REDUCED HOLONOMY, CONES AND BRANES

Based on results of

[1] Acharya–FigueroaO'Farrill–Hull–Spence: Adv Theor Math Phys 2 (1998)

[2] Atiyah–Witten: M-theory dynamics on a manifold with G_2 holonomy, Adv Theor Math Phys 6 (2002)

[3] Acharya–Witten: Chiral fermions from manifolds of G_2 holonomy, hep-th/0109152

and some of my own explicit examples.

Slides at www.ma.ic.ac.uk/˜sms

11-dimensional equations

SUGRA$_{11}$ involves a metric g and reduction to $\mathrm{SO}(10,1)$ and a 4-form $F = dA$ satisfying

$$R_{im} = \tfrac{1}{12}\left(F_{ijkl}F_m{}^{jk\ell} - \tfrac{1}{12}g_{im}F^2\right)$$

$$dF = 0$$

$$d * F = F \wedge F$$

Supersymmetry requires non-zero Killing spinor(s):

$$\nabla_m\eta + \tfrac{1}{288}\left[\Gamma_m^{ijkl} - 8\delta_m^i\Gamma^{jkl}\right]F_{ijkl}\eta = 0$$

and holonomy reduction of a suitable connection.

Let $\nu = \frac{1}{32}\mathrm{dim}(\mathrm{Kspinors})$.

Special solutions

Identifying F with the volume form of a 4-manifold gives an Einstein product $M^4 \times M^7$.

- With $\nu = 1$: $\mathrm{AdS}_7 \times S^4$ or $\mathrm{AdS}_4 \times S^7$.

- M2 brane solution with $\frac{1}{16} \leqslant \nu \leqslant \frac{1}{2}$:

$$\left(1 + \frac{a^6}{r^6}\right)^{-2/3} g_{2,1} + \left(1 + \frac{a^6}{r^6}\right)^{1/3} (dr^2 + r^2 g_7)$$

with $F = \mathrm{vol}_{2,1} \wedge f(r) dr$ and $*F = 6a^6 \mathrm{vol}_7$.
Interpolates between

$$g_{2,1} + (dr^2 + r^2 g_7) \quad \text{on} \quad M_{2,1} \times X^8$$

as $r \to \infty$ and the near horizon limit

$$\left(\frac{r^4}{a^4} g^{2,1} + \frac{a^2}{r^2} dr^2\right) + a^2 g_7 \quad \text{on} \quad \mathrm{AdS}_4 \times Y^7$$

as $r \ll a$.

Metrics with exceptional holonomy

- To ensure that $\nu > 0$, the conical metric

$$dr^2 + r^2 g_7 \quad \text{on} \quad \mathbb{R}^+ \times Y^7$$

must have Hol $\subseteq \mathrm{Spin}7$. This holds iff (Y, g_7) has weak holonomy G_2 (invariant intrinsic torsion).

Examples $\quad Y = S^7,\ S^7_{\mathrm{sq}}\ (\stackrel{S^3}{\to} S^4),\ \dfrac{\mathrm{SO}(5)}{\mathrm{SO}(3)},\ \dfrac{\mathrm{SU}(3)}{\mathrm{U}(1)_{p,q}}$.

- In turn, the conical metric

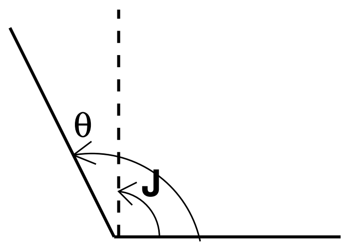
$$dy^2 + y^2 g_6 \qquad \text{on} \quad \mathbb{R}^+ \times Z^6$$

has holonomy $\subseteq \mathrm{G}_2$ iff (Z, g_6) is nearly-Kähler [6].

Examples The 3-symmetric spaces $Z = S^6$,

$$\mathbb{CP}^3, \quad \mathbb{F} = \frac{\mathrm{SU}(3)}{\mathrm{T}^2}, \quad S^3 \times S^3,$$

for which triality $\theta = \frac{1}{2}(-\mathbf{1} + \sqrt{3}J)$ plays a key role.



Weak G_2 metrics with singularities

Suppose that (Z, g_{NK}) is nearly-Kähler. Then

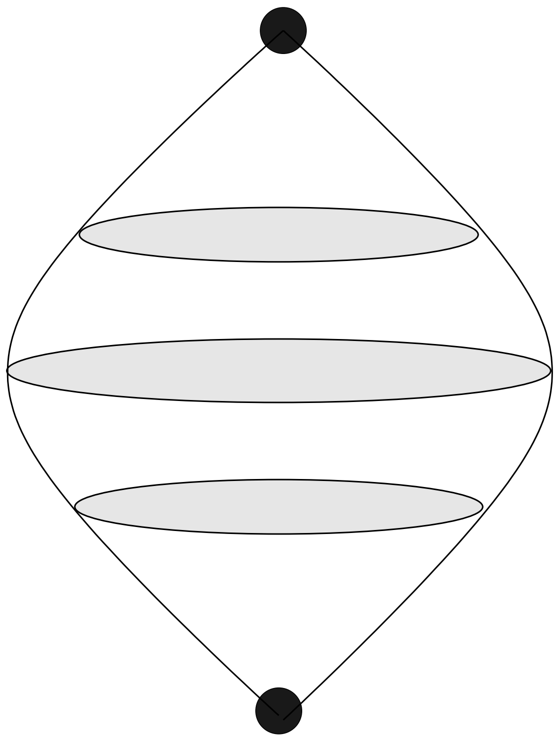
$$\begin{aligned} g_8 &= dx^2 + (dy^2 + y^2 g_{\mathrm{NK}}) \\ &= (dr^2 + r^2 dt^2) + (r \sin t)^2 g_{\mathrm{NK}} \\ &= dr^2 + r^2 (dt^2 + \sin^2 t\, g_{\mathrm{NK}}) \end{aligned}$$

has $\mathsf{Hol}(g_8) \subset \mathrm{Spin}7$.

<u>Corollary</u> [4] The 'spherical metric'

$$dt^2 + (\sin^2 t) g_{\mathrm{NK}}$$

has weak holonomy G_2.



S^1 quotients of G_2 manifolds

If (Y, g_7) has holonomy G_2 then $Q = Y/S^1$ has a symplectic $\mathrm{SU}(3)$ structure with

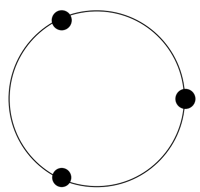
$$g_7 = f^2(dt+\eta)^2 + \frac{1}{f}\,g_{\mathrm{IIA}}$$

in which f is a function on Q (the dilaton).

Theorem [2] Each 3-symmetric space Z admits a S^1 action for which $Z/S^1 \cong S^5$. Therefore, if $Y = \mathbb{R}^+ \times Z$, $Q \cong \mathbb{R}^6$ contains the S^1 fixed points ($f = 0$) as Lagrangian submanifolds.

Example A standard S^1 action on $\mathbb{F} = \frac{U(3)}{U(1)\times U(1)\times U(1)}$ has fixed point set $S^2 \sqcup S^2 \sqcup S^2$, and X/S^1 has three $\mathbb{R}^3$'s intersecting at a point (generating D6 branes). A curved version is inherent in the geometry of

$$S^3 \times S^3 = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{SU}(2)}$$



A tri-Lagrangian example

Let $N^5\ (\stackrel{T^2}{\rightarrow} T^3)$ be a principal torus bundle with base 1-forms e^1, e^3, e^5 and connection 1-forms e^4, e^6 with

$$de^4 = e^{15}, \quad de^6 = e^{13}.$$

Then $N^5 \times S^1$ has a symplectic form

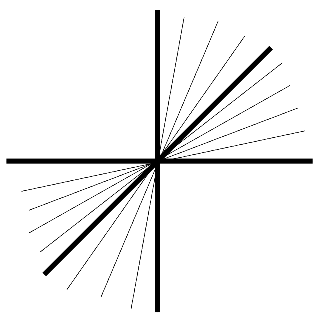
$$\omega = e^{12} + e^{34} + e^{56}.$$

Theorem [10] $N^5 \times S^1$ has a triple of mutually transverse Lagrangian submanifolds through each point.

Follows from the existence of closed simple 3-forms

$$\begin{aligned} \gamma &= e^1 \wedge \theta(e^3) \wedge \theta^2(e^5) \\ \theta(\gamma) &= \theta(e^1) \wedge \theta^2(e^3) \wedge e^5 \\ \theta^2(\gamma) &= \theta^2(e^1) \wedge e^3 \wedge \theta(e^5) \end{aligned}$$

Remark $Sp(n, \mathbb{R})$ acts almost transitively on triples of transverse Lagrangian subspaces of $\mathbb{R}^{2n}$, with stabilizer $O(n)$.



Half flat $\mathrm{SU}(3)$ structures

In 6 dimensions, an $\mathrm{SU}(3)$ structure is characterized by a 2-form ω and a $(3,0)$-form $\psi^+ + i\psi^-$. A G_2 structure is defined in $6+1$ dimensions by setting

$$\begin{aligned} \varphi &= \omega \wedge dt + \psi^+ \\ *\varphi &= \psi^- \wedge dt + \tfrac{1}{2}\omega \wedge \omega. \end{aligned}$$

The holonomy reduces iff

$$\boxed{d\varphi = 0, \quad d * \varphi = 0}$$

implying the half-flat conditions

$$d(\tfrac{1}{2}\omega \wedge \omega) = 0, \quad d\psi^+ = 0.$$

Example $N^5 \times S^1$ has a half-flat structure with

$$\begin{aligned} \psi^+(0) &= e^{135} - e^{146} - e^{236} - e^{245} \\ \tfrac{1}{2}\omega(0) \wedge \omega(0) &= e^{3456} + e^{5612} + e^{1234} \end{aligned}$$

Problem: Find $\omega(t), \psi^+(t)$ satisfying the reduction.

Rich Calabi-Yau geometry

Solution: Let $4t=2u+\sin 2u$ and $x=\cos u,\ y=\sin u$.

$$\begin{array}{ccc} xe^1, & (1+y)^{1/2}e^3, & (1-y)^{1/2}e^5, \\ x^2du, & (1-y)^{-1/2}e^4, & (1+y)^{-1/2}e^6 \end{array}$$

is an ON basis for a metric $g_{\rm CY}$ on $M^6=(-\frac{\pi}{2},\frac{\pi}{2})\times N^5$ with Hol$(g_{\rm CY})=SU(3)$. The Kähler form is

$$\omega = x^3e^1du+e^{45}-e^{63}.$$

<u>Proposition</u> M^6 has a pencil of special Lagrangian submanifolds through each point.

Follows from the existence of closed 3-forms

$$\begin{gathered} e^{146} \\ x(1+y)e^{34}du + x(1-y)e^{56}du + x^2e^{135} \\ xe^{46}du + (1-y)e^{156} + (1+y)e^{134} \\ x^3e^{35}du \end{gathered}$$

spanning the S^1 orbit $\{\exp(sJ)\cdot e^{146}\}$ containing $\psi^\pm$.

Kähler quotients of G_2 metrics

Assumption Y^7 has a metric with Hol $= \mathrm{G}_2$ and the $\mathrm{SU}(3)$ manifold $(Y/S^1, g_{\mathrm{IIA}})$ is Kähler:

$$
\begin{array}{ccc}
 & & Y \\
 & & \downarrow \\
\mu^1(t)/S^1 & \hookrightarrow & Y/S^1,\ g_{\mathrm{IIA}} \\
\downarrow_{S^1} & \swarrow \mathbb{C}^* & \\
M^4 & &
\end{array}
$$

Theorem [5] Suppose that a 4-manifold M has

- a cx structure J_1 and holomorphic 2-form $\omega_2 + i\omega_3$,
- a 1-parameter family of Kähler forms $\tilde{\omega} = \tilde{\omega}(t)$ s.t.

$$\boxed{\tilde{\omega}''(t) = 2i\partial\overline{\partial} f}$$

where $t\tilde{\omega} \wedge \tilde{\omega} = f\omega_2 \wedge \omega_2$. Then a rank 3 bundle over M admits a Ricci-flat metric g with Hol$(g) \subseteq G_2$.

If f is constant on M then $\tilde{\omega}''(t)=0$ and

$$\tilde{\omega}=(p+qt)\omega_0+(r+st)\omega_1,$$

with $(\omega_1,\omega_2,\omega_3)$ a hyperkähler structure and ω_0 an additional closed 2-form.

Example Take

$$\begin{aligned}
\omega_0 &= z dx\wedge dz - dw\wedge dy\\
\omega_1 &= z dx\wedge dy + dz\wedge dw + x dy\wedge dz\\
\omega_2 &= z dx\wedge dz + dw\wedge dy\\
\omega_3 &= dx\wedge dw - x dx\wedge dy + z dy\wedge dz,
\end{aligned}$$

corresponding to a Gibbons-Hawking (HK) metric

$$z(dx^2+dy^2+dz^2)+\frac{1}{z}(dw-x dy)^2$$

conformal to a left-invariant one on $\mathbb{CH}^2$ [8].

Solutions with f non-constant are found starting from

$$\omega_1 = \tfrac{i}{2}(du \wedge d\overline{u} + dv \wedge d\overline{v}), \quad \omega_2 + i\omega_3 = du \wedge dv$$

on $T^4 = \mathbb{C}^2/\mathbb{Z}^4$ and setting $\tilde{\omega} = \omega_1 + i\partial\overline{\partial}\phi$. Then

$$\tilde{\omega} \wedge \tilde{\omega} = \mathcal{M}(\phi)\omega_1 \wedge \omega_1,$$

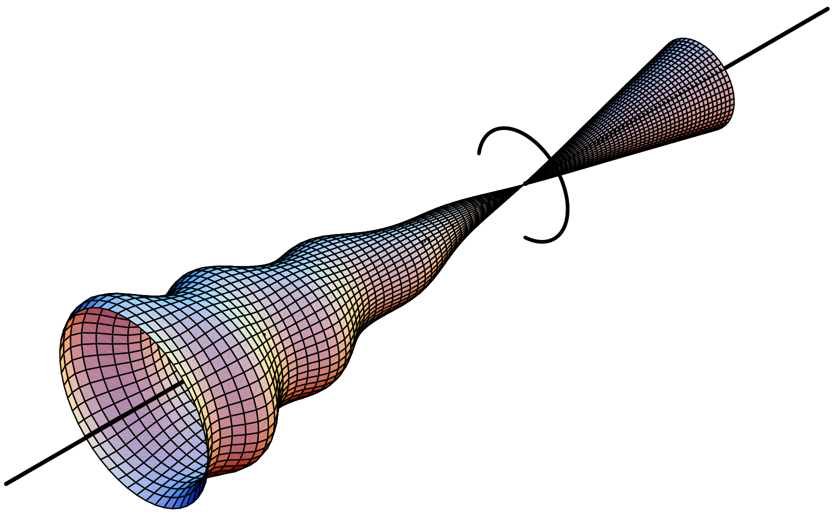
and $t\mathcal{M}(\phi) = f = \frac{1}{2}\phi''$.

<u>Example</u> If $u = x + iy$, there are solutions

$$\phi(t,x,y) = \tfrac{1}{3}t^3 + \ell(t)m(x,y),$$

where $\ell'' - t\ell = 0$ and $\Delta m + m = 0$ such as

$$f(t,x) = t + \tfrac{1}{2}t\,\mathrm{Ai}(t)\sin x.$$



G_2 quotients of hyperkähler metrics

Geometric engineering a partial deformation of $\mathbb{C}^2/\Gamma$ relative to a Hermitian symmetric space $\dfrac{G}{G'\times S^1}$:

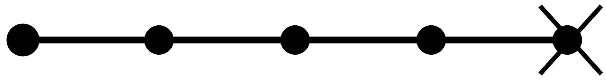
$$
\begin{array}{ccccc}
 & & U & \xleftarrow{K'} & \mu^{-1}(0) \subset \mathbb{H}^{n+1} \\
 & \swarrow & | & & \\
\mathbb{R}^3 \xleftarrow{\pi} U/S^1 & & | & & \\
 & & \downarrow & & \\
 & & M^4 & & \mathbb{HP}^n
\end{array}
$$

- If $K = K'\times S^1$ Kronheimer's construction [12] gives

$$\mathbb{H}^{n+1}/\!/K = \pi^{-1}(\mathbf{x}) = \begin{cases} \mathbb{C}^2/\Gamma, & \mathbf{x}=0 \\ \mathbb{C}^2/\Gamma', & \mathbf{x}\neq 0 \end{cases}$$

- $U = \mathbb{H}^{n+1}/\!/K'$ is a cone over the twistor space of an Einstein self-dual (QK) orbifold M^4 [13].

Example $G = \mathrm{SU}(6)$, $G' = \mathrm{SU}(5)$, $K = T^5$, $K' = T^4$:



U/S^1 is a cone over $\mathbb{WCP}^3_{5,5,1,1}$ and corresponds to five D6 branes intersecting a D6 brane at $0 \in \mathbb{R}^6$.

Conjecture [3] This has a metric with Hol $= \mathrm{G}_2$.

Alternative scenario:

Same definition of U, but with $S^1 \subset Sp(1)$. Then S^1 is not triholomorphic, and U/S^1 is a cone over the twistor space of M.

Example $n = 2$, $K' = U(1)$ and $M = \mathbb{WCP}^2_{p,q,r}$ [9].

Additional references

[4] Acharya–Denef–Hofman–Lambert: Freund–Rubin revisited, hep-th/0308046

[5] Apostolov–Salamon: Comm Math Phys 246 (2004)

[6] Bär: Comm Math Phys 154 (1993)

[7] Chiossi–Salamon: Diff Geom Valencia 2001, World Scientific, 2002

[8] De Smedt–Salamon: Contemp Math 308 (2002)

[9] Galicki–Lawson: Math Ann 282 (1988)

[10] Giovannini: PhD thesis, Turin, 2004

[11] Hitchin: Contemp Math 288 (2001)

[12] Kronheimer: JDG 29 (1989)

[13] Swann: Math Ann 289 (1991)