G_2 METRICS AND M–THEORY

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I Weak holonomy and supergravity

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III G_2 and $\mathrm{SU}(3)$ structures from each other

PART I

The exceptional geometry of conical singularities, based on Alfred Gray's notion of weak holonomy:

- If (Y, g_7) has weak holonomy G_2 then

$$dr^2 + r^2 g_7 \quad \text{on} \quad X = \mathbb{R}^+ \times Y$$

has holonomy in $\mathrm{Spin}\,7$.

Examples: $Y = S^7$, S^7_{sq} $(\stackrel{S^3}{\to} S^4)$

$$\mathrm{Ber}^7 = \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}, \quad \mathrm{AW}^7 = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)_{p,q}}$$

- If (Z, g_6) is nearly-Kähler (weak hol $\mathrm{SU}(3)$) then

$$ds^2 + s^2 g_6 \quad \text{on} \quad \mathbb{R}^+ \times Z$$

has holonomy in G_2.

Examples: The 3-symmetric spaces $Z = S^6$,

$$\mathbb{CP}^3, \quad \mathbb{F} = \frac{\mathrm{SU}(3)}{\mathrm{T}^2}, \quad S^3 \times S^3.$$

11-dimensional equations

Maximal $D = 11$ supergravity involves a metric g of signature $(10,1)$, and a 4-form $F = dA$ satisfying

$$R_{im} = \frac{1}{12}\left(F_{ijkl}F_m^{ijk} - \frac{1}{12}g_{im}F^2\right)$$

$$dF = 0$$

$$d * F = F \wedge F$$

M-theory adds conjectural correction terms that we shall ignore.

Supersymmetry requires Killing spinor(s):

$$\nabla_m \eta + \frac{1}{288}\left[\Gamma_m^{ijkl} - 8\delta_m^i \Gamma^{jkl}\right] F_{ijkl}\eta = 0$$

and holonomy reduction of a suitable connection.
Let $\nu = \frac{1}{32}\dim(\text{K spinors})$.

There are theories of generalized holonomy groups $\mathcal{H} \subset SO(10,1)$ [Duff], [Hull].

First solutions

Identifying F with the volume form of a 4-manifold gives an Einstein product $M^4 \times M^7$.

- With $\nu = 1$: $\mathrm{AdS}_7 \times S^4$ or $\mathrm{AdS}_4 \times S^7$.

- M2 brane solution with $\nu \leqslant \frac{1}{2}$:

$$\left(1 + \frac{a^6}{r^6}\right)^{-2/3} g_{2,1} + \left(1 + \frac{a^6}{r^6}\right)^{1/3} (dr^2 + r^2 g_7)$$

with $F = \mathrm{vol}_{2,1} \wedge f(r) dr$ and $*F = 6a^6 \,\mathrm{vol}_7$.

Interpolates between the asymptotic ($r = \infty$) metric

$$g_{2,1} + (dr^2 + r^2 g_7) \quad \text{on} \quad M_{2,1} \times X^8$$

and the 'near-horizon' limit ($r \ll a$)

$$\left(\frac{r^4}{a^4} g_{2,1} + \frac{a^2}{r^2} dr^2\right) + a^2 g_7 \quad \text{on} \quad \mathrm{AdS}_4 \times Y^7.$$

If g_7 has weak holonomy G_2 then these two limits have $\nu = \frac{1}{16}$ and $\nu = \frac{1}{8}$ respectively.

A weak G_2 orbifold

Suppose that $X^8 = S^1 \times Y^7$ has a product metric

$$g_8 = dx^2 + (dy^2 + y^2 g_6)$$

with holonomy $\{1\} \times \mathrm{G}_2 \subset \mathrm{Spin}\,7$. Setting $x = r\cos t$ and $y = r\sin t$ gives

$$g_8 = dr^2 + r^2(dt^2 + \sin^2 t\, g_6).$$

<u>Corollary</u> [Acharya et al] If (Z, g_6) is nearly-Kähler, the 'spherical' metric

$$dt^2 + \sin^2 t\, g_6$$

has weak holonomy G_2 (so Einstein); its singularities at $t = 0, \pi$ approximate G_2 holonomy cones.

If the nearly-Kähler structure is defined by a 2-form ω and $(3,0)$-form $\psi^+ + i\psi^-$ then the G_2 3-form is

$$\varphi = s^2\,\omega \wedge dt + s^3(c\psi^+ + s\psi^-)$$

where $c = \cos t$ and $s = \sin t$.

PART II

If (Y, g_7) has holonomy G_2 then $Q = Y/S^1$ has a symplectic $\mathrm{SU}(3)$ structure with

$$g_7 = f(q)^2(dx+\theta)^2 + \frac{1}{f(q)}k_6$$

with dilation factor f. The metric k_6 forms part of the dual 10-dimensional 'type IIA' string theory.

In certain cases, (Q, k_6) is Kähler [Apostolov-S].

<u>Theorem</u> [Atiyah-Witten] Each 3-symmetric space Z admits a S^1 action for which $Z/S^1 \cong S^5$. Moreover, $(\mathbb{R}^+ \times Z)/S^1 \cong \mathbb{R}^6$ contains the fixed points ($f = 0$) as Lagrangian submanifolds.

Example: $\mathbb{F} = \dfrac{U(3)}{U(1)\times U(1)\times U(1)} \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \mathbb{CP}^2$.

A standard S^1 action has fixed point set $S^2 \sqcup S^2 \sqcup S^2$, one fibre for each of the three projections to $\mathbb{CP}^2$. Then $(\mathbb{R}^+ \times \mathbb{F})/S^1$ has three $\mathbb{R}^3$'s intersecting transversally at a point.

Nearly-Kähler 6-manifolds as quotients?

Example: $S^3 \times S^3 = \dfrac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{SU}(2)}$ admits an action of $\mathfrak{S}_3 = \langle\sigma,\tau\rangle$. The diagonal S^3 is invariant by σ, and its images by τ give a 120^{o} configuration of S^3's intersecting at z. These give rise to intersecting $\mathbb{R}^4$'s in $\mathbb{R}^+ \times (S^3 \times S^3)$.

Suggests existence of a $\mathrm{Spin}\,7$ total space

$$\begin{array}{ccccc} X & = & \mathbb{R}^+ & \times & Y \\ & & & & \pi \downarrow S^1 \\ & & \mathbb{R}^+ & \times & (S^3 \times S^3) \ni z \end{array}$$

where Y is a weak G_2-orbifold, π maps fixed points to the three S^3's, and $\pi^{-1}(z)$ is the vertex of a cone modelled on $\mathbb{R}^+ \times \mathbb{F}$.

Another topic: metrics with holonomy G_2 on X^8/S^1 via hyper-Kähler reduction [Acharya-Witten].

Lagrangian triality and quaternionic geometry

Let N^5 ($\stackrel{T^2}{\rightarrow} T^3$) be a nilmanifold of type $(0,0,0,12,13)$, so $de^4 = e^{12}$ and $de^5 = e^{13}$. Then $N^6 = N^5 \times S^1$ has a symplectic form

$$\omega = e^{16} + e^{25} + e^{34}$$

with trivial e^6 factor.

Theorem [Giovannini] The compact manifold N^6 has a configuration of invariant Lagrangian submanifolds at 120° through each point.

Proof. Follows from the existence of closed 3-forms

$$\begin{gathered} e^1 \wedge (e^2 - \sqrt{3}e^5) \wedge (e^3 + \sqrt{3}e^4) \\ (e^1 + \sqrt{3}e^6) \wedge e^2 \wedge (e^3 - \sqrt{3}e^4) \\ (e^1 - \sqrt{3}e^6) \wedge (e^2 + \sqrt{3}e^5) \wedge e^3. \end{gathered}$$

Note that $Sp(3,\mathbb{R})$ acts transitively on 'positive-definite' Lagrangian triples (L_1, L_2, L_3) in $\mathbb{R}^6$.

Corollary $N^6 \times T^2$ has a closed 4-form Ω with stabilizer $\mathrm{Sp}(2)\mathrm{Sp}(1)$.

PART III

In 6 dimensions, an $\mathrm{SU}(3)$ structure is characterized by a 2-form ω and a complex $(3,0)$-form $\psi^+ + i\psi^-$. A G_2 structure is defined in $6+1$ dimensions by

$$\begin{aligned} \varphi &= \omega\, dt + \psi^+ \\ *\varphi &= \psi^- dt + \tfrac{1}{2}\omega^2. \end{aligned}$$

The holonomy reduces iff

$$\boxed{d\varphi = 0 \quad \text{and} \quad d * \varphi = 0} \qquad \circledast$$

implying the half-flat conditions

$$d\psi^+ = 0 \quad \text{and} \quad d(\tfrac{1}{2}\omega^2) = 0.$$

Example: $N^6 = N^5 \times S^1$ has a half-flat structure:

$$\begin{aligned} \psi^+(0) &= e^{123} + e^{145} + e^{356} - e^{246} \\ \tfrac{1}{2}\omega(0)^2 &= e^{2345} + e^{1346} + e^{1256}. \end{aligned}$$

Problem: Find $\omega(t), \psi^+(t)$ satisfying $\circledast$. Hamiltonian theory guarantees solution [Hitchin].

Solution: Can split off e^6 to get a Calabi-Yau orbifold:

Theorem $N^5 \times (-\frac{\pi}{2}, \frac{\pi}{2})$ has a metric k_6 with holonomy equal to $\mathrm{SU}(3)$.

Proof. Let $c = \cos t$, $s = \sin t$. Then k_6 has an orthonormal basis

$$ce^1, \qquad (1+s)^{1/2}e^2, \qquad (1-s)^{1/2}e^3,$$
$$(1+s)^{-1/2}e^4, \quad (1-s)^{-1/2}e^5, \quad c^2dt.$$

The closed foms are

$$\Omega = c^3\, e^1dt - e^{24} + e^{35}$$
$$\Psi^+ = -c(1+s)e^{25}dt - c(1-s)e^{34}dt + c^2e^{123} + e^{145}$$
$$\Psi^- = -ce^{45}dt - c^3e^{23}dt - (1-s)e^{134} - (1+s)e^{125}.$$

Since e^{145} and $-c^3e^{23}dt$ are also closed,

$$\exp(\theta J) \cdot e^{145}$$

generates a pencil $\{\mathcal{L}_{\theta,z}\}$ of special Lagrangian submanifolds through each point z, each with phase θ.

Back to M theory

One can seek solutions

$$
\begin{aligned}
g_{10,1} &= e^{A}(y) g_{3,1} + e^{B}(y) g_7 \\
F &= f(y)\mathrm{vol}_{3,1} + F_7 \\
\eta &= \theta \otimes \varepsilon_1 + i\gamma\theta \otimes \varepsilon_2
\end{aligned}
$$

on $\mathrm{AdS}_4 \times Y$ in which the spinors $\varepsilon_1, \varepsilon_2$ determine an $\mathrm{SU}(3)$ structure on $T_y Y = \mathbb{R} \oplus \mathbb{C}^3$.

Equations link F to the intrinsic torsion [Chiossi-S].

<u>Theorem</u> [Lukas-Saffin] (i) $f = 0$ so $F \wedge F = 0$.
(ii) Y is foliated by 6-manifolds on which the induced $\mathrm{SU}(3)$ structure is half-flat.
(iii) if F is $\mathrm{SU}(3)$-invariant ('singlet flux') then

$$g_7 = dt^2 + e^{-2\alpha(t)} g_6$$

is a warped product with a nearly-Kähler metric g_6.

Exact nature of G_2 structures and singularities unclear.

Selected references

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