COMPLEX STRUCTURES AND 2×2 MATRICES

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Part I
Complex structures on $\mathbb{R}^4$

Part II
SKT structures on a 6-manifold

Part I

The standard complex structure

A (linear) complex structure on $\mathbb{R}^4$ is simply a linear map $J\colon \mathbb{R}^4 \to \mathbb{R}^4$ satisfying $J^2 = -1$. For example,

$$J_0 = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & \multicolumn{2}{c}{\smash{\raisebox{.5em}{0}}} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{\smash{\raisebox{.5em}{0}}} & -1 & 0 \end{array}\right)$$

In terms of a basis (dx^1, dx^2, dx^3, dx^4) of $\mathbb{R}^4$,

$$Jdx^1 = -dx^2, \qquad Jdx^3 = -dx^4$$

The choices of sign ensure that, setting

$$z^1 = x^1 + ix^2, \qquad z^2 = x^3 + ix^4,$$

the elements $dz^1, dz^2 \in \mathbb{C}^4$ satisfy

$$Jdz^1 = J(dx^1 + idx^2) = i(dx^1 + idx^2) = idz^1$$
$$Jdz^2 = idz^2.$$

Spaces of complex structures

Any complex structure J is conjugate to J_0, so the set of complex structures is

$$\mathcal{C} \;=\; \{A^{-1}J_0A \;:\; A \in GL(4,\mathbb{R})\}$$

$$\cong\; \frac{GL(4,\mathbb{R})}{GL(2,\mathbb{C})}\,,$$

a manifold of real dim 8, the same dim as $GL(2,\mathbb{C})$.

Taking account of orientation, $\mathcal{C} = \mathcal{C}^+ \sqcup \mathcal{C}^-$.

<u>Theorem</u> $\mathcal{C}^+ \simeq S^2$.

Proof. Any $J \in \mathcal{C}^+$ has a polar decomposition

$$J = SO = -O^{-1}S^{-1} = (O^TS^{-1}O)(-O^{-1}).$$

Thus, $O = -O^{-1} = P^{-1}J_0P$ defines an element

$$U(2)P \in \frac{SO(4)}{U(2)} \cong S^2.$$

Deformations and 2×2 matrices

A complex structure J is determined by its i-eigenspace

$$E = \Lambda^{1,0} = \{v \in \mathbb{C}^4 : \; Jv = iv\},$$

since $E \oplus \overline{E} = \mathbb{C}^4$ and $J = \begin{cases} i & \text{on } E \\ -i & \text{on } \overline{E}. \end{cases}$

For example, $E_0 = \langle dz^1, dz^2 \rangle$ and $\overline{E_0} = \langle d\overline{z}^1, d\overline{z}^2 \rangle$.

Define J by decreeing E to be generated by

$$\begin{aligned} \boldsymbol{\varepsilon}^1 &= dz^1 \qquad\quad + \; a d\overline{z}^1 + b d\overline{z}^2 \\ \boldsymbol{\varepsilon}^2 &= \qquad\quad dz^2 \; + \; c d\overline{z}^1 + d d\overline{z}^2. \end{aligned}$$

Thus $J = J_X$ is determined by

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for suitable $a, b, c, d \in \mathbb{C}$ (not $X = I$!)

Non-degeneracy

If $E = \langle \boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2 \rangle$ then

$$E \cap \overline{E} = \{0\} \;\Leftrightarrow\; \boldsymbol{\varepsilon}^1 \wedge \boldsymbol{\varepsilon}^2 \wedge \overline{\boldsymbol{\varepsilon}}^1 \wedge \overline{\boldsymbol{\varepsilon}}^2 \neq 0.$$

This is easily computed in terms of the volume form $V = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ of $\mathbb{R}^4$:

Proposition $\quad \boldsymbol{\varepsilon}^1 \wedge \boldsymbol{\varepsilon}^2 \wedge \overline{\boldsymbol{\varepsilon}}^1 \wedge \overline{\boldsymbol{\varepsilon}}^2 = 4c_{X\overline{X}}(1)V$.

Here,

$$\begin{aligned} c_{X\overline{X}}(1) &= 1 - \mathrm{tr}(X\overline{X}) + \det(X\overline{X}) \\ &= 1 - (|a|^2 + |d|^2 + 2\mathrm{Re}(b\overline{c})) + |ad - bc|^2 \end{aligned}$$

Example $\quad J_X$ is orthogonal iff X is skew-symmetric ($a = 0 = d$ and $b = -c$). In general,

$$\mathcal{C}^+ = \{J_X : c_{X\overline{X}}(1) > 0\} \;\sqcup\; \mathcal{C}^+_\infty,$$

with $\mathcal{C}^+_\infty \simeq \{-J_0\}$.

Classes of anti-linear maps

Why the appearance of $X\overline{X}$?

J is described by the echelon matrix

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} = \left(I \mid X \right)$$

relative to $\mathbb{C}^4 = E \oplus \overline{E}$.

If $B \in GL(2,\mathbb{C})$ then $B^{-1}JB$ has matrix

$$\begin{aligned} \left(IB \middle| X\overline{B} \right) &\sim B^{-1} \left(IB \mid X\overline{B} \right) \\ &\sim \left(I \mid B^{-1} X \overline{B} \right). \end{aligned}$$

<u>Corollary</u> $GL(2,\mathbb{C})$ acts on $\mathcal{C}$ by

$$B^{-1}(J_X)B = J_{B^{-1}X\overline{B}}.$$

Not surprising, as X is a linear map $E \to \overline{E}$.

Consimilarity

Definition Complex $n \times n$ matrices X, Y are said to be consimilar if

$$Y = B^{-1} X \overline{B}$$

for some invertible matrix B.

Theorem For $n \times n$ matrices of rank n, the map

$$[X]_{\mathrm{cs}} \mapsto [X\overline{X}]_{\mathrm{s}}$$

from consimilarity to similarity classes is injective.

Exercises (i) If $X\overline{X} = I$ then $X = Y^{-1}\overline{Y}$. (Hint: set $Y = \lambda\overline{X} + \overline{\lambda} I$ and compute YX.)

(ii) Any negative eigenvalue of $X\overline{X}$ necessarily has even multiplicity.

Part II

Strong Kähler with Torsion

Let M be a Hermitian manifold of real dimension $2n$. Each tangent space T_mM admits a complex structure J, orthogonal relative to the Riemannian metric g. The associated 2-form ω is defined by

$$\omega(X, Y) = g(JX, Y).$$

Definition M is Kähler iff $d\omega = 0$

This is equivalent to asserting that $\mathrm{Hol}(g) \subseteq U(n)$.

The exterior derivative decomposes as $d = \partial + \overline{\partial}$ and

$$dJd = -2i^{p+1}\partial\overline{\partial} : \Omega^{p,p} \to \Omega^{p+1,p+1}.$$

Definition M is SKT iff $\partial\overline{\partial}\omega = 0$.

Example M is an even-dim compact Lie group.

The Iwasawa manifold

Let H denote the complex Heisenberg group

$$\left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} : z^\alpha \in \mathbb{C} \right\}$$

and Γ the subgroup with $z^\alpha \in \mathbb{Z}[i]$. Then

$$\mathbb{M} = \Gamma \backslash G = \{\Gamma h \ : \ h \in H\}$$

is a compact 6-dimensional nilmanifold.

Since

$$(z^1, z^2, z^3) \sim (z^1 + \gamma^1,\ z^2 + \gamma^2,\ z^3 + \gamma^1 z^2 + \gamma^3),$$

the 1-forms

$$dz^1, \quad dz^2, \quad dz^3 - z^1 dz^2$$

are invariantly defined and define the standard complex structure $\mathbb{J}_0$ on $\mathbb{M}$. There is also a well-defined map sending the above class to

$$(z^1, z^2) \in \frac{\mathbb{R}^4}{\mathbb{Z}^4} = T^4.$$

Complex structures on $\mathbb{M}$

The fibration

$$\begin{array}{ccc} \mathbb{M} & & \mathbb{CP}^3 \\ \pi \downarrow T^2 & \text{is analogous to} & \downarrow S^2 \\ T^4 & & S^4, \end{array}$$

though $\mathbb{M}$ admits no Kähler metric.

<u>Theorem</u> (i) Given an invariant integrable complex structure $\mathbb{J}$ on $\mathbb{M}$ there exists a complex structure J on T^4 such that π is holomorphic.

(ii) If $J = J_X$ with $\det X \neq 0$ then $\mathbb{J}$ is determined by J up to right translation by H (inner automorphism).

(iii) The previous action $J_X \mapsto J_{B^{-1}X\overline{B}}$ is induced from outer automorphisms of H.

The SKT condition

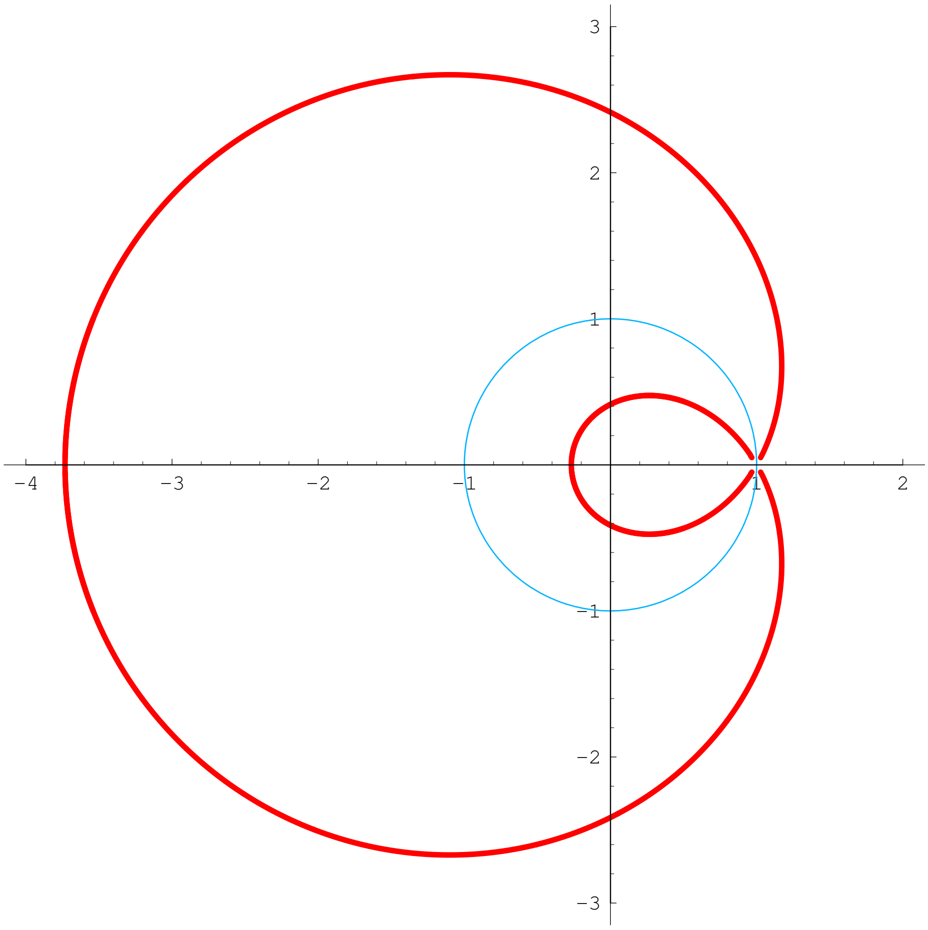
<u>Theorem</u> (i) The vanishing of $\partial\overline{\partial}\omega$ depends only on J, not the choice of invariant Hermitian metric on $\mathbb{M}$.

(ii) It occurs iff the eigenvalues $\{z, \overline{z}\}$ of $X\overline{X}$ satisfy

$$(1+|z|^2)|1+z|^2 = 8|z|^2$$

or $r = -3\cos\theta \pm \sqrt{2+\cos^2\theta}$ with $r = |z-1|$.

Inversion $z \leftrightarrow 1/\overline{z}$ corresponds to $J \leftrightarrow -J$.



Final remarks

Proof. Lift J_X to $\mathbb{J}$ by setting $\mathbb{E} = \langle \boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2, \boldsymbol{\varepsilon}^3 \rangle$ with $\boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2$ as above and

$$\boldsymbol{\varepsilon}^3 = \sigma - (ad - bc)\overline{\sigma}, \qquad \sigma = dz^3 - z^1 dz^2.$$

The SKT condition turns out to be equivalent to

$$(\mathbb{J} d\boldsymbol{\varepsilon}^3) \wedge \overline{d\boldsymbol{\varepsilon}^3} = 0$$

and it suffices to compute $J(dz^1 \wedge dz^2)$.

Corollary Any invariant SKT structure on $\mathbb{M}$ arises from J_X on T^4, where

$$X = B^{-1}\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}\overline{B},$$

with $B \in GL(2,\mathbb{C})$ and $z \neq 1$ on the curve.

The stabilizer in $GL(2,\mathbb{C})$ is $\mathbb{C}^*$ unless $z = -2 \pm \sqrt{3}$, giving a moduli space of real dimension 7.

References

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