ORTHOGONAL COMPLEX STRUCTURES ON DOMAINS OF $\mathbb{R}^4$

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joint work with Jeff Viaclovsky

based on twistor theory of
Penrose (1967)
Atiyah-Hitchin-Singer (1978)
Pontecorvo (1992)

PART ONE

Definition. An OCS on an open set $\Omega \subset \mathbb{R}^4$ is a (C^1) map

$$J\colon \Omega \to \{M \in SO(4) : M^2 = -I\} \cong \mathbb{CP}^1$$

satisfying the usual integrability condition

Problem. Given Ω, classify OCS's on Ω

Conformal invariance:

$$\begin{array}{ccccc} \mathbb{CP}^3 & \supset & \mathbb{R}^4 \times \mathbb{CP}^1 & & \\ \pi \big\downarrow & & \uparrow J & & \\ \mathbb{HP}^1 = S^4 & \supset & \mathbb{R}^4 & \supseteq & \Omega \end{array}$$

Known proposition. An OCS on Ω is the same as a "holomorphic" section $s\colon \Omega \to \mathbb{CP}^3$

Corollary. Any hyperplane $\mathbb{CP}^2$ determines an OCS J on $S^4 \setminus \{p\}$ or $\mathbb{R}^4$. Such a J is "conformally constant"

Projective coordinates

$$
\begin{array}{rl}
p = [1, a, W_1, W_2] & \in \mathbb{CP}^3 \\
& \downarrow \\
\pi(p) = [1 + ja,\ W_1 + jW_2] & \in \mathbb{HP}^1
\end{array}
$$

$$
\begin{aligned}
\pi(p) = [1, z_1 + jz_2] &\in \mathbb{HP}^1 \setminus [0,1] \\
\iff\ & W_1 + jW_2 = q(1 + ja) \\
\iff\ & \begin{cases} W_1 = z_1 - a\overline{z}_2 \\ W_2 = z_2 + a\overline{z}_1 \end{cases}
\end{aligned}
$$

Fix z_1, z_2; then a is a coordinate on the fibre and determines an a.c.s. J_a for which

$$\Lambda^{1,0} = \langle dz_1 - a\,d\overline{z}_2,\ dz_2 + a\,d\overline{z}_1 \rangle$$

An OCS J on $\Omega \subseteq \mathbb{R}^4$ is defined by a function $a \colon \Omega \to \mathbb{C} \cup \infty$ (deforming J_0) satisfying

$$\frac{\partial a}{\partial \overline{z}_1} - a\frac{\partial a}{\partial z_2} = 0, \quad \frac{\partial a}{\partial \overline{z}_2} + a\frac{\partial a}{\partial z_1} = 0.$$

Any C^1 solution is necessarily harmonic

Application to quadrics

Let $\mathscr{Q}$ be the "anti-diagonal" quadric

$$\begin{aligned} 0 &= 1.W_2 - aW_1 \\ &= a^2\overline{z}_2 - a(z_1 - \overline{z}_1) + z_2 \\ &= a^2\overline{z}_2 - 2ia\,\mathfrak{Im}\,z_1 + z_2 \end{aligned}$$

Thus

$$\begin{aligned} \pi^{-1}(z_1, z_2) \subset \mathscr{Q} &\Leftrightarrow z_2 = 0,\ z_1 \in \mathbb{R} \\ &\Leftrightarrow (z_1, z_2) \in S^1 \subset S^4 \end{aligned}$$

Otherwise roots occur in pairs $\{a, -1/\overline{a}\}$ and cannot be coincident in this particular case

Definition. Given a quadric $\mathscr{Q} \subset \mathbb{CP}^3$ and a point $q \in \mathbb{R}^4$,

$$\begin{aligned} q \in D_0 &\Leftrightarrow \pi^{-1}(q) \subset \mathscr{Q} \\ q \in D_1 &\Leftrightarrow \#\pi^{-1}(q) = 1 \end{aligned}$$

The discriminant locus is $D = D_0 \cup D_1$

Two Liouville theorems

Let J be an OCS on an open set $\Omega = \mathbb{R}^4 \setminus K$

1. If $\mathrm{Hm}^1(K) = 0$ then J arises from some hyperplane and is conformally constant

$$\lim \ \uparrow \ \times 2$$

2. If K is a round circle or straight line then $\pm J$ arises from a unique real quadric in $\mathbb{CP}^3$. Any two are conformally equivalent

Idea of the proof of 1:

$$\underset{\dim_{\mathbb{R}} 4}{J(\Omega)} \quad \text{is analytic in} \quad \underset{\text{“}\dim < 3\text{”}}{\mathbb{CP}^3 \setminus \pi^{-1}(K)}$$

Key point is to deduce that $\overline{J(\Omega)}$ is analytic, thus (Chow/Mumford) algebraic of degree 1

Removal of singularities

If $E_i \subset \mathbb{R}^m$, let $r_i = \frac{1}{2}\text{diam}(E_i)$ and

$$v_i^d = \text{vol}(B^d(r_i)) = \frac{(\pi r_i^2)^{d/2}}{\Gamma(\frac{d}{2}+1)}$$

Hausdorff measure:

$$\text{Hm}^d(K) = \lim_{\delta\to 0} \inf\left\{\sum_{i=1}^{\infty} v_i^d : K \subseteq \bigcup_{i=1}^{\infty} E_i,\ r_i < \delta\right\}$$

Shiffman's Theorem (1968). U open in $\mathbb{C}^n$, E closed in U. If $A^{2k} \subset U \setminus E$ is analytic and $\text{Hm}^{2k-1}(E) = 0$, then $\overline{A} \cap U$ is analytic.

This is based on the proof of

Bishop's Theorem (1964). U open in $\mathbb{C}^n$ and B analytic in U. If $A^{2k} \subset U \setminus B$ is analytic and $\text{Hm}^{2k}(\overline{A} \cap B) = 0$ then $\overline{A} \cap U$ is analytic.

Itself a generalization of Remmert-Stein (1953)

PART TWO

A non-degenerate quadric $\mathscr{Q}$ is determined by the reduction

$$\begin{array}{ccc} \mathbb{C}^4 & = & \mathbb{C}^2 \otimes \mathbb{C}^2 \\ SO(4,\mathbb{C}) & \sim & SL(2,\mathbb{C})\times SL(2,\mathbb{C}) \end{array}$$

The symmetric bilinear form g is identified with $\omega_1 \otimes \omega_2$ and $\mathscr{Q} \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ is generated by the rank-one elements $u \otimes v$

The quadric $\mathscr{Q}$ is "real" if it arises from a further reduction

$$\begin{array}{ccc} \mathbb{H}^2 & = & \mathbb{R}^2 \otimes \mathbb{H} \\ SO^*(4) & \sim & SL(2,\mathbb{R})\times SU(2) \end{array}$$

The involution j on $\mathbb{C}^4$ equals $\sigma_1 \otimes \sigma_2$ where $\sigma_i\colon \mathbb{C}^2 \to \mathbb{C}^2$ with ${\sigma_1}^2 = 1$ and ${\sigma_2}^2 = -1$. The real lines on $\mathscr{Q}$ are those of the form $[u] \times \mathbb{CP}^1$ where $\sigma_1(u) = u$, confirming $D_0 \cong \mathbb{RP}^1 \times \mathbb{CP}^1$

'Real' or j-invariant matrices

$$\mathbf{p} = (1, W_1, a, W_2) \in \mathbb{C}^4$$
$$\Rightarrow \mathbf{p}j = (-\overline{a}, -\overline{W}_2, 1, W_1)$$

A matrix $G \in \mathbb{C}^{4,4}$ belongs to $\mathfrak{gl}(2, \mathbb{H})$ iff

$$G = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} = \{A|B\}$$

Then $G \in SL(2, \mathbb{H}) \sim SO_o(1, 5)$ if $\det G = 1$

Any symmetric $Q \in \mathbb{C}^{4,4}$ equals $Q_1 + iQ_2$, $Q_\alpha \in \mathfrak{gl}(2, \mathbb{H})$. If $\mathrm{rank} Q_1 = 4$, we can choose $G \in GL(2, \mathbb{H})$ such that

$$G^\top Q_1 G = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \{0|K\} = Q_0$$

$$G^\top Q_2 G = \{A|B\} = \{L + iM \,|\, bK + iN\}$$

with $b \in \mathbb{R}$ and L, M, N real symmetric.

Singular value decomposition

We may now suppose

$$Q = (1 - ib)Q_0 + iQ_2$$

where the "trace-free part" $Q_2 = \{L + iM \,|\, N\}$ is determined by

$$X = \begin{pmatrix} L_{11}+L_{22} & M_{11}+M_{22} & N_{11}+N_{22} \\ L_{11}-L_{22} & M_{11}-M_{22} & N_{11}-N_{22} \\ 2L_{12} & 2M_{12} & 2N_{12} \end{pmatrix}$$

The stabilizer of Q_0 in $SL(2,\mathbb{H})$ is

$$SL(2,\mathbb{R}) \times SU(2) \sim SO(2,1) \times SO(3)$$

Note that X defines an element of $\Lambda^3_+(\mathbb{R}^{5,1})$

We use SVD to diagonalize X and obtain

$$e^{i\theta} H^\top Q H = \begin{pmatrix} e^{\lambda+i\nu} & 0 & 0 & 0 \\ 0 & e^{\mu-i\nu} & 0 & 0 \\ 0 & 0 & e^{-\lambda+i\nu} & 0 \\ 0 & 0 & 0 & e^{-\mu-i\nu} \end{pmatrix}$$

with $\lambda, \mu, \nu \in \mathbb{R}$.

Quadrics in $\mathbb{CP}^3 \to S^4$

Theorem 3. Let $\mathscr{Q} \subset \mathbb{CP}^3$ be a non-degenerate quadric. There are three cases:

(0) $\mathscr{Q}$ is "real" and $D = D_0 = S^1$

(1) $D = D_1 = S^1 \times S^1$ is a smooth unknotted torus in $\mathbb{R}^4$

(2) D is a torus pinched over two points q_1, q_2 and $D_0 = \{q_1, q_2\}$

(0) occurs if $(\lambda, \mu, \nu) = (0, 0, 0)$
(2) occurs if $(\lambda, \mu, \nu) = (\lambda, \lambda, 0)$

(0) determines an OCS J on $\mathbb{R}^4 \setminus S^1$ and $\pi^{-1}(S^1) = S^1 \times S^2$

(1) determines an OCS J on $\mathbb{R}^4 \setminus K$ where $K = S^1 \times B^2$ is a solid torus and once again $\pi^{-1}(K) = S^1 \times S^2$

Example (1): $(\lambda, \mu, \nu) = (0, 0, \frac{\pi}{4})$

The quadratic form $\mathbf{p}^\top Q \mathbf{p}$ equals

$$e^{i\nu}(1 + a^2) + e^{-i\nu}({W_1}^2 + {W_2}^2)$$
$$= Aa^2 + 2Ba + C$$

in terms of $\mathbf{z} = (z_1, z_2) = (\mathbf{x}, \mathbf{y})$

The discriminant $\Delta = B^2 - AC$ equals

$$i - i|\mathbf{z}|^4 + {z_1}^2 + {\overline{z}_1}^2 + {z_2}^2 + {\overline{z}_2}^2$$

The zero set D of Δ is given by

$$\mathfrak{Im} \quad : \quad 1 = |\mathbf{z}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$
$$\mathfrak{Re} \quad : \quad -1 = |\mathbf{x}|^2 - |\mathbf{y}|^2$$

So $|\mathbf{x}| = |\mathbf{y}| = \frac{1}{2}$ and

$$D = D_1 \cong S^1 \times S^1$$

is a smooth Clifford torus

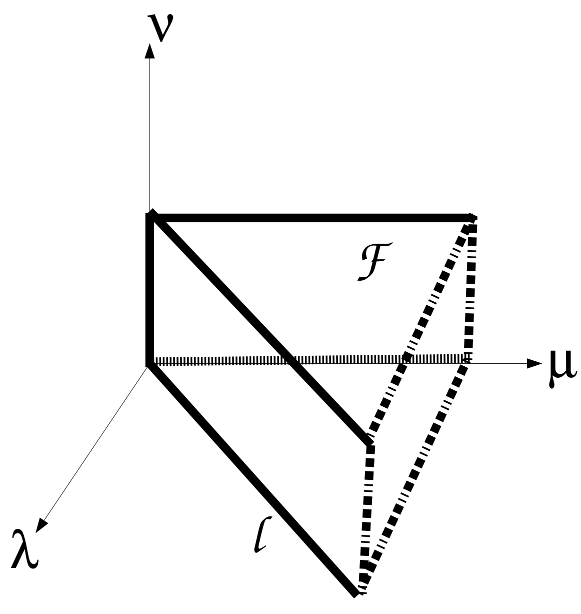
PART THREE

Under the action of the conformal group,

$$\mathcal{Q} \sim \begin{pmatrix} e^{\lambda+i\nu} & 0 & 0 & 0 \\ 0 & e^{\mu-i\nu} & 0 & 0 \\ 0 & 0 & e^{-\lambda+i\nu} & 0 \\ 0 & 0 & 0 & e^{-\mu-i\nu} \end{pmatrix}$$

In view of equivalences such as $\nu \leftrightarrow \frac{\pi}{2}-\nu$, a fundamental domain is

$$\{(\lambda,\mu,\nu) : 0 \leqslant \lambda \leqslant \mu,\ 0 \leqslant \nu < \tfrac{\pi}{2}\}.$$



Let $\ell = \{(\lambda,\lambda,0)\}$ and $\mathcal{F} = \{(0,\mu,\nu)\}$

$\mathbf{0} \in \ell \cap \mathcal{F}$ represents the real quadric

Proof of Theorem 3

This is divided into the following steps:

1. It is straightforward to determine that $D_0 = \emptyset$ outside ℓ, and $\# D_0 = 2$ if $\lambda = \mu > 0$

2. Consider the discriminant

$$\Delta = B^2 - AC : \mathbb{R}^4 \to \mathbb{R}^2$$

rank(grad Δ) < 2 only on $\ell \setminus \{\mathbf{0}\}$ (next slide)

3. $\mathfrak{Im}\,\Delta$ is a smooth 3-sphere in $\mathbb{R}^4$ if $\nu \neq 0$

4. We then use $\chi(D) = 2\chi(S^2) - \chi(\mathscr{Q}) = 0$ to prove that D has no S^2 components, and is in fact connected, at all interior points

5. D is then a smooth torus except on $\ell \setminus \{\mathbf{0}\}$

Example (2): $(\lambda,\mu,\nu)=(\lambda,\lambda,0)$

Set $A=\begin{pmatrix}0 & -1\\ 0 & i\end{pmatrix}$, $B=\begin{pmatrix}1 & 0\\ i & 0\end{pmatrix}$, $G=\frac{1}{\sqrt{2}}\{A|B\}$

Then

$$G^\top Q G = \begin{pmatrix} 0 & 0 & 0 & e^{-\lambda} \\ 0 & 0 & -e^{\lambda} & 0 \\ 0 & -e^{\lambda} & 0 & 0 \\ e^{-\lambda} & 0 & 0 & 0 \end{pmatrix}$$

and $\mathscr{Q}$ has equation $2e^{-\lambda}W_2 - 2e^{\lambda}aW_1 = 0$

$D=\{\Delta=0\}$ is given by

$$2|z_1|^2 + 4|z_2|^2 = e^{2\lambda}{z_1}^2 + e^{-2\lambda}{\overline{z}_1}^2$$

$$\mathfrak{Im} \; : \qquad x_1 y_1 = 0$$

$$\mathfrak{Re} \; : \quad 2({x_2}^2+{y_2}^2) = (c-1){x_1}^2$$

If $c=\cosh\lambda>1$ then D is a cone with vertex at 0 (and $\infty\in S^3$) and $D_0=\{0,\infty\}$

Higher degree

What are the possible maximal domains of definition $\Omega = \mathbb{R}^4 \setminus K$ for OCS's?

Let $\mathscr{H}$ be an irreducible hypersurface of $\mathbb{CP}^3$ of degree $d \geqslant 2$ with discriminant locus D

If $p \in \pi^{-1}(D) \in \mathbb{CP}^3$ then $\mathrm{rank}(d\pi_p) = 2$ and it follows that $\dim D \leqslant 2$

Theorem 4. If $\mathscr{H}$ contains the graph of a single-valued OCS J on $S^4 \setminus K$ then $K \supseteq D$ and $\mathscr{H} \setminus \pi^{-1}(K)$ is disconnected. Moreover, $\mathrm{Hm}^3(K) \neq 0$ unless $\mathscr{H}$ is a real quadric

If $d > 2$, D_0 consists of finitely many points. If $\mathscr{H}$ is j-invariant then $\# D_0 \leqslant d^2$

Quartics in $\mathbb{CP}^3 \to S^4$

The hypersurface $\mathscr{K}_c$ with equation

$$1 + a^4 + W_1^4 + W_2^4 + 6ca^2 = 0$$

is "real" and non-singular for $c \neq \pm\frac{1}{3}$

Proposition: $\mathscr{K}_c$ contains no $\mathbb{CP}^1$ fibres unless $c = -1, 0, 1$, in which cases it has 8.

Generically, $D(\mathscr{K}_c)$ is given by

$$\begin{cases} 6ABC = 4B^3 - 2\overline{B}A^2 \\ A\overline{B}^2 = \overline{A}B^2 \end{cases}$$

where $A = 1 + z_1^4 + z_2^4$, $B = -z_1^3\overline{z}_2 + z_2^3\overline{z}_1$ and $C = z_1^2\overline{z}_2^2 + z_2^2\overline{z}_1^2 + c$

If $c \notin \{-1, 0, 1\}$, consider $\mathscr{E} = \mathscr{K}_c/\mathbb{Z}_2$ and

$$\mathscr{E} \setminus \pi^{-1}(D) \xrightarrow{2:1} S^4 \setminus D$$

Then $\chi(D) = -8$ and D must be singular