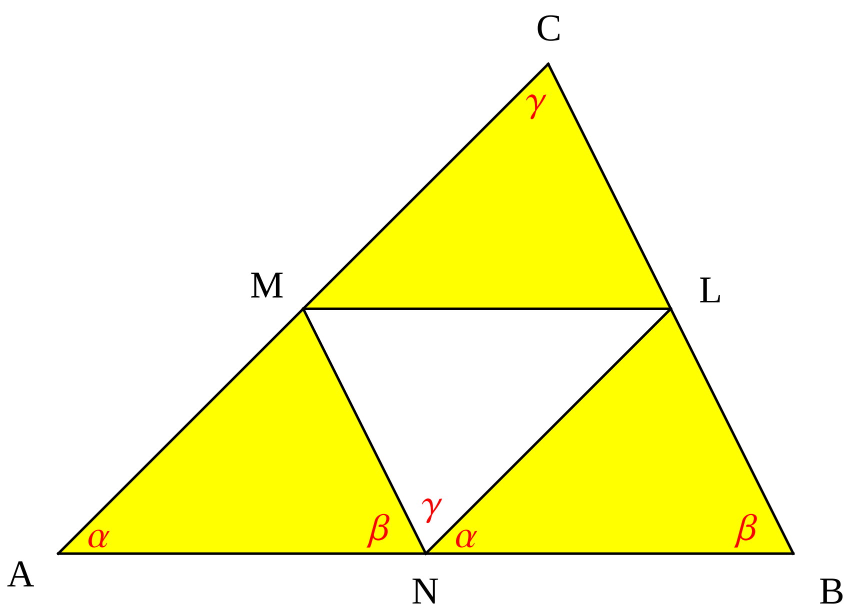
Proof. Consider $\triangle ABC$. Let ℓ, m, n denote the perpendicular bisectors of BC, CA, AB. Consider first ℓ and m. They can't be parallel (why?) and therefore meet meet in a point O. By the lemma in §4.2,

$$|OA| = |OB| = |OC|.$$

Thus, O is at the same distance r from all three vertices, and it follows (again from §4.2) that $O \in n$. More to the point, the circle with centre O and radius r passes through A, B, C. $\square$

Definition. The circle $\mathscr{C}$ constructed by this theorem is called the *circumcircle* of $\triangle ABC$. Its centre (the common intersection point O of the perpendicular bisectors) is called the *circumcentre* of $\triangle ABC$.

Recall the following diagram from §3.4, in which L, M, N are midpoints, and all four small triangles are congruent, and similar to the big one. It follows from §6.3 that all the lines that look parallel *are* parallel!



In particular, the perpendicular bisector n of AB (that passes through N but is not shown) intersects ML at right angles and is an altitude of $\triangle LMN$. Similarly, the perpendicular bisectors ℓ, m of BC, CA are also altitudes of $\triangle LMN$. It follows from the previous definition that the altitudes of $\triangle LMN$ intersect in the same point, namely the circumcentre O of $\triangle ABC$!

We can also go backwards. Given *any* triangle LMN, construct a line parallel to each side passing through the opposite vertex. For example, the new line through L is perpendicular to the altitude linking L to MN. These new lines will define $\triangle ABC$ housing 4 "mini-triangles" including $\triangle LMN$. Because transversals make the same angles with parallel lines, it is easy to see that the 4 mini-triangles are all similar. Because parallograms have opposite sides of equal length (sheet 5), tha 4 triangles are also congruent. It follows that L, M, N are the midpoints of BC, CA, AB, and we recover the picture above in which the altitudes of $\triangle LMN$ are the perpendicular bisectors of the sides of the bigger $\triangle ABC$.

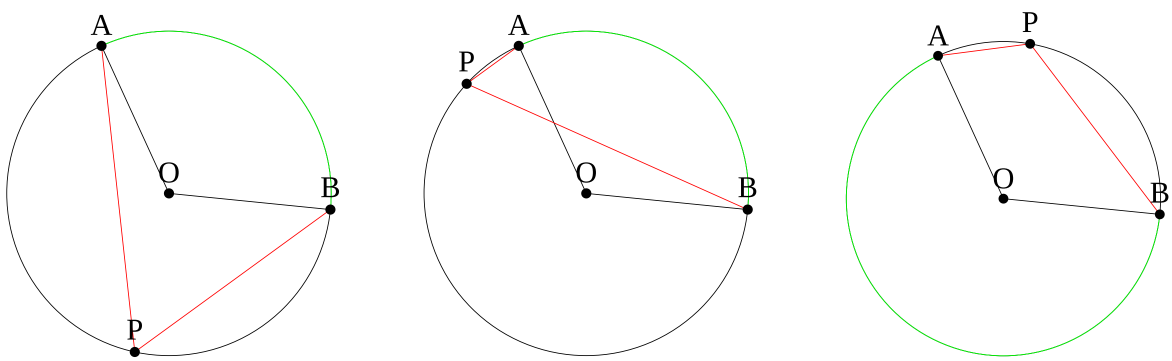
This confirms that *the altitudes of any triangle LMN are concurrent.* Their common point of intersection is called the *orthocentre* of $\triangle LMN$ and is often denoted H (though it is the circumcentre of $\triangle ABC$).

9 More circles and triangles

9.1. Thales' theorem

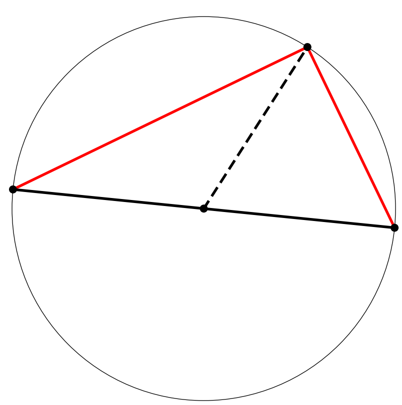
Let $\mathscr{C}$ be a circle with centre O. Fix two points A, B on $\mathscr{C}$, and let P be a third point free to move on $\mathscr{C}$. Join A and B to O and P.

Definition. $\angle APB$ is called the angle *inscribed* or *subtended* at P by the chord AB (which is not drawn). The arc (shown in green) joining A to B not containing P defines the corresponding *central angle* at O. There are various cases:



Theorem (on inscribed angles). *An inscribed angle is half the corresponding central angle, and is therefore independent of P, provided P stays on one side of AB.*

Theorem (Thales, c. 600 BC). *A diameter subtends an angle of $\pi/2$ at any point of $\mathscr{C}$:*



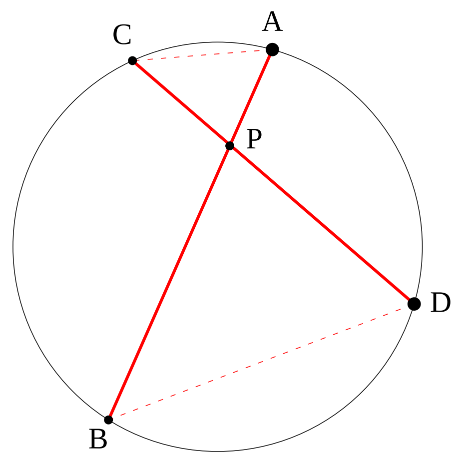
To deduce the second theorem from the first, interpret the straight angle formed by the diameter as that associated to the top inscribed angle. But it is easier to prove it directly as follows.

Both visible triangles are isosceles with angles α, α (left triangle) and β, β (right triangle) at points on the circle. The central angles are therefore $\pi - 2\alpha$ and $\pi - 2\beta$. But (by B4) these add up to π, so $\alpha + \beta = \frac{\pi}{2}$. $\square$

This proof also tells us that the right central angle equals 2α, that is twice the left inscribed angle. This is another special case of the theorem in which one chord passes through the centre.

Proof of the first theorem. In the each of the three diagrams, we can add the diameter extending PO, and apply what we have just said (and the diagram at the foot of the previous page) separately to each side of this diameter. $\square$

Corollary. *If two chords AB, CD intersect in P, then $|AP||PB| = |CP||PD|$.*



Proof. Having added the chord AD, the angles at B and C are both inscribed relative to AD, and therefore equal. Since $\triangle APC$ and $\triangle DPC$ already share opposite angles at P, they are similar by the AA rule. It follows that

$$\frac{|AP|}{|DP|} = \frac{|CP|}{|BP|},$$

and we can cross multiply to get the result. $\square$

9.2. More concurrence results

There are various triples of lines associated to a triangle, and we list four such sets. Three sets pass through the vertices, two pass through the midpoints of the sides, and two are perpendicular to the sides:

- the *angle bisectors* bisect each interior angle;
- the *medians* connect each vertex to the midpoint of the opposite side;
- the *altitudes* connects each vertex to the opposite side perpendicularly;
- the *perpendicular bisectors of the sides*.

Master Theorem. *The three lines defined in each triple above are concurrent.*

We have already shown that the perpendicular bisectors of the sides are concurrent using the lemma in §4.2. The concurrence of the angle bisectors can be proved in a similar way, using Sheet 5, q. 6(ii) (and its converse, which is easy to prove).

Definition. The common intersection I of the angle bisectors is called the *incentre*. The common intersection G of the medians is called the *centroid* or *barycentre*.

The three sides of a triangle are all tangent to a circle with centre at I (why?).

If the vertices of the triangle are represented by complex numbers or vectors z_1, z_2, z_3, it is well known (see Sheet 7, q. 2) that the centroid is their artithmetic mean

$$\tfrac{1}{3}(z_1 + z_2 + z_3).$$

Here is a proof of the existence of the orthocentre H (the common intersection of the altitudes) using complex numbers. Represent the vertices of $\triangle A_1A_2A_3$ by complex numbers z_1, z_2, z_3 in the Argand plane. We shall suppose that the origin O (i.e. $z = 0$) is the circumcentre of $\triangle A_1A_2A_3$, and (by choosing an appropriate unit of measurement) that the radius of the circumcircle equals 1. It follows that z_1, z_2, z_3 all have modulus one: $|z_k| = 1$ for $k = 1, 2, 3$.

We can also use a complex number like $z_1 = x_1 + iy_1$ to represent the vector

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \vec{OA_1}.$$

The dot product

$$x_1x_2 + y_1y_2 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

then equals the real part of

$$z_1\overline{z_2} = (x_1 + iy_1)(x_2 - iy_2).$$

Proposition. *With the above assumptions, the orthocentre H of $\triangle A_1A_2A_3$ is represented by the complex number $z_1 + z_2 + z_3$.*

Warning. The triangle inequality implies that $|z_1 + z_2 + z_3| < 3$ (why $<$ and not $\leqslant$?), but $|z_1 + z_2 + z_3| > 1$ when the triangle is obtuse and H is outside of it.

Proof. Let $w = z_1 + z_2 + z_3 \in \mathbb{C}$, and let W denote the corresponding point in the Argand plane. Then $w - z_1$ represents the vector $\vec{A_1W}$, and $z_2 - z_3$ represents $\vec{A_3A_2}$. Observe that

$$(w - z_1)(\overline{z_2 - z_3}) = (z_2 + z_3)(\overline{z_2} - \overline{z_3}) = 1 + (z_3\overline{z_2} - z_2\overline{z_3}) - 1$$

has zero real part. Thus A_1W is perpendicular to A_3A_2, and thus A_1W *is* an altitude. It follows that W lies on all three altitudes, so $W = H$. □

10 Isometries in the plane

10.1. Rotations, reflections and translations

Let $\mathscr{E}$ stand for the set of points in the plane. Once we have chosen an origin and a parallel network of lines, we obtain a bijection between $\mathscr{E}$ and $\mathbb{R}^2$, but we prefer not to make this choice (of Cartesian coordinates) for the moment.

Thanks to Postulates B1 and B2, we can measure the *distance* $|PQ|$ between any two points $P, Q \in \mathscr{E}$. At times it will be convenient to write $d(P, Q) = |PQ|$ for this distance. The resulting function

$$d: \mathscr{E} \times \mathscr{E} \to [0, \infty)$$

satisfies the triangle inequality $d(P, Q) \leqslant d(P, R) + d(R, Q)$, with strict inequality (meaning $<$ in place of $\leqslant$) if P, Q, R are the vertices of a triangle.

Definition. A mapping $f: \mathscr{E} \to \mathscr{E}$ is called an *isometry* if it preserves distances in the sense that $d(f(P), f(Q)) = d(P, Q)$.

The d notation makes this concept clearer, though we shall often set $f(P) = P'$ and $f(Q) = Q'$ so as to write $|P'Q'| = |PQ|$.

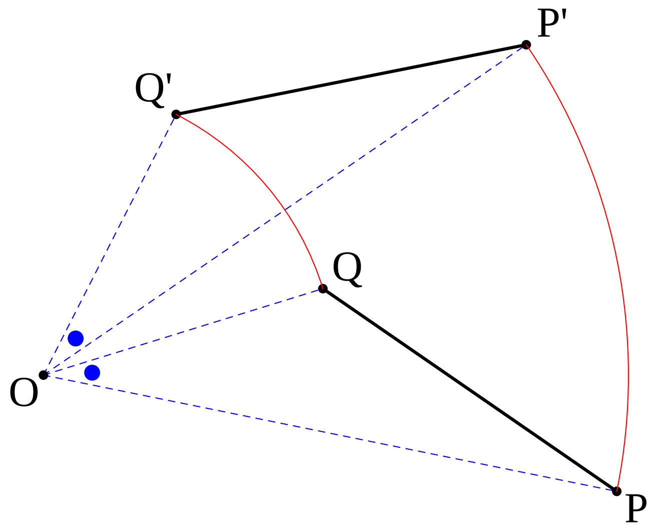
Example 1. The *identity* (that maps every point P to itself) is obviously an isometry. We denote it by id.

Observation. It follows that any isometry f is necessarily injective because

$$f(P) = f(Q) \Rightarrow d(f(P), f(Q)) = 0 \Rightarrow d(P, Q) = 0 \Rightarrow P = Q.$$

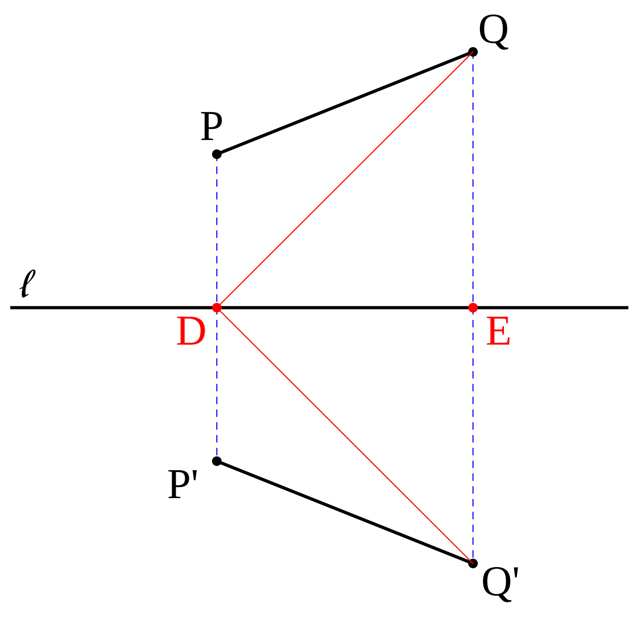
We shall see later any isometry f has an inverse, and is therefore a bijection.

Example 2. A *rotation* $f = \mathrm{Rot}_{O,\theta}$ about a point O through an angle θ is an isometry. How is such a rotation defined rigorously? To determine $f(P) = P'$ we join O to P and constuct a ray from O making an angle θ with $\overrightarrow{OP}$ in the anti-clockwise direction (*anti* by convention). We then measure off P' so that $|OP'| = |OP|$. Note that O is a fixed point of f since $f(O) = O$.



Given the diagram, the condition $|P'Q'| = |PQ|$ follows from Postulate B5 (SAS), having first verified that the two angles indicated are equal.

Example 3. A *reflection* Ref_ℓ in a line ℓ. The image $f(P) = P'$ is found by dropping a perpendicular from P to ℓ with foot D, and extending it to the "other side" of ℓ and measuring off P' so that $|DP'| = |DP|$.

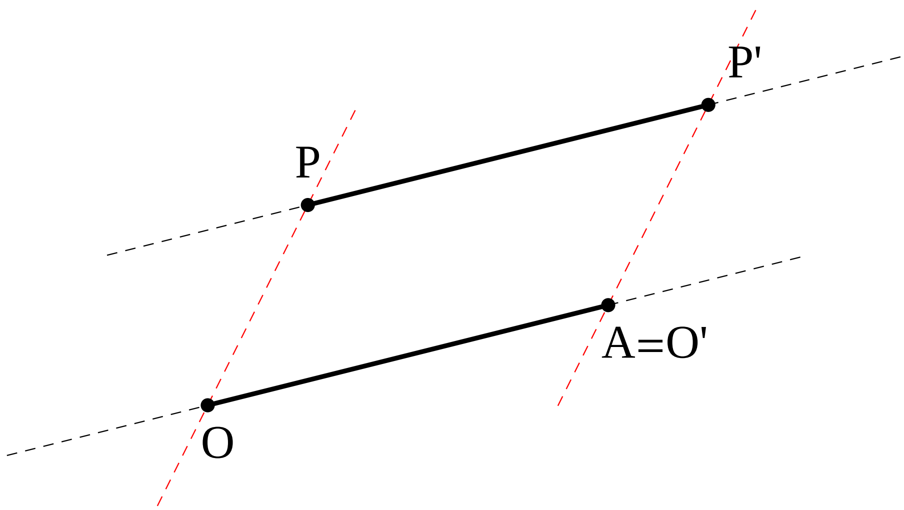


Note that every point on ℓ is fixed, and $f(P) = P \Leftrightarrow P \in \ell$. This time, to prove that Ref_ℓ *is* an isometry, we apply SAS to $\triangle DEQ$ and its image to deduce that

$$|DQ| = |DQ'| \quad \text{and} \quad \angle QDE = \angle Q'DE.$$

Then again to $\triangle PDQ$ and its image to deduce that $|P'Q'| = |PQ|$.

Example 4. A translation $f = \mathrm{Tran}_{OA}$ determined by a segment OA. The image of P is obtained by constructing the unique line through P parallel to $\overleftrightarrow{OA}$. and measuring off $f(P) = P'$ so that AP' is parallel to OP. In this way we obtain a parallelogram, and $|OA|$ will equal $|PP'|$ (recall Sheet 5, q. 4).



Given another point Q and its image $Q' = f(Q)$, we obtain a parallelogram $PQQ'P'$, and it follows that $|PQ| = |P'Q'|$. Thus the translation is an isometry.

The order in which we have listed these familiar isometries corresponds to the way in which Euclidean geometry was developed in the first half of the course. For example, whilst a translation is perhaps the easiest isometry to imagine, its definition requires the concept of parallel lines that arose relatively late in our treatment.

10.2. Orientation and composition

If f is an isometry and P, Q, R are the vertices of a triangle, then

$$f(P) = P', \quad f(Q) = Q', \quad f(R) = R'$$

are the vertices of a congruent triangle, by the SSS rule. Moreover, rotations and translations (and the identity, which strictly speaking is a special case of both) all preserve the *orientation* of the triangles: if P, Q, R occur in clockwise order around the triangle, so do P', Q', R'.

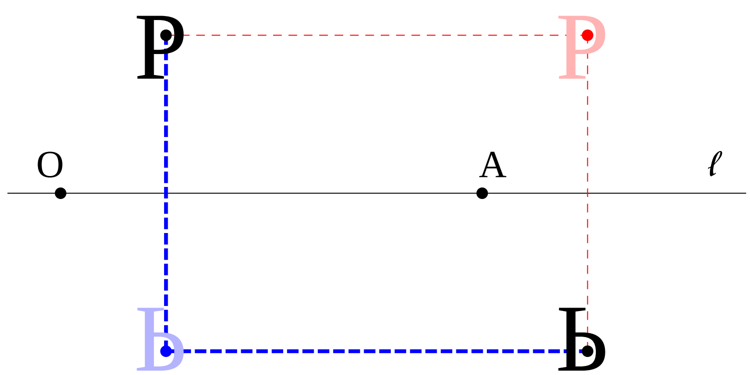
One way to see this for a rotation is to slowly increase the angle of the rotation from 0 to θ. When the angle is 0, the isometry is the identity and the triangle does not change. Subsequently, the orientation of the triangle cannot suddenly jump from clockwise to anticlockwise. (This argument can be made rigorous, but requires the notion of a *continuous* function.)

By contrast, reflections are more "brutal", and change the orientation of all triangles from clockwsie to anti-clockwise and vice versa. We shall say that rotations and translations *preserve orientation* or are "even", whilst reflections *reverse orientation* or are "odd". In this way, we speak of the *parity* of an isometry.

Example 5. Let OA be a line segment, and let $\ell = \overleftrightarrow{OA}$. A *glide reflection* $f = \mathrm{Gref}_{OA}$ is equal to a reflection in ℓ followed by the translation Tran_{OA}, and (by Sheet 7, q. 4) the order does not matter:

$$\mathrm{Gref}_{OA} = \mathrm{Tran}_{OA} \circ \mathrm{Ref}_\ell = \mathrm{Ref}_\ell \circ \mathrm{Tran}_{OA}.$$

A glide reflection is an odd isometry:



As we did in Example 5, we may *compose* any two isometries f and g to obtain new mappings

$$f \circ g \colon \mathscr{E} \to \mathscr{E}, \qquad (f \circ g)(P) = f(g(P)),$$
$$g \circ f \colon \mathscr{E} \to \mathscr{E}, \qquad (g \circ f)(P) = g(f(P)).$$

These are both isometries, the first because

$$d(f(g(P)), f(g(Q))) = d(g(P), g(Q)) = d(P, Q).$$

Moreover, the parity of $f \circ g$ is obtained in the obvious way:

	even	odd
even	even	odd
odd	odd	even

The five examples provide us with the table

isometry:	identity	rotation	reflection	translation	glide reflection
parity:	even	even	odd	even	odd
fixed points:	$\mathscr{E}$	O	ℓ	$\varnothing$	$\varnothing$

10.3. Classification by fixed points

Main theorem. *Any isometry of the plane is one of the ones in the table, namely: the identity, a rotation, a translation, a reflection, or a glide reflection.*

Let us first record two consequences of this theorem.

Corollary 1. *The composition of any two rotations, or of any two reflections, must be a rotation, a translation or the identity.*

Corollary 2. *Any isometry* $f: \mathscr{E} \to \mathscr{E}$ *is a bijective mapping and therefore has an inverse* $f^{-1}: \mathscr{E} \to \mathscr{E}$.

This inverse can be described explicitly case by case, and is always another isometry. For example, $(\mathrm{Rot}_{O,\theta})^{-1} = \mathrm{Rot}_{O,-\theta}$ (or $\mathrm{Rot}_{O,2\pi-\theta}$ if one prefers to use angles between 0 and 2π), whereas $(\mathrm{Ref}_\ell)^{-1} = \mathrm{Ref}_\ell$ is self-inverse.

We shall prove the main theorem by breaking it down into a number of simpler steps.

Propositon 1. *If* A, B *are distinct fixed points of an isometry* f *then* f *is the identity or a reflection in the line* $\ell = \overleftrightarrow{AB}$.

Proof. We shall first prove that f fixes every point of ℓ. Let $P \in \ell$. Construct circles, one with centre A, and one with centre B, passing through P. The two circles must *touch* at P, meaning that they intersect in a single point. If not, there are two points of intersection P, Q and the perpendicular bisector of the chord PQ must pass though both centres, contradicting the fact that $P \in \ell$. (Recall that there is a unique circle through 3 non-collinear points, proving the well-known fact that two circles cannot meet in more than 2 points!) We now know that

$$d(f(P), A) = d(f(P), f(A)) = d(P, A),$$

and similarly with B in place of A. Therefore $f(P)$ lies on both circles, and $f(P) = P$.

Suppose that there exists a point not on ℓ such that $f(P) = P$. Drop a perpendicular from P to ℓ with foot $D \in \ell$. Then f fixes every point on $m = \overleftrightarrow{PD}$. But now every point $Q \in \mathscr{E}$ can be connected by a line that meets ℓ and m in different points (any line through Q will do except for three). By the argument above, Q is also fixed. Therefore f is the identity.

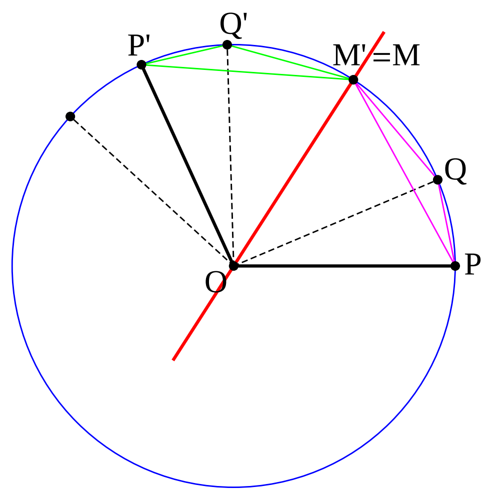
Finally, suppose that f is not the identity, and take P to be a point not on ℓ. Construct circles though P with centres A and B and apply the same trick as before. This time, our assumption forces $f(P) = Q$, where PQ is a chord whose perpendicular bisector is ℓ. It follows that $f(P)$ is reflection of P in ℓ, and $f = \mathrm{Ref}_\ell$. $\square$

Corollary 3. An isometry is uniquely determined by its action on three non-collinear points.

Proof. Suppose that two isometries f, g both move P, Q, R to P', Q', R'. Then $h = f^{-1} \circ g$ fixes these points. Assuming R does not lie on the same line as P and Q, it follows that $h = \mathrm{id}$ and so $f = g$. $\square$

Proposition 2. Let $O \in \mathscr{E}$. Any isometry f such that $f(O) = O$ and for which O is the *only* fixed point is a rotation: $f = \mathrm{Rot}_{O,\theta}$ for some angle θ (not a multiple of 2π).

Sketch proof. A point P distinct from O maps to $f(P) = P'$. Both P and P' lie on a circle $\mathscr{C}$ with centre O. A third point $Q = \mathrm{Rot}_{O,\alpha}(P)$ on $\mathscr{C}$ maps to $f(Q) = Q'$, which is either $\mathrm{Rot}_{O,\alpha}(P')$ or $\mathrm{Rot}_{O,-\alpha}(P')$; suppose the latter:

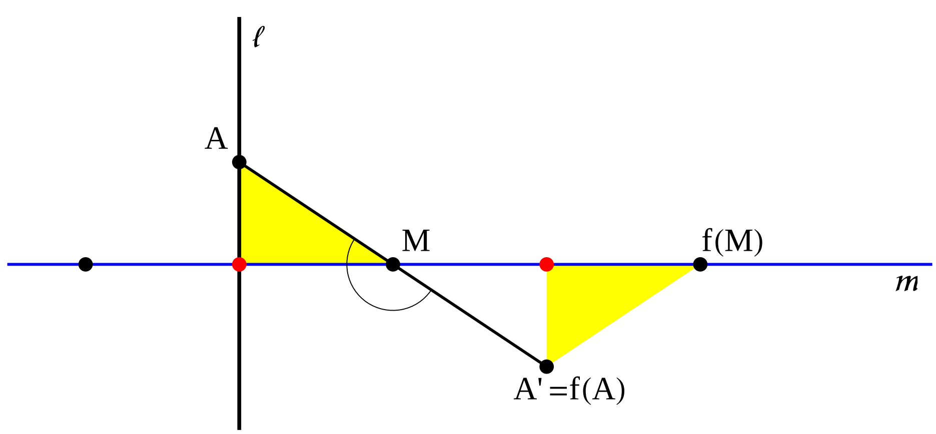


Let M be the point of $\mathscr{C}$ such that $\overrightarrow{OM}$ bisects $\angle QOQ'$, and $M' = f(M)$ its image. By considering $\triangle PQM$, congruent to $\triangle P'Q'M'$, one proves that $M' = M$ is a fixed point, contrary to hypothesis. □

Proposition 3. Any isometry without fixed points is a glide reflection or a translation.

Proof. Given such an f, fix $A \in \mathscr{E}$ and let M be the midpoint of AA' where $A' = f(A)$. Let $g = \mathrm{Rot}_{M,\pi}$ so that A is a fixed point of $h = g \circ f$. By the previous propositions, h is the identity, a reflection or a rotation. *Case 0.* It cannot be the identity for then $f = g^{-1} = g$ has a fixed point.

Case 1. $h = g \circ f$ is a reflection in a line ℓ. The latter cannot contain M for otherwise $f = g \circ h$ would fix M. Drop a perpendicular m from M to ℓ, and consider the effect of $f = g \circ h$ on M. In fact, f is now acting on the yellow triangle as a glide reflection G. By Corollary 3, f must equal G :



Case 2. h is a rotation. It follows that $f = g \circ h$ is an even isometry preserving the orientation of every triangle. We now go back to the drawing board: replace g by $g' = \mathrm{Ref}_m$ where m is the perpendicular bisector of AA'. Then $h' = g' \circ f$ again fixes A, is an odd isometry and must be a reflection. It follows that $f = g' \circ h'$ is the *composition of two reflections*. We shall see that such a composition is rotation about the point of intersection of the mirrors, unless they are parallel in which case (by q. 5 of Sheet 7) it is a translation. □

11 Groups of Isometries

11.1. The concept of a group

Let G denote the set of isometries $f\colon \mathscr{E} \to \mathscr{E}$ of the Euclidean plane $\mathscr{E}$. We have defined a way of *multiplying* two isometries f, g, regarded as abstract symbols, to form their composition $f \circ g$. This operation satisfies the so-called *associative law*

$$(f \circ g) \circ h = f \circ (g \circ h),$$

which asserts that brackets are not essential. The equation is true because both sides applied to P yield the same point $f(g(h(P)))$.

There is also an *identity* element for the multiplication, namely the "lazy isometry" id that satisfies

$$\mathrm{id} \circ f = f = f \circ \mathrm{id}, \qquad \forall f \in G.$$

Moreover, given any isometry $f \in G$ we know that there exists an *inverse* $f^{-1} \in G$ characterized by the equations

$$f \circ f^{-1} = \mathrm{id} = f^{-1} \circ f.$$

They key point here is that f^{-1} exists not just as a mapping $\mathscr{E} \to \mathscr{E}$ of sets, but it is also an isometry. If there were any doubt of this last fact, we merely observe that

$$d(f^{-1}(P), f^{-1}(Q)) = d(f(f^{-1}(P)), f(f^{-1}(Q))) = d(P, Q),$$

the first equality because f itself preserves distance.

A set G with the above structure is called a *group*, and this is a key concept that occurs in many different branches of mathematics and other subjects, including the physics of elementary particles and developmental psychology.

The set G of isometries (we can now say the "group of isometries") obviously has infinitely many elements. A simple example of a finite group consists of the set

$$F = \{1,\ i,\ -1,\ -i\} \subset \mathbb{C}$$

("F" because it has 4 elements), with the usual multiplication of complex numbers. All the group rules are satsified. For example, the inverse of i is $i^{-1} = 1/i = -i$, and -1 is self-inverse.

Another infinite example is the set of 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with non-zero determinant $\Delta = ad - bc$. We can multiply them since $\det(AB) = (\det A)(\det B)$ remains non-zero. Matrix multiplication obeys the associative law (though this is not obvious), and there is an identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The inverse of A is given by

$$\frac{1}{\Delta}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & d/\Delta \end{pmatrix}.$$

Denoting this matrix by A^{-1}, it is easy to verify that both AA^{-1} and $A^{-1}A$ equal I. We shall show that this example is relevant to the study of isometries, at least those that fix a given point.

Let us return to isometries, and choose a point O in the plane. Let f be an isometry, and set $A = f(O)$, and let $h = \mathrm{Tran}_{OA}$. Then $h^{-1} = \mathrm{Tran}_{AO}$ and $g = h^{-1} \circ f$ fixes O. It follows that any element of G can be expressed as $h \circ g$, where g belongs to the set

$$G_O = \{f \in G : f(O) = O\}$$

of isometries that fix O.

The set G_O is a group just like G is: just observe that if g and g' fix O so does $g \circ g'$, and $g^{-1}(O) = g^{-1}(g(O)) = O$. Indeed, G_O is an example of a *subgroup* of G, being a subset "closed" under the group operations. Another subgroup is the set H of all translations, and G is in some sense formed by combining H and G_O.

In practice, we already know what the elements of G_O are: the identity, all rotations centred at O and all reflections whose mirror contains O. We can also assert that the identity and the set of rotations forms a subgroup. However, at this juncture we wish to describe elements of G_O more analytically.

11.2. Linear isometries

Choose perpendicular axes centred at O, so that we can represent any point P of $\mathscr{E}$ by its Cartesian coordinates (x, y), or equivalently by the vector

$$\mathbf{v} = \vec{OP} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that

$$d(O,P)^2 = |OP|^2 = x^2 + y^2 = \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}.$$

In the same way, we denote by $f(\mathbf{v})$ the column vector version of $f(P)$.

Theorem. *Any element of* G_O *is a* linear *mapping, that is* $f(\mathbf{v}+\mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w})$ *and* $f(\lambda\mathbf{v}) = \lambda f(\mathbf{v})$ *for* $\lambda \in \mathbb{R}$.

Proof. The isometry condition is $d(f(P), f(Q)) = d(P, Q)$. In vector language this translates into the equation

$$\|f(\mathbf{v}) - f(\mathbf{w})\| = \|\mathbf{v} - \mathbf{w}\|,$$

which when squared and expanded becomes

$$\|f(\mathbf{v})\|^2 - 2f(\mathbf{v}) \cdot f(\mathbf{w}) + \|f(\mathbf{w})\|^2 = \|\mathbf{v}\|^2 - 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2.$$

But we also know that

$$\|f(\mathbf{v})\|^2 = d(O, f(P))^2 = |OP|^2 = \|\mathbf{v}\|^2,$$

and similarly for $\mathbf{w}$ in place of $\mathbf{v}$. Therefore,

$$\boxed{f(\mathbf{v}) \cdot f(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}}$$

Now consider the unit vectors

$$\mathbf{v}_1 = \mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

parallel to the axes. These two form an *orthonormal basis*, meaning that

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases}$$

and an arbitrary vector equals

$$\mathbf{v} = x\mathbf{v}_1 + y\mathbf{v}_2, \quad \text{with} \quad x = \mathbf{v} \cdot \mathbf{v}_1, \ \ y = \mathbf{v} \cdot \mathbf{v}_2.$$

It follows from the boxed formula that $f(\mathbf{v}_1)$ and $f(\mathbf{v}_2)$ also form an orthonormal basis, and so

$$f(\mathbf{v}) = Xf(\mathbf{v}_1) + Yf(\mathbf{v}_2),$$

where $X = f(\mathbf{v}) \cdot f(\mathbf{v}_1) = \mathbf{v} \cdot \mathbf{v}_1 = x$, and $Y = y$. Thus,

$$f(x\mathbf{v}_1 + y\mathbf{v}_2) = xf(\mathbf{v}_1) + yf(\mathbf{v}_2),$$

and the fact that f is linear follows immediately (well, almost!). □

If we set

$$f(\mathbf{v}_1) = \begin{pmatrix} a \\ c \end{pmatrix}, \qquad f(\mathbf{v}_2) = \begin{pmatrix} b \\ d \end{pmatrix},$$

then the last equation of the proof shows that f maps

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}.$$

This means that to apply the isometry f, we need to multiply the column vector $\mathbf{v}$ by the matrix A defined by the last equality.

But we also know that the columns of A are orthonormal: they satisfy

$$a^2 + c^2 = 1 = b^2 + d^2 \quad \text{and} \quad ab + cd = 0.$$

This can be expressed by the equation

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad A^T A = I.$$

Definition. A square matrix A that satisfies $A^T A = I$ is called *orthogonal*.

It follows from the theory of determinants that if A is an orthogonal matrix then $\det A = \pm 1$. This in turn tells us that the inverse matrix A^{-1} exists and equals A^T, so it is also true that $AA^T = I$.

11.3. Rotations and reflections revisited

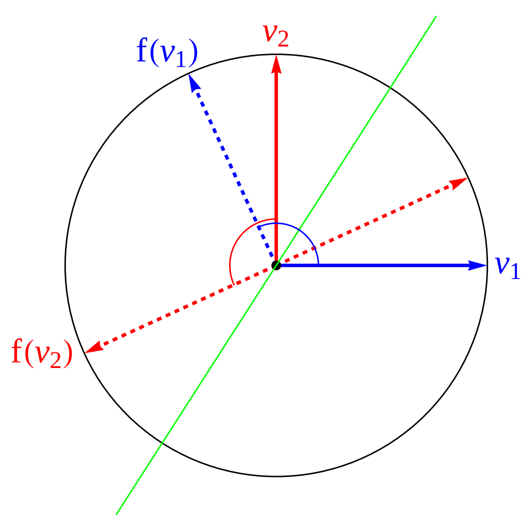
We are now in a position to classify 2×2 orthogonal matrices. Given such a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we know that the first column is a unit vector, and so there is a unique angle θ with $0 \leqslant \theta < 2\pi$ for which $a = \cos\theta$ and $b = \sin\theta$. The second column is also a unit vector, but it is perpendicular to the first, so there are two choices, one minus the other:

$$\begin{aligned} & b = -\sin\theta, \quad c = \ \cos\theta, \\ \text{or} \quad & b = \ \sin\theta, \quad c = -\cos\theta. \end{aligned}$$

They correspond to rotating the first column by $\pi/2$ and $-\pi/2$ respectively.



Both situations are illustrated above, but $f(\mathbf{v}_1)$ is labelled only for the first, which obviously corresponds to an overall rotation though an angle θ of about 120^{o}. The second corresponds to a reflection in the (green) line making an angle of $\theta/2$ with $\mathbf{v}_1$. We saw a similar diagram in the proof of Proposition 2 in §10.3.

The corresponding rotation and reflection matrices are

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

They are easily distinguished by their determinants: $\det R_\theta = 1$ whilst $\det S_\theta = -1$. The sign of the determinant therefore coincides with the partity of the linear isometry.

Let $f, g \in G_O$ be two linear isometries with associated matrices A, B. The composition $f \circ g$ is represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\left(B\begin{pmatrix} x \\ y \end{pmatrix}\right) = AB\begin{pmatrix} x \\ y \end{pmatrix},$$

so composition of isometries translates into multiplication of matrices!

Since R_θ represents a rotation by an angle θ, we can now assert that $R_\theta R_\phi$ represents a rotation by $\theta + \phi$. Indeed,

$$R_\theta R_\phi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = R_{\theta+\phi}.$$

A very similar calculation gives

Lemma. $S_\theta S_\phi = R_{\theta-\phi}$.

This confirms that the composition of *any* two distinct reflections whose mirrors are not parallel is a rotation. (Just choose O to be the point of intersection of the mirrors.)

𝔚𝔞𝔯𝔫𝔦𝔫𝔤. Remember that θ in S_θ is twice the mirror angle (q 4 on Sheet 8 also justifies this interpretation).

11.4. 3 × 3 orthogonal matrices

Let us begin with the following observation. If $\mathbf{v}, \mathbf{w}$ are column vectors (now with 3 entries), the matrix product $\mathbf{v}^T\mathbf{w}$ (of a row with a column) equals the dot product $\mathbf{v} \cdot \mathbf{w}$. (This is not quite true since $\mathbf{v}^T\mathbf{w}$ is strictly speaking a 1×1 matrix, so first we have to throw its brackets away!) In particular, $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{v}$.

Now let A be a 3×3 orthogonal matrix, so (by definition) $A^TA = I$. If we denote the columns of A by the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, the rows of A^T are the corresponding row vectors $\mathbf{v}_1^T, \mathbf{v}_2^T, \mathbf{v}_3^T$, and it follows from above that

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases}$$

This means that the columns of A form an *orthonomal* triple: they are unit vectors that are mutually perpendicular. ($\mathfrak{Warning}$. "Orthogonal" means "perpendicular", so calling A orthogonal is a misnomer. The word "normal" in this context means "normalized", i.e. of unit length. But to add to the confusion, "normal" also means "perpendicular" in other contexts!)

Lemma. If A is an $n \times n$ orthogonal matrix, its determinant equals 1 or -1.

Proof. We have already verified this when $n = 2$. In general,

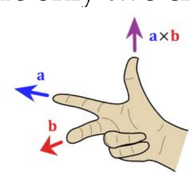
$$\det(A^T A) = \det I = 1,$$

and the left-hand side equals $\det(A^T)\det(A) = (\det A)^2$, by two standard properties of the determinant. □

Since $\det A \neq 0$, the orthogonal matrix A has an inverse A^{-1} which must therefore equal its transpose A^T. This in turn implies that $AA^T = I$ which allows one to repeat the discussion above using rows instead of columns, so the rows of an orthogonal matrix must also be orthonormal.

When $n = 3$, the two signs of the determinant correspond to the two different ways of "orienting" the triple above with thumb, first and second fingers (use of the left hand or the right hand). Moreover, once we know $\mathbf{v}_1$ and $\mathbf{v}_2$, there are only two choices for $\mathbf{v}_3$, namely plus or minus the cross product:

$$\det A = 1 \quad \Rightarrow \quad \mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3 \qquad \text{`right-handed' :}$$
$$\det A = -1 \quad \Rightarrow \quad \mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_3 \qquad \text{`left-handed'.}$$



12 Some 3-dimensional geometry

12.1. Rotations in space

Let A be a 3×3 matrix. Then

$$f \colon \mathbf{v} \mapsto A\mathbf{v}$$

defines a mapping of points in space that fixes the origin O. It is also linear because, for example, $A(\mathbf{v}+\mathbf{w}) = A\mathbf{v} + A\mathbf{w}$. It is a fact that any linear mapping $\mathbb{R}^3 \to \mathbb{R}^3$ arises in this way (we proved the analogous result for $\mathbb{R}^2$).

If A is also orthogonal (meaning $A^T A = I$), then f preserves the dot product in the sense that

$$\boxed{f(\mathbf{v}) \cdot f(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}}$$

To see this, recall that the dot product can be computed by multiplying a row by a column, so if $\mathbf{v}, \mathbf{w}$ are column vectors,

$$f(\mathbf{v}) \cdot f(\mathbf{w}) = (A\mathbf{v})^T (A\mathbf{w}) = \mathbf{v} A^T A\, \mathbf{w} = \mathbf{v}^T \mathbf{w} = \mathbf{v} \cdot \mathbf{w}.$$

Taking $\mathbf{v}=\mathbf{w}$, we get $\|f(\mathbf{v})\|^2=\|\mathbf{v}\|^2$. Moreover,

$$\|f(\mathbf{v})-f(\mathbf{w})\|^2=\|f(\mathbf{v}-\mathbf{w})\|^2=\|\mathbf{v}-\mathbf{w}\|^2,$$

so f is in fact an isometry. (In a previous lecture, we stook these steps backwards, starting from an isometry that fixes O, but arguing forwards is easier!)

Here is an example:

$$A=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}=\left(\begin{array}{c|c} R_\theta & 0\\ \hline 0 & 1\end{array}\right).$$

Since

$$A\begin{pmatrix}0\\0\\z\end{pmatrix}=\begin{pmatrix}0\\0\\z\end{pmatrix},$$

this column vector is an *eigenvector* of A with *eigenvalue* 1 (we should take $z\neq 0$ since the definition of eigenvector are normally requires it to be non-zero). It means that f fixes every point on the z-axis. On the other hand, A acts on the xy-plane as a rotation with angle θ.

Lemma. *Let A be a 3×3 orthogonal matrix with $\det A=1$. Then $\det(A-I)=0$.*

Proof. This depends on various properties of the determinant. We have

$$\begin{aligned}\det(A-I)=\det(A-A^TA) &= \det((I-A^T)A)\\ &= \det(I-A^T)(\det A)\\ &= \det((I-A)^T)\\ &= \det(I-A)\\ &= -\det(A-I),\end{aligned}$$

the last equality because $\det(\lambda A)=\lambda^3 A$. $\square$

This result tells us that the matrix $A-I$ does not have an inverse, or that the *rank* of $A-I$ is less than 3. It follows from the theory of linear equations that the equation

$$(A-I)\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad\text{or}\quad (A-I)\mathbf{v}=\mathbf{0}$$

has a non-trival solution (meaning not all of x,y,z are zero). Thus, there exists a non-zero vector $\mathbf{v}$ such that $A\mathbf{v}=\mathbf{v}$. It is again an eigenvector with eigenvalue 1. Since $\mathbf{v}\neq\mathbf{0}$, we may also suppose (by dividing by its norm) that it is a unit vector.

It follows that f fixes every point on the line ℓ through O parallel to $\mathbf{v}$. Notice also that

$$\mathbf{v}\cdot\mathbf{w}=0\ \Rightarrow\ \mathbf{v}\cdot f(\mathbf{w})=f(\mathbf{v})\cdot f(\mathbf{w})=\mathbf{v}\cdot\mathbf{w}=0.$$

This means that if $\mathbf{w}$ is perpendicular to $\mathbf{v}$ so is $f(\mathbf{w})$. Hence, f maps the plane perpendicular to $\mathbf{v}$ to itself, and therefore acts as an isometry in this plane. Using the fact that $\det A = 1$, one can show that this isometry is even, and so a rotation (or the identity). Since f is linear, its action on *any* vector is now determined, and we can conclude by stating the

Theorem. *The linear mapping f associated to a 3×3 orthogonal matrix with determinant 1 is a rotation in space about a line ℓ.*

12.2. Equation of a plane

This subsection and the next is mainly revision of Linear Methods material, but will be essential in our study of triangles on the sphere. Consider first a plane Π that passes through the origin O. It consists of all points P whose position vectors

$$\vec{OP} = \mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{are perpendicular to some fixed vector} \quad \mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The equation is therefore

$$ax + by + cx = 0.$$

In general, Π will not pass through O but we can suppose that $P_0 = (x_0, y_0, z_0) \in \Pi$. If the plane is still perpendicular to $\mathbf{n}$ then the equation is

$$(\mathbf{v} - \mathbf{v}_0) \cdot \mathbf{n} = 0 \quad \text{or} \quad ax + by + cz = d,$$

where $d = \mathbf{v}_0 \cdot \mathbf{n} = ax_0 + by_0 + cz_0$ is a constant.

The vector $\mathbf{n}$ is said to be *normal* to the plane. We can, if we wish, suppose it is a unit vector, so that $1 = \|\mathbf{n}\|^2 = a^2 + b^2 + c^2$.

Two planes

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

are parallel if and only if their normals $\mathbf{n}_1, \mathbf{n}_2$ are parallel. This means that

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \times \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{pmatrix}$$

is the zero vector.

If the planes are not parallel, they must intersect in a line. In this case, $\mathbf{n}_1 \times \mathbf{n}_2$ is parallel to the line formed by the intersection of the two planes (because a vector in such a line must be perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$, and so parallel to $\mathbf{n}_1 \times \mathbf{n}_2$). A worked problem will make these statements clear:

Example. Describe the intersection of the planes

$$\begin{cases} x + y + z = 1, \\ x + 2y + 3z = 0, \end{cases}$$

We can solve these equations by treating (for example) z as a constant, and finding x and y in terms of z:

$$y + 2z = -1 \;\Rightarrow\; y = -1 - 2z; \qquad x - z = 2 \;\Rightarrow\; x = 2 + z.$$

Setting $z = t$, we see that the intersection consists of the line $(x, y, z) = (2+t, -1-2t,\ t)$ with $t \in \mathbb{R}$, or, in column vector format,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

The third column gives the direction of the line and, as predicted, is proportional (indeed, equal) to $\mathbf{n}_1 \times \mathbf{n}_2$.

12.3. Angles between lines and planes

Two distinct lines in space are related to one another in one of three ways: (i) they are parallel, (ii) they intersect, or (iii) they do not lie in a common plane. These possibilities are mutually exclusive, because if two lines lie in a plane, then either (i) or (ii) must apply. In case (iii), the lines are said to be *skew*.

If two lines intersect, then they lie in a plane, and the angle between them is defined. Actually, there are two angles, since we cannot distinguish between θ and $\pi - \theta$ (apart from choosing the smaller one, which is unfair on the other!).

The situation is sightly different for rays (half-lines) because this time two valid angles are θ and $2\pi - \theta$. In practice, if we choose vectors in the direction of the rays, this can be computed using the dot product. We have (from Linear Methods) the

Proposition. *The angle θ between two unit vectors $\mathbf{u}_1, \mathbf{u}_2$ satisfies*

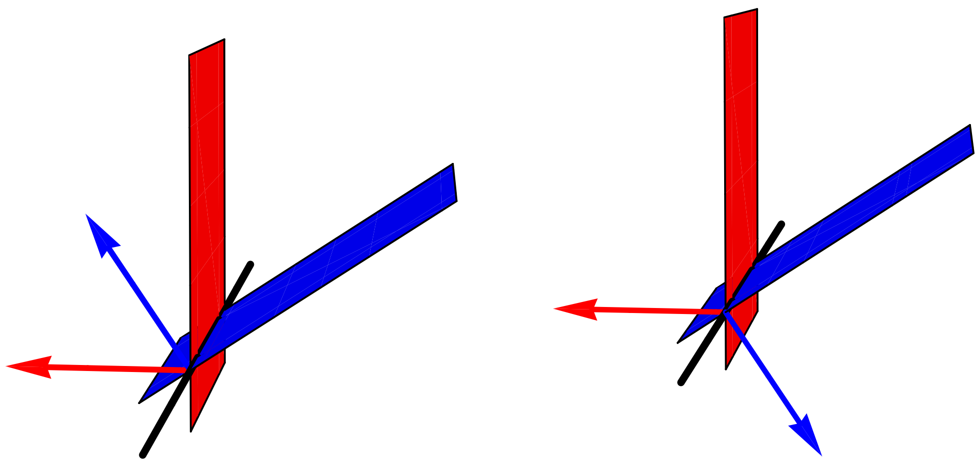
$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \cos\theta = \cos(2\pi - \theta).$$

It two planes Π_1, Π_2 intersect, necessarily in a line ℓ, then the angle between them is defined by imagining their cross-sections with a third plane perpendicular to ℓ. These cross-sections will form a pair of intersecting lines m_1, m_2, which will define a pair of angles ϕ, $\pi - \phi$ from the discussion above. Since m_1, m_2 make angles of $\pm\pi/2$ with the normal vectors $\mathbf{n}_1, \mathbf{n}_2$, we have have another result from Linear Methods:

Proposition. *The angle ϕ between two planes Π_1, Π_2 with* unit *normals $\mathbf{n}_1, \mathbf{n}_2$ satisfies*

$$\pm\mathbf{n}_1 \cdot \mathbf{n}_2 = \cos\phi \text{ or } \cos(\pi - \phi), \quad \text{with} \quad 0 < \phi < \pi.$$

We cannot distinguish one sign or other on the left-hand side since $-\mathbf{n}_i$ is as good a normal as $\mathbf{n}_i$. But that ambiguity is consistent with the fact that the two angles ϕ and $\pi - \phi$ are equally valid and yet their two cosines differ by a sign:



13 Spherical geometry

Let $\triangle ABC$ be a triangle in the Euclidean plane. From now on, we indicate the interior angles $\angle A = \angle CAB$, $\angle B = \angle ABC$, $\angle C = \angle BCA$ at the vertices merely by A, B, C. The sides of length $a = |BC|$ and $b = |CA|$ then make an angle C. The *cosine rule* states that

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$$

if $C = \pi/2$ it reduces to Pythagoras' theorem. It is easily proved by constructing (say) the altitude AA' of length $a' = h$. (Take BC to be the "base" of the triangle so that h is the height, and draw the picture.) Now apply Pythagoras to $\triangle AA'B$ and $\triangle AA'C$ to get

$$c^2 = h^2 + (a - b\cos C)^2, \qquad b^2 = h^2 + (b\sin C)^2.$$

The rule follows by eliminating h^2.

The *sine rule* states that

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

It can also be proved using the altitude AA', since

$$b\sin C = h = c\sin B,$$

and the rest follows by symmetry.

It is important to realize that the sine rule can also be deduced algebraically from the cosine rule. The latter tells us that

$$\left(\frac{\sin C}{c}\right)^2 = \frac{1-\cos^2 C}{c^2} = \frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2b^2c^2}.$$

The numerator on the right-hand side, when expanded, is symmetric in $a, b, c,$ and it follows that we can replace c, C by a, A or b, B on the left. The sine rule follows because $\sin C/c > 0$.

The aim of this section is to prove analogous formulas for spherical triangles.

13.1. Spherical triangles: the vertices and sides

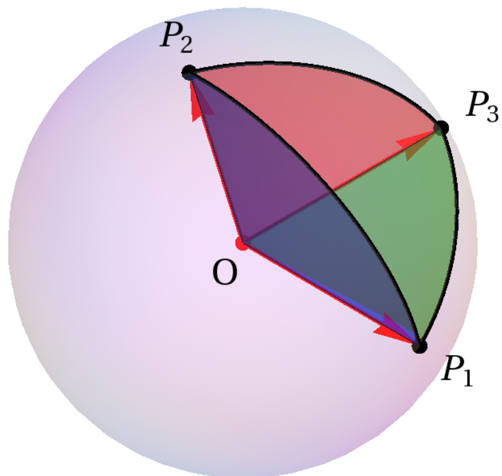
Fix Cartesian coordinates in space, with origin $O = (0, 0, 0)$. Consider the sphere

$$\mathscr{S} = \{P \ : \ d(O, P) = 1\} = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

centred at O. The word "sphere" in geometry refers exclusively to the *surface*, not the inside! The position vector

$$\mathbf{v} = \vec{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

of any point $P \in \mathscr{S}$ is a unit vector, i.e. a vector of norm one: $\|\mathbf{v}\| = 1$.



Now suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are three *unit* vectors representing points on $\mathscr{S}$. The corresponding points P_1, P_2, P_3 will be the vertices of a spherical triangle provided the 'unit-column' matrix

$$V = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix},$$

is invertible, or equivalently

$$\begin{aligned}\det V &= (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 = (\mathbf{v}_2 \times \mathbf{v}_3) \cdot \mathbf{v}_1 = (\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_2 \\ &= -(\mathbf{v}_2 \times \mathbf{v}_1) \cdot \mathbf{v}_3 = -(\mathbf{v}_3 \times \mathbf{v}_2) \cdot \mathbf{v}_1 = -(\mathbf{v}_1 \times \mathbf{v}_3) \cdot \mathbf{v}_2\end{aligned}$$

is non-zero. In this case, O, P_1, P_2, P_3 are not coplanar, and the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a *basis* of $\mathbb{R}^3$. The "sides" of the triangle are then the segments of great circles (of radius 1) through the vertices.

Two of our vectors, say $\mathbf{v}_1$ and $\mathbf{v}_2$, generate a (blue) plane Π_3 that passes through O; the intersection $\Pi_3 \cap \mathscr{S}$ is a circle, and the arc from P_1 to P_2 is the side of the triangle opposite P_3. The lengths of the three sides are *equal* to the angles (in radians)

$$\theta_1 = \angle P_2 O P_3, \quad \theta_2 = \angle P_3 O P_1, \quad \theta_3 = \angle P_1 O P_2,$$

whose cosines are

$$c_1 = \cos\theta_1 = \mathbf{v}_2 \cdot \mathbf{v}_3, \quad c_2 = \cos\theta_2 = \mathbf{v}_3 \cdot \mathbf{v}_1, \quad c_3 = \cos\theta_3 = \mathbf{v}_1 \cdot \mathbf{v}_2.$$

These quantities feature in the symmetric matrix

$$V^T V = \begin{pmatrix} 1 & c_3 & c_2 \\ c_3 & 1 & c_1 \\ c_2 & c_1 & 1 \end{pmatrix}.$$

We shall assume that the three angles/lengths are no greater than π.

13.2. Spherical law of cosines[1]

Set $\Delta = \det V \neq 0$, and define

$$\mathbf{w}_1 = \frac{1}{\Delta}\mathbf{v}_2 \times \mathbf{v}_3, \quad \mathbf{w}_2 = \frac{1}{\Delta}\mathbf{v}_3 \times \mathbf{v}_1, \quad \mathbf{w}_3 = \frac{1}{\Delta}\mathbf{v}_1 \times \mathbf{v}_2.$$

Then $\mathbf{w}_1 \cdot \mathbf{v}_1 = 1$, $\mathbf{w}_1 \cdot \mathbf{v}_2 = 0$ and so on, indeed: $\mathbf{w}_i \cdot \mathbf{v}_j = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$

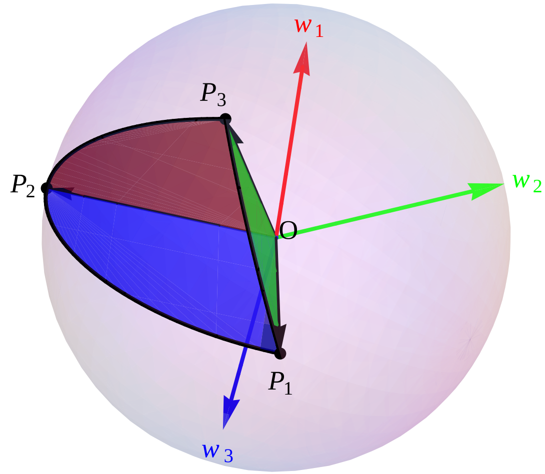
It follows that the inverse of V is the matrix

$$W^T = \begin{pmatrix} \leftarrow \mathbf{w}_1^T \rightarrow \\ \leftarrow \mathbf{w}_2^T \rightarrow \\ \leftarrow \mathbf{w}_3^T \rightarrow \end{pmatrix}.$$

If V is an orthogonal matrix, the original basis is orthonormal and $W = V$. In general, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is the *reciprocal basis* to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

[1]The following approach due to W. P. Thurston, 1946–2012

The vector $\mathbf{w}_1 = \frac{1}{\Delta}\mathbf{v}_1 \times \mathbf{v}_2$ is normal to the plane containing O, P_2, P_3. Moreover, a triple like $\mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1$ has a right-handed orientation, which makes the normal $\mathbf{w}_3$ point 'outwards' from the triangular solid. We previously defined the angle of the spherical triangle at say P_1 to be the angle between the tangents to the two arcs meeting at P_1. But these tangents are both perpendicular to the radial line OP_1, and we are therefore speaking of the angle (defined in §12.3) between the two planes meeting along $\overleftrightarrow{OP_1}$.



The interior angles ϕ_1, ϕ_2, ϕ_3 of the spherical triangle each measure $180^{\rm o}$ minus the angles between the normals pictured overleaf, and so

$$\cos\phi_1 = -\frac{\mathbf{w}_2\cdot\mathbf{w}_3}{\|\mathbf{w}_2\|\|\mathbf{w}_3\|}, \qquad \cos\phi_2 = -\frac{\mathbf{w}_3\cdot\mathbf{w}_1}{\|\mathbf{w}_3\|\|\mathbf{w}_1\|}, \qquad \cos\phi_3 = -\frac{\mathbf{w}_1\cdot\mathbf{w}_2}{\|\mathbf{w}_1\|\|\mathbf{w}_2\|}.$$

We now apply these calculations to a spherical triangle. Since $V^{-1} = W^T$ and (for any matrix, $(V^{-1})^T = (V^T)^{-1}$), we have $(V^TV)^{-1} = V^{-1}(V^T)^{-1} = W^T(V^{-1})^T = W^TW$, and so

$$W^TW = \frac{1}{\Delta^2}\begin{pmatrix} 1-c_1^2 & c_1c_2-c_3 & c_1c_3-c_2 \\ c_1c_2-c_3 & 1-c_2^2 & c_2c_3-c_1 \\ c_1c_3-c_2 & c_2c_3-c_1 & 1-c_3^2 \end{pmatrix}.$$

Each row of this symmetric matrix is the cross product of the columns of V^TV, and the entries of the matrix are the cofactors of V^TV; this is how they were written down.

It follows that

$$\mathbf{w}_1\cdot\mathbf{w}_1 = \frac{1}{\Delta^2}(1-c_1^2), \qquad \mathbf{w}_1\cdot\mathbf{w}_2 = \frac{1}{\Delta^2}(c_1c_2-c_3).$$

The first equation confirms that $\|\mathbf{w}_1\| = s_3/|\Delta|$, where $s_3 = \sin\theta_3$, something we already know as $\|\mathbf{v}_1\times\mathbf{v}_2\| = s_3$. The second yields

$$-\cos\phi_3 = \frac{\mathbf{w}_1\cdot\mathbf{w}_2}{\|\mathbf{w}_1\|\|\mathbf{w}_2\|} = \frac{c_1c_2-c_3}{s_1s_2}.$$

Rearranging, and writing this out in full,

$$\cos\theta_3 = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi_3.$$

Using A, B, C for the vertices and their interior angles, and a, b, c for the lengths of he opposite sides, we have proved the

Theorem. *The cosine of the length of the third side of a spherical triangle is given by*

$$\boxed{\cos c = \cos a\cos b + \sin a\sin b\cos C}$$

13.3. Applications

If the triangle is very small compared to the unit radius of the sphere (as is the case on the surface of the earth), we may reasonably use approximations

$$\cos x = 1 - \tfrac{1}{2}x^2, \qquad \sin x = x$$

for $x = a, b, c,$ given by Taylor's theorem. Then

$$1 - \tfrac{1}{2}c^2 = (1 - \tfrac{1}{2}a^2)(1 - \tfrac{1}{2}b^2) + ab\cos C,$$

and to order 2 we obtain the Euclidean cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

We do not approximate $\cos C$ as there is no assumption that C be small. Pythagoras' theorem is the special case in which $C = \pi/2$ so $\cos C = 0$. The spherical version of "Pythagoras" is therefore

$$\cos c = \cos a\cos b.$$

The spherical sine rule

$$\boxed{\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}}$$

can be deduced from the cosine rule as we did in the for the Euclidean version

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The fact that the spherical sine rule can also be expressed as

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

suggests that one might be able to interchange the side lengths a, b, c (the θ_i) and vertex angles A, B, C (the ϕ_i) in the cosine formula. This is almost true, because we can switch the roles of the matrices V and W in the proof. But remembering the minus signs in front of $\mathbf{w}_i \cdot \mathbf{w}_j$, gives one overall sign change:

Theorem. *The third angle of a spherical triangle is given by*

$$\boxed{\cos C = -\cos A \cos B + \sin A \sin B \cos c}$$

In all these formulas, we assume that the quantities a, b, c and A, B, C are all less than π. In particular, the mapping $C \mapsto \cos C$ is a *bijection* $[0, \pi) \to (-1, 1]$.

Once one angle C and the lengths a, b of adjacent sides are known, the (first) cosine rule can be used to determine the third side. The same rule (with sides switched) can then be used to find the remaining angles B and C. This tells us that the SAS rule applies to spherical triangles provided we restrict to $k = 1$ to get congruent triangles:

Corollary. *If two spherical triangles have one angle equal and the lengths of the corresponding adjacent sides equal, then all corresponding sides and angles are equal:*

$$a = a',\ b = b',\ C = C' \quad \Rightarrow \quad A = A',\ B = B',\ c = c'.$$

The second cosine rule gives us a property that is *not* true for Euclidean triangles:

Corollary. *The lengths of a spherical triangle are determined by its angles.*

Let us "grade" spherical geometry according to our initial postulates B1–B5:

B1 fails because lines (meaning, great circles) are not infinite in extent, and the distance between any two points on $\mathscr{S}$ is at most π.

B2 fails because opposite (the correct word is *antipodal*) points lie on infinitely many lines.

B3 is valid because at any point (think of it as the north pole) there is a great circle leaving at any angle.

B4 is valid because we use tangent vector to measure angles.

B5 only works for the scaling factor $k = 1$.

[Not examinable: The failure of B2 is not in itself serious. B2 *will* apply if we merely declare that antipodal points are in fact equal. This gives a new type of plane:

Definition. The *real projective plane* $\mathscr{P}$ is the set of all straight lines passing through the origin O in $\mathbb{R}^3$; these are the *points* of $\mathscr{P}$. The set of all such lines in a given plane through O defines a subset of $\mathscr{P}$, called a *line*.

Each straight line though O intersects $\mathscr{S}$ in two antipodal points, so a single *point* of $\mathscr{P}$ corresponds to a pair of antipodal points of $\mathscr{S}$. A line of $\mathscr{P}$ corresponds to a great circle in $\mathscr{S}$. Since B5 still fails, the parallel postulate is not valid in $\mathscr{P}$. But the situation is rather satisfactory: *any two lines meet in exactly one point*!]

13.4. The area of a spherical triangle

In this course, we have said little about area. But we take it for granted that the area of a sphere of radius 1 is known to be 4π. In view of the last corollary, it is reasonable to suppose that the area of a spherical triangle $\mathscr{T}$ is completely determined by its three angles A, B, C. This means that we can write

$$\text{area}(\mathscr{T}) = f(A, B, C),$$

where f is a positive function whose value depends synmetrically on three variables.

If we divide the solid sphere into segments like those of an orange, we obtain a region on the surface consisting of just two great circles that meet at antipodal points at an equal angle of (say) C. The area of this region will be proportional to C and therefore equal to $(C/2\pi)4\pi = 2C$. We can divide the region into two triangles by choosing a third great circle that cuts the other two, giving us angles $A, \pi - A$ and $B, \pi - B$. Hence the equation

$$f(A, B, C) + f(\pi - A,\ \pi - B,\ C) = 2C$$

must hold for all $A, B, C < \pi$. The obvious guess

$$f(A, B, C) = A + B + C - \pi$$

is in fact the correct formula for f, though the proof of this fact is postponed until the second year[2].

Theorem (on spherical excess). *The area of a spherical triangle equals the sum of its angles minus π.*

Corollary. *The angles of any spherical triangle add up to a number greater than π.*

Example. An equilateral triangle (meaning $a = b = c$) must have all angles greater than $\pi/3$. We can check this from the cosine rule, which tells us that $A = B = C$ satisfies

$$\cos a = \cos^2 a + (\sin^2 a)\cos A.$$

Set $t = \tan(a/2)$ so that $\sin a = 2t/(1 + t^2)$ and $\cos a = (1 - t^2)/(1 + t^2)$. Then

$$(1 - t^2)(1 + t^2) = (1 - t^2)^2 + 4t^2 \cos A,$$

and $\cos A = \frac{1}{2}(1 - t^2) < \frac{1}{2}$ so $A > \pi/3$.

14 The hyperbolic plane

The cosine rule for triangles on a sphere of radius r is

$$\cos\frac{c}{r} = \cos\frac{a}{r}\cos\frac{b}{r} + \sin\frac{a}{r}\sin\frac{b}{r}\cos C.$$

[2] Later Dr Dmitri Panov told me of a simple proof based on similar ideas

To modify the proof in §13.2, we just have to replace the arc lengths a, b, c by the angles a/r, b/r, c/r that these arcs subtend at the centre of the sphere, whilst the vertex angles A, B, C are unchanged. The quantity $1/r^2$ is called the *curvature* of the sphere. The Euclidean plane corresponds to a sphere with $r = \infty$ and has zero curvature.

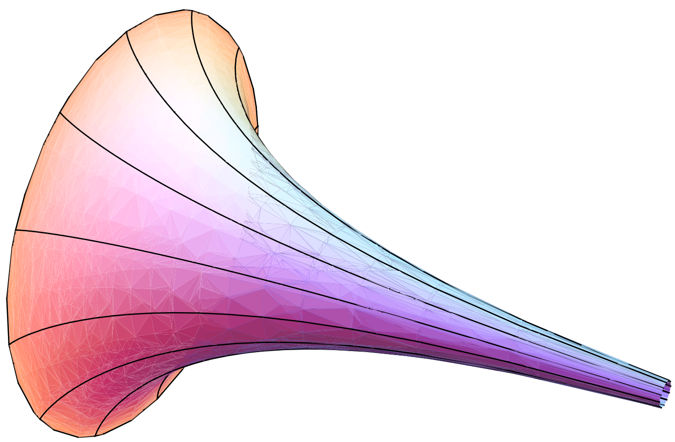
If we substitute $r = i = \sqrt{-1}$, the trigonometric functions become hyperbolic since

$$\cos(-ia) = \cosh a, \qquad \sin(-ia) = -i \sinh a$$

(see §14.2 below). In this way, we obtain the so-called *hyperbolic cosine rule*

$$\boxed{\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C}$$

This formula is valid on a surface of revolution called a *pseudosphere*, obtained by rotating a tractrix curve:



The sides of the triangle must be *geodesics*, the paths of least length between two points, but it is not so easy to say what these are in general (though the tractrix curves shown are geodesics).

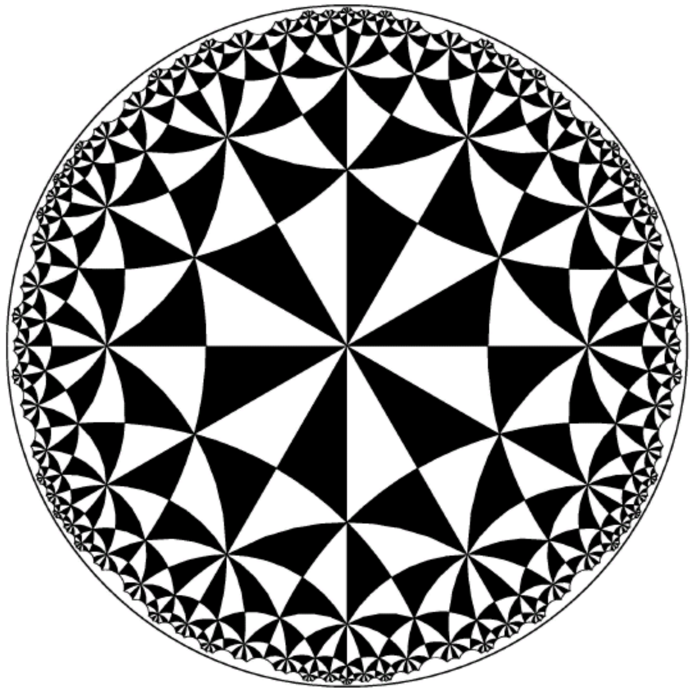
14.1. The Poincaré disc

We shall now present a more abstract model in which the hyperbolic cosine rule is valid. It will also be a model for our geometrical postulates, one which satisfies B1 (unlike the real projective plane), B2, B3, B4, but not B5. There are different ways of presenting this, and we shall adopt the so-called Poincaré disc model:

Definition. The *hyperbolic plane* is the set of points inside the unit circle C, excluding the boundary:

$$\mathscr{H} = \{(x, y) : x^2 + y^2 < 1\}.$$

A *line* in $\mathscr{H}$ (also called *h-line*) is either (i) a diameter, or (ii) the arc of another circle that intersects C at right angles:



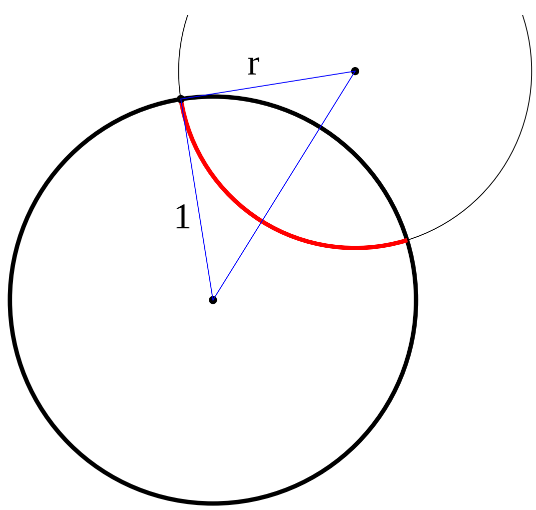
Proposition. *Any line in* $\mathscr{H}$ *has an equation of the form*

(i) $ax + by = 0$ *where* a, b *are not both zero, or*

(ii) $x^2 + y^2 + 2dx + 2ey + 1 = 0$ *where* $d^2 + e^2 > 1.$

Proof. The equation in (i) is obviously that of a diameter.

The equation in (ii) was the one we used for a circle, centre $(-d, -e)$, and radius r where $r^2 = d^2 + e^2 - 1$. We have to explain why this constraint between radius and centre is exactly the condition that it meets the unit circle C at right angles.



This is illustrated by the diagram: the circles meet at right angles, thus so do their radii. The line segment joining their centres is then the hypotenuse of a right-angled triangle with other sides of length 1 and r. □

Proposition. *Given any two points $P_1, P_2 \in \mathscr{H}$, there exists a unique line ℓ in $\mathscr{H}$ such that $P_1, P_2 \in \ell$.*

Proof. To make this precise, let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, and suppose they both lie on a circular arc

$$x^2 + y^2 + 2dx + 2ey + 1 = 0.$$

Then we have to solve the linear system

$$\begin{cases} 2x_1 d + 2y_1 e = k_1 \\ 2x_2 d + 2y_2 e = k_2, \end{cases}$$

where k_1, k_2 are constants. This is of course possible unless $\begin{vmatrix} 2x_1 & 2y_1 \\ 2x_2 & 2y_2 \end{vmatrix} = 0.$

Exercise (complete the squares). The unique solution satisfies $d^2 + e^2 > 1$, as required.

If the determinant is zero, there exist a, b, not both zero, such that

$$a(x_1, y_1) + b(x_2, y_2) = 0,$$

and P_1, P_2 lie on a unique diameter. □

In the case of hyperbolic geometry the failure of the parallel postulate B5 is dramatic:

Observation. *Given a line ℓ and a point P not on ℓ, there exist infintely many lines through P that do not intersect ℓ.*

However this failure only becomes significant after we have defined *lengths and angles* in the hyperbolic plane and verified postulates B1–B4, so that we have revealed a true non-Euclidean geometry.

14.2. Revision of hyperbolic functions

The functions $\sinh a = \frac{1}{2}(e^a - e^{-a})$ and $\cosh a = \frac{1}{2}(e^a + e^{-a})$ have properties that are analogous to those of sine and cosine, which can be defined in a similar way by inserting $i = \sqrt{-1}$ into the exponents. As in ordinary trigonometry, it is convenient to use substitutions involving

$$t = \tanh(a/2) = \frac{\sinh(a/2)}{\cosh(a/2)} = \frac{e^{a/2} - e^{-a/2}}{e^{a/2} + e^{-a/2}} = \frac{e^a - 1}{e^a + 1}.$$

This quantity takes values between -1 and 1. The red curve below is the graph of $a \mapsto t$ with a playing the role of the x-coordinate.

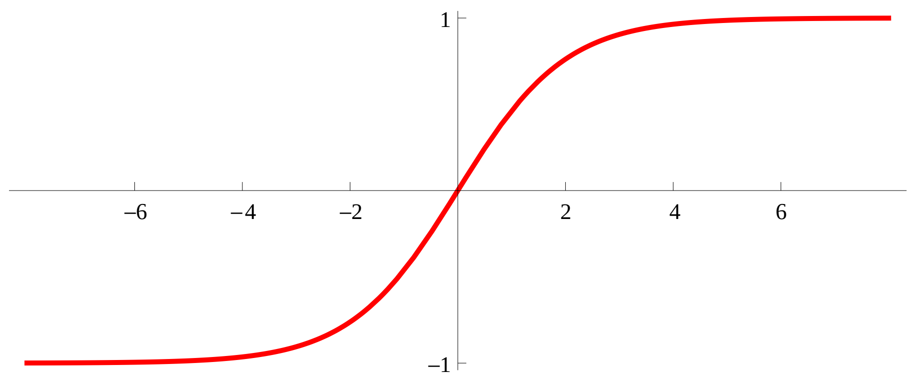
We can determine the inverse function explicitly:

$$t(e^a + 1) = e^a - 1 \quad \text{and} \quad e^a = \frac{1+t}{1-t}.$$

Hence

$$a = 2\tanh^{-1}(t) = 2\text{arctanh}(t) = \log\frac{1+t}{1-t}.$$

The function $t \mapsto a$ defines a bijection $(-1, 1) \to \mathbb{R}$.



Exercise. The following identities are valid:

$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b},$$

$$\tanh^{-1}(s) + \tanh^{-1}(t) = \tanh^{-1}\Big(\frac{s+t}{1+st}\Big).$$

Moreover, if $t = \tanh(a/2)$ then

$$\cosh a = \frac{1+t^2}{1-t^2}, \qquad \sinh a = \frac{2t}{1-t^2}.$$

14.3. Hyperbolic angles and distance

For the next definition, we adopt complex numbers, and represent a point $P = (x, y)$ in $\mathscr{H}$ by a complex number $z = x + iy$, so that $|z| < 1$.

Definitions. The *hyperbolic angle* between two lines meeting in $P \in \mathscr{H}$ is equal to the ordinary Euclidean angle between their tangents at P (like on the sphere).
The *hyperbolic distance* between two points z_1, z_2 ($z_k \in \mathbb{C}$ and $|z_k| < 1$) is given by

$$d(z_1, z_2) = 2\tanh^{-1}\left|\frac{z_1 - z_2}{1 - \overline{z_1} z_2}\right|.$$

Exercise (on Sheet 10). The absolute value on the right is less than $1,$ so the definition makes sense.

If we move along the x-axis from the origin $O=(0,0)$ towards the boundary point $(1,0)$ stopping at $P=(x,0)$ then

$$d(O,P)=2\tanh^{-1}(x),\quad x\geqslant 0.$$

The boundary C is infinitely far from any point of the hyperbolic plane, and (using minus the distance for points to the left of O) there is a bijection between the diameter $\{y=0\}\cap\mathscr{H}$ and $\mathbb{R}$. What's more, this bijection preserves distances: if $P=(-s,0)$ and $Q=(t,0)$ with $s,t>0$ then

$$d(P,O)+d(O,Q)=d(P,Q),$$

since

$$2\tanh^{-1}(s)+2\tanh^{-1}(t)=2\tanh^{-1}\Big(\frac{t+s}{1+st}\Big).$$

The last statement is essentially B1: it tells us that distances add up in the usual way when measured along a line.

Pythagoras' Theorem. The sides of a hyperbolic triangle with a right angle at C satisfy

$$\cosh c=\cosh a\cosh b.$$

Proof. We shall only work out the special case in which $C=O,\ A=(x,0)$ and $B=(0,y)$. These vertices are the complex numbers $0,x,iy,$ so

$$\tanh(a/2)=y,\quad \tanh(b/2)=x,\quad \tanh(c/2)=\left|\frac{x-iy}{1-ixy}\right|=\sqrt{\frac{x^2+y^2}{1+x^2y^2}}.$$

From the half-angle identities quoted above, we have

$$\cosh a=\frac{1+x^2}{1-x^2},\quad \cosh b=\frac{1+y^2}{1-y^2}.$$

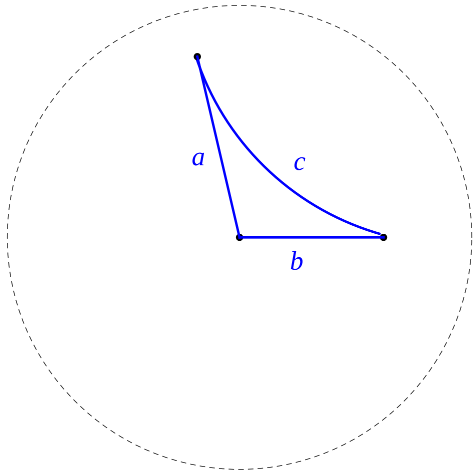
After a calculation

$$\cosh c=\frac{1+x^2y^2+x^2+y^2}{1+x^2y^2-x^2-y^2}=\frac{(1+x^2)(1+y^2)}{(1-x^2)(1-y^2)},$$

as required. □

14.4. Exercises

1. The triangle below has vertices $z_0=0,\ z_1=x,$ and $z_2=e^{i\theta}y,$ where $0<x,y<1,$ so that $\theta=\angle C.$ It follows that $x=\tanh(b/2)$ and $y=\tanh(a/2)$ (rotation about the origin does not affect how one measures distances along a diameter). Use the definition of hyperbolic distance $d(z_1,z_2)$ to verify the hyperbolic cosine rule.



2. Suppose that ABC is an isosceles triangle with $a = b = 1$ and a right angle at C. We can compute the length c of the hypotenuse in three different geometries, using the different versions of Pythagoras.

- On the sphere $\mathscr{S}$ of radius 1 (or in $\mathscr{P}$), $c = \arccos[(\cos 1)^2] = 1.27\ldots$
- In the Euclidean plane $\mathscr{E}$, we obviously have $c = \sqrt{2} = 1.41\ldots$
- In the hyperbolic plane $\mathscr{H}$, $\cosh c = \operatorname{arccosh}[\frac{e^2+2+e^{-2}}{4}] = 1.51\ldots.$

The third side of the triangle is "longest" in the hyperbolic case. Moreover, only in $\mathscr{E}$ can we double the answer to find out what happens when $a = b = 2$ (by SAS).

3. Using the technique of question 6, Sheet 10, deduce the hyperbolic sine rule

$$\boxed{\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}}$$

14.5. Hyperbolic isometries and tessellations [not examinable]

Fix $\alpha, \beta \in \mathbb{C}$, and consider the matrix $A = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$. Observe that

$$A\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha z + \beta \\ \overline{\beta} z + \overline{\alpha} \end{pmatrix}.$$

If we interpret the column vectors as *ratios*, then A defines a mapping

$$f\colon\ z = \frac{z}{1} \ \mapsto\ \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}.$$

This is an example of a *Möbius transformation*, and actually defines a bijection

$$\mathbb{C} \setminus \{-\overline{\alpha}/\overline{\beta}\} \longrightarrow \mathbb{C} \setminus \{\alpha/\overline{\beta}\}.$$

Its inverse is given by A^{-1}, though we may ignore the factor $\det A = |\alpha|^2 - |\beta|^2$, so

$$f^{-1}\colon\, w \;\mapsto\; \frac{\overline{\alpha}w - \beta}{-\overline{\beta}w + \alpha}.$$

It is known that a transformation like f maps Euclidean lines/circles to lines/circles (mixing the two classes up) and preserves angles between them. On the other hand, the lengths of a hyperbolic triangle are determined by their angles.

Theorem. *Suppose that $|\alpha| > |\beta|$. Then f defines bijections $C \to C$ and $\mathscr{H} \to \mathscr{H}$, and preserves hyperbolic distance:*

$$d(f(z_1), f(z_2)) = d(z_1, z_2), \qquad z_1, z_2 \in \mathscr{H}.$$

Exercise (in place of a proof). Set $\alpha = 1$ and $\beta = -w$. Then $f(z) = \dfrac{z - w}{1 - \overline{w}z}$, and $f(w) = 0$. As predicted by the theorem,

$$d(f(z), f(w)) = d(f(z), 0) = 2\tanh^{-1}|f(z)| = d(z, w),$$

which helps to understand the definition we gave of hyperbolic distance.

The isometries above preserve orientation, whereas the map $z \mapsto \overline{z}$ is a hyperbolic isometry reversing orientation. Altogether, there are four different types of hyperbolic isometries (excluding the identity), as there were in the Euclidean plane. For example, the composition

$$g\colon\, z \mapsto \frac{2\overline{z} - i}{i\overline{z} + 2}$$

is a "reflection" that fixes every point satisfying $2\overline{z} - i = z(i\overline{z} + 2)$, which is the h-line

$$x^2 + (y + 2)^2 = 3.$$

This is shown red in the Somerset House illustration at the top of the Geometry I page on Keats: g maps the rectangular image to the curved one, so the two images are *congruent* in $\mathscr{H}$.

Finally, consider a tessellation of the plane in which k regular congruent polygons with n sides meet at each vertex. In the Euclidean plane, this is possible only if

$$k\,\frac{(n-2)\pi}{n} = 2\pi \quad\Rightarrow\quad \frac{1}{n} + \frac{1}{k} = \frac{1}{2}.$$

The only solutions are $(k, n) = (4, 4), (6, 3), (3, 6),$ giving the familiar tessellations made up of squares, triangles or hexagons.

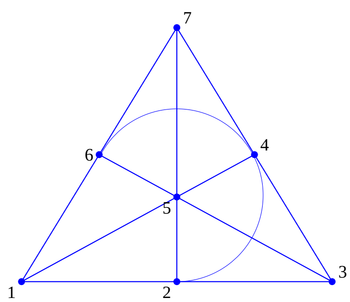
Because of the failure of SAS, there is much more flexibility in the hyperbolic plane, where the bigger the polygon, the smaller the sum of its angles. We can always find a tessellation if $1/n + 1/k < 1/2$. The picture on the front cover of these lecture notes has $n = 5$ and $k = 4$, and is based on right-angled pentagons repeatedly reflected in their sides! Other hyperbolic tilings can be found by following the link on Keats.

Problem sheet 1

1. Let $\mathscr{A} = \{1,2,3,4\}$. We know that there are $4! = 24$ bijections $f\colon \mathscr{A} \to \mathscr{A}$ (permutations). We say that an element a of $\mathscr{A}$ is a *fixed point* of f if $f(a) = a$. How many of the 24 bijections do *not* have a fixed point? (Hint: If f does not have a fixed point then $f(1) = 2, 3$ or 4. If $f(1) = 2$ then $f(2) = 1$ or $f(2) = 3$ or 4 ...)

2. Let $\mathbb{N} = \{1,2,3,\ldots\}$ be the set of natural numbers. Explain why a mapping $f\colon \mathbb{N} \to \mathbb{R}$ is really the same thing as an infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers. Provide an example of a mapping f in which the image $f(\mathbb{N})$ is a finite set of size 4, and another example in which $f(\mathbb{N})$ is an infinite set contained in the interval $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$.

3. The *Fano plane* $\mathscr{F}$ consists of just 7 points that we can label with numbers, so $\mathscr{F} = \{1,2,3,4,5,6,7\}$. The following subsets:



$$\ell_1 = \{1,2,3\}, \quad \ell_2 = \{3,4,7\},$$
$$\ell_3 = \{2,4,6\}, \quad \ell_4 = \{2,5,7\},$$
$$\ell_5 = \{1,6,7\}, \quad \ell_6 = \{3,5,6\},$$
$$\ell_7 = \{1,5,4\} = \{1,4,5\}$$

are called "lines" (the sketch may help). Verify that any two points lie on a unique line and that any two lines meet in one point. How many lines contain a given point? How many sets of 3 lines are there that do not all pass through one point?

4. Let x be the real number such that $x^x = 2$. (You may assume that x exists and is greater than 1). Prove that x is irrational. (Hint: as for $\sqrt{2}$, suppose that $x = p/q$ where p, q are positive integers without a common factor and $p > q$.)

5. In the Euclidean (usual) plane, Let AB and CD be two line segments intersecting at a point O that lies on both segments. Show that the lines bisecting the angles $\angle AOC, \angle BOC$ are perpendicular, writing down the argument so that in the end it can be followed without including a sketch.

6. Draw two triangles $\triangle ABC$, $\triangle A'B'C'$ such that $\angle BAC = \angle B'A'C'$ (equal angles) and $|AC| = |A'C'|$, $|CB| = |C'B'|$ (equal pairs of sides), but that $|AB| \neq |A'B'|$. (So the last pair of sides are not equal, and the triangles are not similar).

Problem sheet 2

In questions 2 and 3 you may assume that the three angles of any triangle add up to π radians. Question 6 takes for granted Pythagoras' theorem. We shall prove both these results soon in lectures.

1. Given a triangle ABC, let L, M, N be the midpoints of BC, CA, AB respectively, so (for example) $|BL| = |LC|$. Deduce from Postulate B5 (two triangles are similar if an angle of one equals an angle of the other and the adjacent sides are proportional) that the triangles ABC, ANM, NBL, MLC are similar.

2. Continuing question 1, prove that $\triangle LMN$ is also similar to $\triangle ABC$.

3. Suppose that A, B, C, D are four distinct points such that the midpoints of the segments AC, BD coincide. Using B5, show that $\angle ACD$ and $\angle BAC$ are equal. Deduce that the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel. [Hint: If not, these two lines meet in a point E forming a triangle ACE.]

4. Continuing question 3, the same argument shows that $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{DA}$. Thus $ABCD$ is a *parallelogram*, a 4-sided figure consisting of two pairs of parallel line segments. If A, B, C, D are four distinct points such that $|AB| = |CD|$ and $|AD| = |BC|$, is $ABCD$ necessarily a parallelogram?

5. As explained in lectures, a great circle is the circle formed by intersecting the surface of a sphere $\mathscr{S}$ with any plane that passes through the centre of $\mathscr{S}$. It is known that any three points in space lie on a unique plane, unless the three points already lie on a straight line. Deduce from this fact that any two points $A, B \in \mathscr{S}$ lie on a unique great circle unless A and B are directly opposite each other. [The deduction is easy, but write out your argument carefully and logically.]

6. Let a, b, c be fixed real numbers with $b \neq 0$ and consider the line

$$\ell = \{(x, y) : ax + by + c = 0\}$$

in the Cartesian plane. If $P = (x, y)$ and $Q = (x', y')$ belong to ℓ express the distance

$$|PQ| = \sqrt{(x' - x)^2 + (y' - y)^2}$$

in terms of $|x' - x|$. Deduce that there exists $\lambda > 0$ such that the mapping $f\colon \ell \to \mathbb{R}$ defined by $(x, y) \mapsto \lambda x$ satisfies the requirement of B1, namely $|PQ| = |f(P) - f(Q)|$. Are other definitions of f possible?

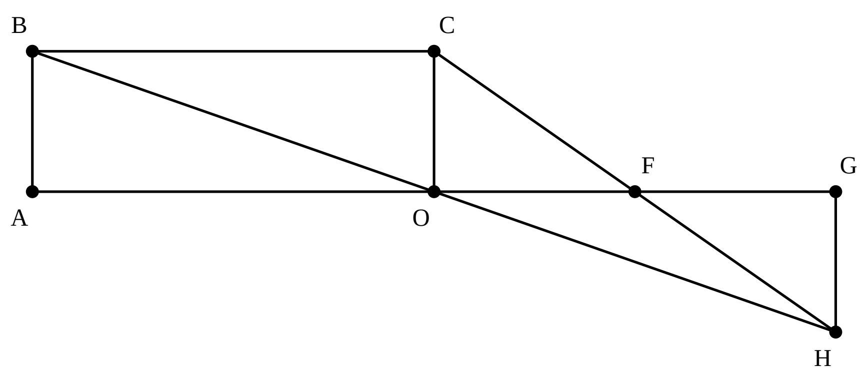
Problem sheet 3

1. A triangle $\triangle ABC$ has equal angles $\angle A = \angle B$. Use the AA rule to deduce that $|AC| = |BC|$, so that the triangle is isosceles.

2. Is B5 (the SAS rule) valid on the surface of a sphere, in which the sides of a "triangle" are segments of great circles (as discussed previously)? [Hint: consider a small isosceles triangle with one vertex at the north pole N and a right-angle at that vertex. Now extend the two rays from N until they meet the equator.]

3. A *kite* is a quadrilateral $ABCD$ (with vertices "in order" and interior angles each less than π) such that $|AB| = |AD|$ and $|CB| = |CD|$. Draw one. Use the SSS rule to deduce that $\triangle ABC \sim \triangle ADC$. Now prove that the diagonals AC and BD are perpendicular (you may assume that they meet at a point O).

4. In the diagram, you may assume that all lines are straight, $\angle OAB, \angle OGH, \angle COG$ are right angles, and $|AB| = |CO|$. Use the AA rule to show that $\triangle OAB \sim \triangle OGH$, $\triangle FOC \sim \triangle FGH$, and that both pairs of triangles are similar with the same factor $k > 0$. If $|AO| = 2$ and $|OF| = 1$, prove that $|FG| = 1$.



5. Suppose that the sides of $\triangle ABC$ satisfy $|AB|^2 = |BC|^2 + |CA|^2$. Prove that $\angle C = \pi/2$. [Hint: construct a second triangle $A'B'C'$ with $\angle C' = \pi/2$ and the sides adjacent to $\angle C'$ of the same lengths as the first. You may assume Pythagoras' theorem.]

6. Using Cartesian coordinates (x, y), the distance between $(0, 0)$ and (x, y) equals $\sqrt{x^2 + y^2}$ by Pythagoras' theorem. Assuming x, y are both non-zero, is this quantity less or greater than

(i) $\max(|x|, |y|)$,

(ii) $|x| + |y|$?

Justify your two answers using only algebraic formulas.

Problem sheet 4

1. Let ℓ be a line and P a point not on ℓ. Choose *two* points A, B on ℓ, and P' so that ℓ bisects $\angle PAP'$ and $|AP| = |AP'|$. Show that $\triangle PAB \sim \triangle P'AB$. Deduce [from the lemma in §4.2] that both A and B lie on the perpendicular bisector of PP', and so PP' meets ℓ at $\pi/2$. [This is a bit simpler than the proof of the theorem in §4.2.]

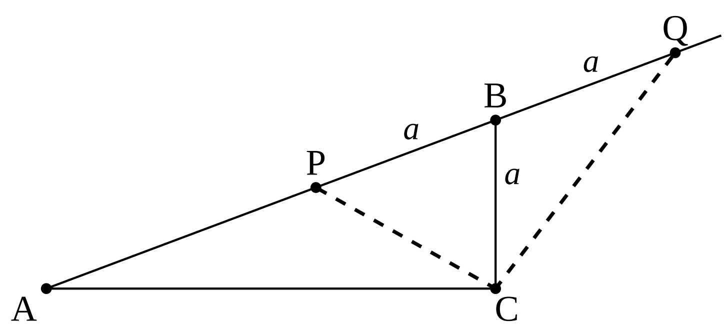
2. Suppose that $d(A, B)$ represents the distance between two points A, B in the plane. Use the triangle inequality [from §4.3] to deduce that

$$|d(A,B) - d(A,C)| \leqslant d(B,C),$$

for any three points A, B, C.

3. Let $s = \sin(\pi/12)$. Show that $1 - 2s^2 = \sqrt{3}/2$. Deduce that $s = (\sqrt{6} - \sqrt{2})/4$. [Hint: work backwards!]

4. Triangle ABC has a right-angle at C, and lengths $a = |BC|,\ b = |CA|,\ c = |AB|$. Points P, Q are constructed either side of B so that $|PB| = |BQ| = a$:



Use isosceles triangles to show that $\angle PCQ = \pi/2$. Now prove that $\triangle APC$ and $\triangle AQC$ are similar [they already have a common angle at A]. Deduce that $(c+a)/b = b/(c-a)$, which implies that $c^2 = a^2 + b^2$.

5. A *rectangle* is a 4-sided figure with four equal angles of $\pi/2$. A *rhombus* is a 4-sided figure with four equal sides. (In both cases the sides meet only in the 4 vertices.) Let P, Q, R, S be the midpoints (taken in clockwise order) of the sides of a rectangle. Prove that $PQRS$ is a rhombus, without resorting to Pythagoras' theorem.

6. Suppose that $a = p^2 - q^2$, $b = 2pq$, $c = p^2 + q^2$. Express $\cos\theta$ and $\sin\theta$ in terms of $t = \tan(\theta/2)$ and verify that

$$(a,\, b) = c(\cos\theta,\, \sin\theta), \quad \text{where} \quad t = q/p.$$

Deduce that there are infinitely many points (x, y) on the circle $x^2 + y^2 = 1$ whose coordinates x, y are rational numbers.

Problem sheet 5

1. The table in §5.3 displays 13 coprime Pythagorean triples (a, b, c). How many others are there with $c < 100$?

2. Fermat proved (around 1640) that *there are no solutions of the equation* $x^4 - y^4 = z^2$ *with* x, y, z *positive integers.*

(i) Show that this statement implies Fermat's last theorem for the case $n = 4$.

(ii) A triangle ABC has a right angle at C. Its sides a, b, c are all integers, and its area is the square of an integer d. By considering $(a \pm b)^2$, verify that $(a^2 - b^2)^2 = c^4 - 16d^4$, and deduce that this is impossible.

3. Suppose that segments AB and BC are perpendicular at B. Construct a line m perpendicular to AB at A, and a line n perpendicular to BC at C. Show that m and n meet at a point D. [Hint: if not, then $\overleftrightarrow{BC}$ and n are both parallel to m.] Deduce that $ABCD$ is a rectangle. So, with our postulates, rectangles exist!

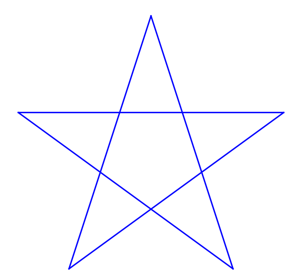
4. A parallelogram is the figure formed by two pairs of parallel lines intersecting each other. Let $ABCD$ be a parallelogram with diagonals AC and BD. Regarding AC as a transversal, show that $\angle BAC = \angle ACD$ and $\angle BCA = \angle CAD$. Use the AA rule to deduce that $|AB| = |DC|$ and $|BC| = |AD|$. Prove also that AC and BD bisect one another.

5. Let ℓ, m, n be three distinct parallel lines. Prove the theorem on ratios in §6.3 in the case in which the two transversals k, k' are parallel. [Hint: First construct the segment LN' and show that $\triangle NLN' \sim \triangle L'N'L$.]

6. (i) Let ℓ be a line and P a point not on ℓ. Let D be the foot of the perpendicular from P to ℓ. Show that, if Q is a point that varies on ℓ, then $|PQ|$ is least when $Q = D$. The number $|PD|$ is called the *distance from* P *to* ℓ.

(ii) Let r and s be two rays (half-lines) meeting at a point O. Let b be a ray from O that bisects the angle that r and s make at O (it exists by B3). Show that, for any point P on b, the distance from P to r equals the distance from P to s.

7. A "pentagram" is formed by extending the sides of a regular pentagon as shown. Find the angles of each triangle outside the pentagon. If the pentagon has side 1, what is the length of each long segment that can be seen joining a pair of outer vertices?



Problem sheet 6

1. Describe the set of points defined by the Cartesian equations $xy=0,$ and $x^2-y^2=0.$ Now sketch the hyperbola $x^2-y^2=1,$ observing that it passes though $(\pm 1,0).$

2. A parabola is often defined as the set of points P in the plane such that the distance from P to a fixed line d (the *directrix*) equals the distance from P to a fixed point F (the *focus*). If $F=(0,0)$ is the origin of a system of Cartesian coordinates and d is the line $x=1$ (not the usual choice!), find the equation satisfied by $P=(x,y).$

3. Suppose that two circles $\mathscr{C}_1,\mathscr{C}_2$ with centres O_1,O_2 meet in two points $P,Q.$ Prove that $\triangle O_1PO_2$ is congruent to $\triangle O_1QO_2,$ and that O_1O_2 is the perpendicular bisector of $PQ.$

4. An ellipse is sometimes defined as the set of points P such that the sum $|PF_1|+|PF_2|$ of the distances from P to two given points (the *foci*) equals a constant, $c.$ If $F_1=(-1,0),\ F_2=(1,0)$ and $c=4,$ show that $P=(x,y)$ satisfies an equation of the form

$$\sqrt{A}+\sqrt{B}=4, \quad \text{where} \quad A-B=4x,$$

and A and B are functions of x and $y.$ Deduce that $x=\sqrt{A}-\sqrt{B}$ and that $4A=(x+4)^2.$ Simplify the last equation in terms of x and $y.$

5. Let $\triangle ABC$ be a right-angled triangle with $\angle A=\pi/2,$ and let P be the midpoint of $BC.$ Show that $|AP|=|PB|.$ (Hint: add the midpoint Q of AC and consider $\triangle APQ.$) Deduce that P is the circumcentre of $\triangle ABC.$ Where is the orthocentre of $\triangle ABC$?

6. (i) Prove that if a quadrilateral $ABCD$ is inscribed in a circle $\mathscr{C}$ (meaning that A,B,C,D lie in order on $\mathscr{C}$) then

$$\angle A+\angle C=\pi=\angle B+\angle D.$$

(Hint: join the vertices to the centre of the circle and consider the resulting triangles.)

(ii) Prove that if a circle $\mathscr{C}'$ is inscribed in a quadrilateral $ABCD$ (meaning that the sides AB,BC,CD,DA are all tangents to $\mathscr{C}'$) then

$$|AB|+|CD|=|BC|+|AD|.$$

7. Explain why it is possible to fit six circles around a seventh, all of the same radius, so that each circle touches at least three others. (Try it with 70p, but then think of the centres of the circles.)

Problem sheet 7

1. A complex number $z = x + iy$ is represented by the point (x, y) in the plane, and we shall use z as a symbol for this point. Use similar triangles to verify that $\frac{1}{2}(z_1 + z_2)$ is the midpoint of the segment joining z_1 to z_2. Show, any way you want, that an arbitrary point on the same segment equals

$$z + k(w - z) = (1 - k)z + kw,$$

for some $k \in [0, 1]$. (This is a question as much about vectors as complex numbers.)

2. Represent the vertices A, B, C of a triangle by complex numbers z_1, z_2, z_3, and hence the midpoints L, M, N of the sides of $\triangle ABC$ by $\frac{1}{2}(z_2 + z_3),\ \frac{1}{2}(z_3 + z_1),\ \frac{1}{2}(z_1 + z_2)$. Let $w = \frac{1}{3}(z_1 + z_2 + z_3)$. Find k such that $w = (1 - k)z_1 + k\frac{1}{2}(z_2 + z_3)$. Deduce that w lies on all three medians $AL,\ BM,\ CN$. It is therefore the centroid G of $\triangle ABC$.

3. With the same notation as in q. 2, suppose (as in lectures) that $|z_i| = 1$ for $i = 1, 2, 3$, so $z = 0$ is the circumcentre O of $\triangle ABC$ and $z_1 + z_2 + z_3$ represents its orthocentre H. Let A', B', C' be the feet of the altitudes of $\triangle ABC$ (so $A' \in \overleftrightarrow{BC}$ etc.) and A'', B'', C'' the midpoints of the segments AH, BH, CH. Let P be the midpoint of OH. Show that (i) $|PA''| = \frac{1}{2}$ and (ii) $|PL| = \frac{1}{2}$. Deduce (by applying q. 5 on Sheet 6 to $\triangle LA'A''$) that (iii) $|PA'| = \frac{1}{2}$. It follows that P is the centre of a circle that passes through the nine points

$$L,\ M,\ N;\ \ A',\ B',\ C';\ \ A'',\ B'',\ C''.$$

Show that P, H, O, G all lie on a line (called the *Euler line* of $\triangle ABC$).

---

4. (i) Let f be a translation parallel to a line ℓ, and g reflection in ℓ. By tracing the effect of $f \circ g$ and $g \circ f$ on an arbitrary point P, show that these two compositions are equal.

(ii) Let h be a glide reflection. Given any point P, let $P' = h(P)$. Prove that the midpoint of PP' lies on the line defining the reflection. Deduce that no points are fixed by h.

5. Let ℓ, m be two parallel lines, and f, g the respective reflections in these lines. Given a point P, let n be the line perpendicular to ℓ and m passing though P, and let L, M be the points of intersection of n with ℓ and m. Show that $f \circ g$ shifts P along n in a fixed direction by a distance $2|LM|$, irrespective of the position of P relative to ℓ, m. Thus, $f \circ g$ is a translation. Is it true that $f \circ g = g \circ f$?

6. Let f, g be 180° rotations centred at two distinct points P, Q respectively. Let R be a third point such that $\angle PQR = \pi/2$. Show that $f \circ g$ acts on each of P, Q, R as the same translation, and deduce that $f \circ g$ *is* a translation. [You may assume that an isometry that fxes 3 non-collinear points must be the identity.]

Problem sheet 8

1. Study and write out the proof of Proposition 2 (asserting that any isometry with exactly one fixed point is a rotation) in your own words, following the sketch in §10.3.

2. Let G be the group of isometries $f\colon \mathscr{E} \to \mathscr{E}$ of the Euclidean plane. Which of the following, with the identity added, are subgroups of G? [You need to decide whether the composition of any two, and the inverse of any one, remains in the set.]

(i) the set of all translations,

(ii) the set of all rotations (about any point) and translations,

(iii) the set of all reflections,

(iv) the set of all rotations about a fixed point O through $n\pi/5$ radians, where n is a positive integer.

3. Fix Cartesian coordinates in the plane with origin O. Let $f = \mathrm{Rot}_{O,\pi/2}$ be rotation by 90^{o} about O. Let ℓ be the x-axis and let $g = \mathrm{Ref}_\ell$ be the reflection $(x,y) \mapsto (x,-y)$.

(i) Verify that $f^{-1} = f \circ f \circ f$. Verify that $f \circ g = g \circ f^{-1}$ [consider the effect of both sides on two points other than O.]

(ii) Describe the isometries $f \circ f \circ g$ and $g \circ f$. Show that the set

$$\mathscr{D} = \{\mathrm{id}, \quad f, \quad f \circ f, \quad f^{-1}, \quad g, \quad f \circ g, \quad f \circ f \circ g, \quad g \circ f\}$$

is a subgroup of G. [You need to show that the composition of any two of the 8 in either order belongs to $\mathscr{D}$, as does the inverse of any one.]

4. Let

$$S = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}.$$

What isometries (fixing the origin) do S and T represent? Compute the matrix products S^2 (that is, SS), ST, $S\mathbf{v}$, $S\mathbf{w}$, simplifying the answers. Interpret all these results geometrically [the Linear Methods notes may be helpful].

5. Simplify $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$ and interpret the result.

6. By considering each of the five types (listed at the end of §10.2) in turn, show that *any* isometry of the plane is the composition of at most three reflections.

Problem sheet 9

1. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the vectors determined by the columns of $M = \begin{pmatrix} c & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$, where $c \in \mathbb{R}$. Compute the vectors

$$\mathbf{v}_2 \times \mathbf{v}_3, \quad \mathbf{v}_3 \times \mathbf{v}_1, \quad \mathbf{v}_1 \times \mathbf{v}_2.$$

If these are the columns (in order) of a matrix L, verify that $L^T M$ is a multiple of the identity matrix. If $c = 0$, does L^{-1} exist ?!

2. Describe the isometries (that fix an origin in space) given by the following matrices:

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Write down the inverse matrices A^{-1}, B^{-1}, C^{-1} and compute A^3, B^3, C^3.

3. Consider the two planes

$$\Pi_1\colon\ x + 2y + 5z = 8, \qquad \Pi_2\colon\ -x - y + 2z = 3.$$

(i) Write down normal vectors $\mathbf{n}_1, \mathbf{n}_2$ to each one and compute $\mathbf{n}_1 \times \mathbf{n}_2$. Find just one point that lies on both planes (this is easier than fully solving the equations). Hence write down the line $\ell = \Pi_1 \cap \Pi_2$ in parametric form as in §12.2.

(ii) Write down the equation of any one plane Π_3 that does not intersect ℓ (and is therefore *parallel* to ℓ).

4. Let Π be the plane that passes through the points $(1,0,0)$, $(0,0,1)$ and $(0,0,1)$. Find the angle that Π makes with the xy-plane $z = 0$.

5. Verify that the four points

$$(0, \sqrt{2}, 1), \quad (0, -\sqrt{2}, 1), \quad (\sqrt{2}, 0, -1), \quad (-\sqrt{2}, 0, -1)$$

all lie on a sphere centred at the origin, and that the distances between any two of them are all equal. What figure do they form?

6. The set of 3×3 orthogonal matrices with determinant equal to 1 (as opposed to -1) is sometimes denoted by $SO(3)$ (to stand for *Special Orthogonal*). Use properties of the transpose and determinant of a matrix to verify that

(i) if $A, B \in SO(3)$ then $AB \in SO(3)$;

(ii) if $A \in SO(3)$ then $A^{-1} \in SO(3)$.

Since we already know that matrix multiplication is associative and the identity matrix I belongs to $SO(3)$, this means that $SO(3)$ forms a group. Explain why (i) tells us that the composition of two rotations in space whose axes intersect is another rotation.

Problem sheet 10 [not assessed]

1. Let z_1, z_2 be complex numbers, both of modulus less than 1. Prove that

$$|z_1 - z_2| < |1 - \overline{z_1} z_2|$$

by expanding the squares of both sides. (We need this inequality in order to be able to define the hyperbolic distance between z_1, z_2 as $2\tanh^{-1}$ of $|z_1 - z_2|/|1 - \overline{z_1} z_2|$.)

2. Let $t = \tanh(a/2)$. Verify that $\cosh a = \dfrac{1+t^2}{1-t^2}$ and $\sinh a = \dfrac{2t}{1-t^2}$.

3. Suppose that ABC is an isosceles triangle with $a = b = 1/2$ and a right angle at C. Compute numerically the length c of the hypotenuse:

(i) on a sphere of radius 1,

(ii) in the Euclidean plane,

(iii) in the hyperbolic plane.

Compare the answers with each other and also with those for $a = b = 1$ (found in §14.4).

4. Let $\mathscr{T}$ be a triangle in the hyperbolic plane with sides of hyperbolic lengths a, b, c. Use the hyperbolic cosine rule to prove that $\cosh c < \cosh(a + b)$. Deduce the triangle inequality that asserts that the length c of one side is less than the sum $a + b$ of the other two.

5. Let P_1 and P_2 be two points on the surface of the earth, assumed to be a sphere of radius 1 unit, and let N be the north pole. Let a, b, c denote the arc lengths of the sides of the spherical triangle P_1P_2N opposite P_1, P_2, N respectively. Explain why $b = \pi/2 - l_1$ and $a = \pi/2 - l_2$, where l_1, l_2 are the latitudes of P_1, P_2 (measured in radians), and $C = \angle P_1NP_2 = |m_1 - m_2|$, where m_1, m_2 are their longitudes. Now set $H(x) = 1 - \cos x$. Show that the *Haversine formula*

$$H(c) = H(l_1 - l_2) + \cos l_1 \cos l_2 \, H(m_1 - m_2)$$

(once used for navigation) reduces to the spherical cosine rule

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

6. Use the spherical cosine rule immediately above to show that $\sin^2 a \sin^2 b \sin^2 C$ can be expressed as a symmetric function of $\cos a, \cos b, \cos c$. Deduce the spherical sine rule

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$