The six-sphere and projective geometry

(1) 4 by 4 matrices

Set $z = (z_1, z_2, z_3)$ and $M(z) := \begin{pmatrix} 0 & -z_1 & \cdot & \cdot \\ z_1 & 0 & \cdot & \cdot \\ z_2 & \overline{z}_3 & 0 & \cdot \\ z_3 & -\overline{z}_2 & \overline{z}_1 & 0 \end{pmatrix} \in \mathfrak{o}(4, \mathbb{C})$.

Proposition $\{A \in \mathfrak{o}(4, \mathbb{C}) \cap SU(4) : \mathrm{Pf}\, A = 1\} = \{M(z) : \|z\| = 1\}$.

Note that the columns of $M(z)$ are unitary and $\det M(z) = \|z\|^4$.

The mapping $z \mapsto M(z)$ is an inclusion of $\mathbb{R}^6$ in $\mathfrak{o}(4, \mathbb{C}) \cong \Lambda^2\mathbb{C}^4$, which is a real representation $\mathbb{R}^6 \oplus i\mathbb{R}^6$ relative to $SU(4) \stackrel{2:1}{\to} SO(6)$.

$M(z)$ is Clifford multiplication of $z \in \mathbb{R}^6$ on spinors, and $AM(z)A^\top = M(z')$.

(2) The 6-quadric

Let $\xi, \eta \in \mathbb{C}^4$ be column vectors. Write $\xi \cdot \eta = \xi^\top \eta$ for the symmetric inner product. Consider $\mathbb{P}^7 = \{[\xi, \eta] : \xi, \eta \in \mathbb{C}^4, (\xi, \eta) \neq (0, 0)\}$, so

$$Q^6 = \{[\xi,\ \eta] \in \mathbb{P}^7 : \xi \cdot \eta = 0\}.$$

is a non-singular quadric hypersurface.

Proposition $([\xi],\ z) \mapsto [\xi,\ M(z)\xi]$ embeds $\mathbb{P}^3 \times \mathbb{R}^6 \hookrightarrow Q^6$.

Proof: By skewness, $[\xi,\ M(z)\xi] \in Q^6$. If $(\xi,\ M(z)\xi) = (\xi',\ M(z')\xi')$ then $\xi = \xi'$ and

$$M(z - z')\,\xi = 0 \ \ \Rightarrow \ \ \|z - z'\|^4 = 0 \ \ \Rightarrow \ \ z = z'.$$

We obtain a fibration $\pi\colon Q^6 \to \mathbb{R}^6 \cup \infty = S^6$, with fibres

$$\mathbb{V}_z = \pi^{-1}(z) = \{[\xi,\ M(z)\xi]\}, \qquad \mathbb{V}_\infty = \pi^{-1}(\infty) = \{[0,\ \eta]\}$$

all isomorphic to $\mathbb{P}^3$. In homogeneous terms,

$$Q^6 \cong \frac{SO(8)}{SO(2) \times SO(6)} \cong \frac{SO(8)}{U(4)} \cong \frac{SO(7)}{U(3)} \longrightarrow \frac{SO(7)}{SO(6)} \cong S^6.$$

Moreover, $\mathrm{Conf}(S^6) = SO(7,1) \subset SO(8, \mathbb{C}) = \mathrm{Aut}(Q^6)$.

(3) Linear subspaces

There are two families of $\mathbb{P}^3$'s in Q^6 each parametrized by another Q^6 (via triality).

The fibres $\mathbb{V}_z$ with $z \in S^6$ form part of one family that also contains

$$[a,b,c,d,0,0,0,0] \rightsquigarrow [a,b,\lambda c,\lambda d,0,0,\mu d,-\mu c] \rightsquigarrow [a,b,0,0,0,0,d,-c],$$

generating the twistor fibration $\mathbb{P}^3 \to S^4 = \{z_1 = 0\}$.

Another family consists of

$$\mathbb{H}_\xi = \{[\xi,\ M(z)\xi] : z \in S^6\},$$

These $\mathbb{P}^3$'s project $1:1$ to $\mathbb{R}^6$. For example, $\mathbb{H}_0 = \{[1,0,0,0,0,z_1,z_2,z_3]\} \cup \mathbb{P}^2$.

In general, $\mathbb{V}_0 \cap \mathbb{H}_\xi = \{[\xi]\}$ and $\mathbb{V}_\infty \cap \mathbb{H}_\xi = \xi^\perp \cong \mathbb{P}^2$.

Definition Let's say that a 3-fold X in Q^6 is *horizontal over* $\mathbb{R}^6$ if $X \setminus \mathbb{V}_\infty$ is smooth and projects $1:1$ to $\mathbb{R}^6$.

$H_6(Q^6,\mathbb{Z})$ is generated by $[\mathbb{V}_0]$ and $[\mathbb{H}_0]$, and X will have bidegree $(1,p)$.

(4) Orthogonal complex structures

Theorem If a 3-fold $X \subset Q^6$ is horizontal it induces an *orthogonal* complex structure (OCS) on $\mathbb{R}^6$.

The proof is based on the fact that the Kähler metric on Q^6 projects to the round metric on S^6. There can be no 3-fold that is $1:1$ over S^6 since $b_2(S^6)=0$.

Examples If $\mathbb{H}_0$ defines a standard complex structure $\mathbb{R}^6 = \mathbb{C}^3$. More generally, $\mathbb{H}_\xi$ defines a *constant* complex structure J on $\mathbb{R}^6$. If $\xi = (a,1,0,0)$ then $\mathbb{H}_\xi$ consists of

$$[\xi,\ M(z)\xi] = [a,\ 1,\ 0,\ 0,\ -z_1,\ az_1,\ az_2+\overline{z}_3,\ az_3-\overline{z}_2].$$

and J has $\Lambda^{1,0} = \langle dz_1,\ a\,dz_2 + d\overline{z}_3,\ a\,dz_3 - d\overline{z}_2\rangle$, an isotropic subspace.

Now let $a = a(u)$ be a polynomial of degree p in $u = z_1$. Then

$$\Gamma = \{[a(u),\ 1,\ 0,\ 0,\ -u,\ a(u)u,\ v,\ w] : u,v,w \in \mathbb{C}\}$$

is still horizontal over $\mathbb{R}^6$. The associated complex structure has the form

$$(J_0,\ J_{a(u)}) \qquad \text{on} \qquad \mathbb{R}^6 = \mathbb{C} \oplus \mathbb{R}^4,$$

This is an example of a twisted or *warped* OCS. How can we characterize $\overline{\Gamma}$?

(5) Divisors

$X=\overline{\Gamma}$ lies in the singular 4-quadric

$$Q_s^4=\{[x_1,x_2,0,0,x_5\ldots,x_8]\in\mathbb{P}^7 : x_1x_5+x_2x_6=0\}$$

It satisfies the homogeneous equation

$$a(-x_5/x_2)x_2^p=x_1x_2^{p-1}\quad\Rightarrow\quad f:=A(x_2,x_5)-x_1x_2^{p-1}=0,$$

but does not contain $\{x_2=0=x_5\}\cap Q_s=\mathbb{H}_0$. Thus

$$[f]=(p-1)[\mathbb{H}_0]+[X].$$

Any Weil divisor in Q_s of bidegree $(1,p)$ has this form for some polynomial f.

Exercises (i) If f does not involve $x_7,x_8,$ then $X=\bigcup_{p\in C,\ q\in L}\ell_{p,q}$ is a *double cone* where C is a curve in Q^2 and L is a line.
(ii) But if $f=x_5x_7+x_2x_8$ then $[f]=D\cup[\mathbb{H}_0]$ where $D=\sigma(\mathbb{P}^1\times\mathbb{P}^2)\subset\mathbb{P}^5$ is *smooth*.

Lemma If X^3 is horizontal over $\mathbb{R}^6$ then $X\cap\mathbb{V}_\infty$ contains a linear $\mathbb{P}^2$ and (if $p>0$) points that are singular in X.

(6) Classification

Theorem [BSV] If $X^3\subset Q^6$ is horizontal over $\mathbb{R}^6$ then it is equivalent (under $SO(8,\mathbb{C})$) to the divisor $\overline{\Gamma}$ above.

Corollary Any orthogonal complex structure J on $\mathbb{R}^6$ such that $\int_{S^6\setminus\infty}\|\nabla J\|^6<\infty$ is a warped OCS on $\mathbb{C}\oplus\mathbb{R}^4$.

The finite energy condition guarantees that $\overline{\Gamma}$ is analytic (by Bishop's theorem).

Theorem [BV] Any 3-fold of bidegree $(1,p)$ in Q^6 is one of

- (i) a linear subspace $\mathbb{H}_\xi$ with $p=0$
- (ii) a smooth quadric Q^3 with $p=1$
- (iii) the cone over a Veronese surface in Q^4 with $p=3$
- (iv) a Weil divisor in Q_s^4 with $p\geq 1$.

Cases (i), (ii), (iv) are all contained in a $\mathbb{P}^5$, but (ii), (iii) contain no $\mathbb{P}^2$.

The smooth 3-folds are (i), (ii) and the Segre instance of (iv) with $p=2$.

(7) Sketch proofs

Let X be a 3-fold in Q^6 of bidegree $(1,p)$.

It has degree $p+1$ in $\mathbb{P}^7$ so $X \subset \mathbb{P}^{p+3}$ (only useful if $p \le 3$).

[BSV] can assume that X is $1:1$ over $\mathbb{R}^6$. Consider $X \cap \mathbb{V}_\infty$. It must contain a $\mathbb{P}^2$ and at least one point x which is singular in X.

Then $X \subset \mathrm{Ann}(x) = Q^5_s$, and we can project

$$X \subset Q^5_s \setminus \{x\} \longrightarrow Q^4 = \mathbb{G}\mathrm{r}_2(\mathbb{C}^4).$$

If the image has dimension 3, it has to be in a $\mathbb{P}^4$, so $X \subset \mathbb{P}^5$. If the latter is false, the image is the secant variety of a twisted cubic [Ran], the Veronese surface:

$$v \otimes v \in S^2(S^2\mathbb{C}^2) \cong \Lambda^2(S^3\mathbb{C}^2).$$

What is the open set of $\mathbb{R}^6$ over which this is $1:1$?

[BV] need to prove if X is smooth with $p > 3$ then it is still true that $X \subset \mathbb{P}^5$.
If $p \neq 1$, then $Q^6 \cap \mathbb{P}^5 = Q^4_s$ cannot have rank 5 or 6.

(8) More examples

For an example of J on $\mathbb{R}^6$ without analytic extension, take $a = \wp(u)$ above:

$$J = (J_0, J_{\wp(u)}) \qquad \text{on} \qquad \mathbb{R}^6 = \mathbb{C} \oplus \mathbb{R}^4.$$

This gives a non-standard OCS on $T^6 = \mathbb{R}^6/\mathbb{Z}^6$.

As remarked, no 3-fold can be $1:1$ over S^6, which admits no OCS. Its standard almost-complex structure defines a section s of Q^6 whose fibrewise complement is a *holomorphic* $\mathbb{P}^2$-bundle over S^6 with total space

$$\frac{G_2}{U(2)} \cong \frac{SO(7)}{SO(2) \times SO(5)} \cong Q^5.$$

The Q^3 in (ii) can be chosen to lie in this Q^5 and is $1:1$ over

$$S^6 \setminus S^2 = \mathbb{R}^6 \setminus \mathbb{R}^2 = S^3 \times H^3.$$

It is the 'blow-up' of S^6 in which S^2 is replaced by $Q^2 \cong S^2 \times \mathbb{P}^1$.

By analogy, a suitable quadric Q^2 in $\mathbb{P}^3 \to S^4$ defines an OCS on $\mathbb{R}^4 \setminus \mathbb{R}$ that is conformally Kähler. But an OCS on $\mathbb{R}^4 \setminus K$ with $\dim K < 1$ is constant [SV].