COMPLEX STRUCTURES ON $\mathbb{R}^6$

Simon Salamon

joint work with Lev Borisov & Jeff Viaclovsky

Based on theory of

$$\begin{cases} \text{Penrose (1967)} \\ \text{Atiyah-Hitchin-Singer (1978)} \\ \text{Pontecorvo (1992)} \\ \text{Slupinski (1996)} \end{cases}$$

GLAM, in honour of Gallot, Rome, Sept 2008

Orthogonal complex structures

Definition: An OCS on an open set Ω of $\mathbb{R}^{2n}$ or S^{2n} is a complex structure J compatible with the Euclidean or round metric:

$$g(JX, JY) = g(X, Y)$$

Equivalently, an OCS is a (C^1) map $J\colon \Omega \to Z_n$ satisfying the usual integrability condition, where

$$Z_n = \{M \in SO(2n) : M^2 = -I\} \cong \frac{SO(2n)}{U(n)}.$$

Problem: Given $\Omega \subseteq \mathbb{R}^{2n}$, classify OCS's on Ω up to conformal equivalence.

What exceptional sets $\Lambda = \mathbb{R}^{2n} \setminus \Omega$ occur?

$\delta = \dim_{\mathrm{Hf}}(\Lambda)$ plays a key role.

The twistor space of S^{2n}

…provides an inductive definition of the spaces of linear complex structures:

$$
\begin{array}{rcl}
 & \dfrac{SO(2n+1)}{U(n)} = \frac{SO(2n+2)}{U(n+1)} = Z_{n+1} & \\
 & \Big\downarrow Z_n & \\
S^{2n} = & \dfrac{SO(2n+1)}{SO(2n)} & \qquad\qquad \Omega
\end{array}
$$

Each fibre $\pi^{-1}(x) \cong Z_n$ is a complex submanifold of the total space Z_{n+1}.

$$\boxed{\text{Proposition: An OCS on } \Omega \subset S^{2n} \text{ is the same as a holomorphic section } \tilde{J}\colon \Omega \to Z_{n+1}.}$$

So look at algebraic n-folds X in Z_{n+1}.

Facts in four dimensions

$$\begin{array}{rcl} & & Z_3 \;=\; \mathbb{P}^3 \\ & & \big\downarrow Z_2 = \mathbb{P}^1 \\ \infty & \in & S^4 \end{array} \qquad\qquad \mathbb{P}^k = \mathbb{P}(\mathbb{C}^{k+1})$$

Any hyperplane $\mathbb{P}^2$ of Z_3 contains exactly one fibre $\pi^{-1}(x)$ and defines an OCS on $S^4 \setminus \{x\}$.

There is a $\mathbb{P}^1$ worth of such OCS's containing $\pi^{-1}(\infty)$, and these are the constant OCS's on $\mathbb{R}^4$. They are in fact the only OCS's globally on $\mathbb{R}^4$.

Moreover, any OCS on Ω where $\mathrm{Hf}^1(\mathbb{R}^4 \setminus \Omega) = 0$ is associated to a hyperplane $\mathbb{P}^2$ [VS].

A "real" quadric in Z_3 gives an OCS on $S^4 \setminus S^1$. A generic quadric gives an OCS on the complement of a solid torus [VS].

Examples in six dimensions

$$Z_4 \overset{*}{=} Q^6 = \{x_1x_5 + x_2x_6 + x_3x_7 + x_4x_8 = 0\} \subset \mathbb{P}^7$$

$$\downarrow Z_3 = \mathbb{P}^3$$

$$\infty \in \ S^6$$

This time, any constant OCS on $\mathbb{R}^6$ arises from a "horizontal" $\mathbb{P}^3$ that intersects $\pi^{-1}(\infty)$ in a $\mathbb{P}^2$. These OCS's are parametrized by $(\mathbb{P}^3)^* \cong Z_3$.

There exists a global section $\tilde{J}\colon S^6 \to Q^6$ such that $\mathrm{Aut}(S^6, J) \cong G_2$ and $\tilde{J}^{\perp} = Q^5$ is holomorphic!

But there is no OCS J on S^6 because $\tilde{J}(S^6)$ cannot be a Kähler submanifold of Q^6 [L].

There does not exist a complex structure J on S^6 for which $\mathrm{Aut}(S^6, J)$ has an open orbit [HKP]. A hypothetical complex structure on S^6 gives a 1-para family of exotic complex structures on $\mathbb{P}^3$.

“Warped product” structures

Consider an almost complex structure J on

$$\begin{aligned} \mathbb{R}^6 &= \quad \mathbb{C} \oplus \mathbb{R}^4 \\ J &= J_0 + K(z), \end{aligned}$$

where $K(z)$ is a constant OCS on $\mathbb{R}^4$ depending on $z \in \mathbb{C}$. If $K\colon \mathbb{C} \to \mathbb{P}^1$ is holomorphic then J is integrable.

If K is rational then the graph $\Gamma = \tilde{J}(\mathbb{R}^6)$ has “finite energy” in the sense that $\mathrm{Hf}^6(\Gamma) < \infty$. In this case $\overline{\Gamma}$ is an algebraic 3-fold in Q^6 [B].

Moreover, $\overline{\Gamma} \cap \pi^{-1}(\infty) = \mathbb{P}^2$ but (unless $K=$ const) this fibre contains a singular line $L \cong \mathbb{P}^1$ and

$$\overline{\Gamma} \subset Q^4_s = \{x_1x_2 + x_3x_4 = 0\} \subset \mathbb{P}^5.$$

$\boxed{\text{Theorem [BSV]: Any finite-energy OCS on } \mathbb{R}^6 \text{ arises from a rational function } K \text{ as above.}}$

Explicit coordinates

Let $[x_1 \ldots, x_8] = [\mathbf{x}, \mathbf{y}]$, so that

$$Q^6 = \{[\mathbf{x}, \mathbf{y}] \in \mathbb{P}^7 : \mathbf{x}^\top \mathbf{y} = 0\}.$$

Suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{C}^4$ are both non-zero.

$$\boxed{\begin{array}{l} \text{Lemma: } [\mathbf{x}, \mathbf{y}] \in Q^6 \text{ if and only if } \mathbf{x} = M\mathbf{y}\text{, where} \\ \quad M = \begin{pmatrix} 0 & -z_3 & -z_2 & -z_1 \\ z_3 & 0 & -\overline{z}_1 & \overline{z}_2 \\ z_2 & \overline{z}_1 & 0 & -\overline{z}_3 \\ z_1 & -\overline{z}_2 & \overline{z}_3 & 0 \end{pmatrix}. \\ \text{Moreover, } \frac{1}{\|\mathbf{z}\|} M \in SU(4) \cap \mathfrak{so}(4, \mathbb{C}). \end{array}}$$

The twistor projection is given by

$$[M\mathbf{y}, \mathbf{y}] \ \stackrel{\pi}{\mapsto}\ \mathbf{z} \in \mathbb{C}^3 \cong \mathbb{R}^6,$$

with fibre parametrized by $[\mathbf{y}] \in \mathbb{P}^3$. It is easy to identify the action on Q^6 of the conformal group $SO(7,1)$. The latter contains the transformations

$$[\mathbf{x}, \mathbf{y}] \mapsto [A\mathbf{x}, \overline{A}\mathbf{y}], \quad A \in SU(4),$$

acting as $SO(6)$ on $\mathbb{R}^6$ via $M \mapsto AMA^\top$.

Spinors and triality

Let $\Delta_\pm$ be the spin representations of $\mathrm{Spin}(8)$. We can identify

$$Z_4 = \{\text{pure spinor classes } [\xi] \in \mathbb{P}(\Delta_+)\},$$

since any such ξ defines a max iso subspace

$$\Lambda^{1,0} = \{v \in \mathbb{C}^8 : v \cdot \xi = 0\}.$$

Now reduce to $U(4)$ by fixing $J \in Z_4$, w.r.t. which

$$\Delta^+ = \Lambda^{0,0} \oplus \Lambda^{2,0} \oplus \Lambda^{4,0}, \quad \dim = 1 + 6 + 1.$$

Then a generic pure spinor has the form

$$e^\omega = 1 + \omega + \tfrac{1}{2}\omega \wedge \omega, \quad \omega \in \Lambda^{2,0},$$

confirming that Z_4 is a non-singular 6-quadric $Q_+ \subset \mathbb{P}(\Delta_+)$.

Altogether we have three 6-quadrics

$$Q_+ \subset \mathbb{P}(\Delta_+)$$
$$Q_- \subset \mathbb{P}(\Delta_-)$$
$$Q_0 \subset \mathbb{P}(\mathbb{C}^8).$$

Bidegree

Altogether we have three 6-quadrics

$$Q_+ \subset \mathbb{P}(\Delta_+)$$
$$Q_- \subset \mathbb{P}(\Delta_-)$$
$$Q_0 \subset \mathbb{P}(\mathbb{C}^8).$$

Q_+, Q_- parametrize max iso subspaces of $\mathbb{C}^8$

Q_0, Q_- parametrize max iso subspaces of Δ_+, giving rise to two families of $\mathbb{P}^3$'s in the twistor space $Q^6 = Q_+$ of S^6:

"vertical" ones, either fibres $\pi^{-1}(x)$ or twistor spaces of conformal S^4's;

"horizontal" ones, each $1:1$ outside some $x \in S^6$.

One from each family generates

$$H_6(Q_+, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}.$$

Corollary: A finite-energy OCS J on $\mathbb{R}^6$ gives an algebraic 3-fold $\overline{\Gamma}$ in Q_+ of bidegree $(1, p)$.

Classification of 3-folds of order one

Theorem [BV]: An irreducible 3-fold X in Q^6 of bidegree $(1,p)$ is one of:

(i) a horizontal $\mathbb{P}^3$ $(p=0)$,

(ii) a smooth 3-quadric Q^3 $(p=1)$,

(iii) the cone over a Veronese $\mathbb{P}^2 \subset Q^4$ $(p=3)$,

(iv) a Weil divisor in a rank 4 quadric Q^4_s $(p \geqslant 1)$.

Example of (iii):

$$\begin{array}{ccccc} Q^2 & \subset & Q^3 & & \\ S^2\downarrow & & \downarrow & & \\ S^2 & \subset & S^6 \subset & \mathbb{R}^3 \oplus \mathbb{R}^4 = \mathrm{Im}\,\mathbb{O} \end{array}$$

But we require $\pi : X \to S^6$ to be $1:1$ except over ∞, and the exceptional fibre $\pi^{-1}(\infty)$ must in fact contain a $\mathbb{P}^2$. This rules out (ii) and (iii).

Working in the singular 4-quadric

In case (iv), take

$$\begin{aligned} \mathbb{P}^5 &= \{[x_1,\dots,x_6,0,0]\} \subset \mathbb{P}^7, \\ Q^4_s &= \{x_1x_5+x_2x_6=0\} \subset \mathbb{P}^5, \\ L &= \{[0,0,x_3,x_4,0,0]\} \subset Q^4_s. \end{aligned}$$

Example: Taking $x_3x_6+x_4x_5=0$ defines

$$\mathrm{Segre}(\mathbb{P}^1\times\mathbb{P}^2)\cup\mathbb{P}^3 \subset Q^4_s \subset \mathbb{P}^5.$$

For a non-constant OCS,

$$X=\overline{\Gamma}\subset Q^4_s, \quad L\subset X\cap\pi^{-1}(\infty)\cong\mathbb{P}^2,$$

and we get a different subcase of (iv). Let

$$P_\lambda=\{[ax_1,ax_2,x_3,x_4,bx_2,-bx_1,0,0]\}\cong\mathbb{P}^3,$$

with $\lambda=b/a\in\mathbb{P}^1$.

$\boxed{\text{Lemma: Each } X\cap P_\lambda\cong\mathbb{P}^2 \text{ defines the fibre of a projection } X\setminus L\longrightarrow\mathscr{C}\subset\mathbb{P}^1\times\mathbb{P}^1.}$

It follows that $X\setminus P_0$ is a graph $\tilde{J}$ over $\mathbb{R}^6$, and J is a warped product.

Conclusions

$\boxed{\text{Theorem v2 [BSV]: A finite-energy OCS on } S^6 \text{ minus a finite set of points is a warped product arising from a rational function } K\colon \mathbb{C} \to \mathbb{P}^1.}$

Counterexample: If $K = \wp$ is doubly-periodic then $\mathrm{Hf}^6(\Gamma) = \infty$, but J induces a non-constant OCS on the torus T^6.

Other examples include $S^6 \setminus S^2 \cong S^3 \times H^3$.

The generalization to $\mathbb{R}^{2n}$ with $n \geqslant 4$ is unclear, but an algebraic OCS J on $\mathbb{R}^{2n}$ defines an n-fold in Z_{n+1} such that

$$\overline{\Gamma} \cap \pi^{-1}(\infty) \subset Z_n,$$

and (if we are lucky) an OCS on $\mathbb{R}^{2n-2}$.

Example: If J is "asymptotically constant" then

$$\overline{\Gamma} \cap \pi^{-1}(\infty) = \mathbb{P}^{n-1},$$

and J must in fact be conformally constant.