SOME TRI-LAGRANGIAN STRUCTURES

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Special geometries

. . . defined by Lie groups and differential forms are all interrelated!

dim	structure group
8	$Sp(2)Sp(1)$
$\downarrow$	
6	$Sp(3,\mathbb{R}),\ SO(3)$
$\downarrow$	
7	G_2
$\updownarrow$	
6	$SU(3)$
6	$SO(3)\times SO(2)$

4-forms in 8 dimensions

$$\mathbb{R}^8 = \mathbb{H}^2 = \langle e^1, e^3, e^5, e^7 \rangle \oplus \langle e^2, e^4, e^6, e^8 \rangle$$

$$\begin{aligned} \sigma^1 &= e^{13} + e^{57} + e^{24} + e^{68} \\ \sigma^2 &= e^{15} + e^{73} + e^{26} + e^{84} \\ \sigma^3 &= e^{17} + e^{35} + e^{28} + e^{46} \end{aligned}$$

$$\begin{aligned} \Omega &= \sigma^1 \wedge \sigma^1 + \sigma^2 \wedge \sigma^2 + \sigma^3 \wedge \sigma^3, \\ \Omega' &= \sigma^1 \wedge \sigma^1 + \sigma^2 \wedge \sigma^2 - \sigma^3 \wedge \sigma^3. \end{aligned}$$

form(s)	stabilizer	geometry
$\sigma^1,\ \sigma^2$	$Sp(2,\mathbb{C})$	cx symplectic
$\sigma^1, \sigma^2, \sigma^3$	$Sp(2)$	hyper-Kähler
Ω	$Sp(2)Sp(1)$	quat-Kähler
Ω'	$Spin7$	exceptional

Closed versus parallel

A 4-form of type Ω' on an 8-manifold M determines a Riemannian metric g. Let ∇ be the associated Levi-Civita connection, so that $\nabla\Omega' = 0 \Rightarrow d\Omega' = 0$.

Proposition If $d\Omega' = 0$ then $\nabla\Omega' = 0$ and therefore hol$(g) \subseteq Spin7$.

A 4-form of type Ω on M determines a metric and a rank 3 subbundle $V = \langle\sigma^1, \sigma^2, \sigma^3\rangle$ of $\bigwedge^2 T^*M$.

Proposition If $d\Omega = 0$ AND d maps V into $V \wedge T^*M$ then $\nabla\Omega = 0$ and M is quaternion-Kähler.

There exist 8-manifolds with Ω closed and non-parallel. A partial analogue is a Kodaira surface $\Gamma\backslash\mathbb{C}^2$ that is (holomorphic) symplectic but not Kähler.

Reduction from 8 to 6

Distinguishing e^7, e^8,

$$\tfrac{1}{2}\Omega = -\omega \wedge e^{78} + \alpha \wedge e^7 + \beta \wedge e^8 - \tfrac{1}{2}\omega^2,$$

where

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56} \\ \alpha &= 3e^{135} + e^{146} + e^{236} + e^{245} \\ \beta &= 3e^{246} + e^{235} + e^{136} + e^{145} \end{aligned}$$

form(s)	stabilizer in $GL^+(6,\mathbb{R})$
α	$SL(3,\mathbb{R})\times SL(3,\mathbb{R})$
α, β	$SL(3,\mathbb{R})$
ω, α	$SL(3,\mathbb{R})'$
ω, α, β	$SO(3)$

Linear circle actions

Consider the action of $SO(2)=S^1$ on

$$\mathbb{R}^6 = \mathbb{C}^3 = \langle e^1, e^2\rangle \oplus \langle e^3, e^4\rangle \oplus \langle e^5, e^6\rangle$$

commuting with $SO(3)$. The simple 3-forms

$$\gamma(t) = (xe^1 + ye^2)\wedge(xe^3 + ye^4)\wedge(xe^5 + ye^6),$$

with $x = \cos t$, $y = \sin t$, form an S^1 orbit spanning a 4-dim subspace U of $\bigwedge^3\mathbb{R}^6$.

<u>Lemmas</u>

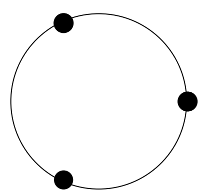
(i) If $\psi^+ + i\psi^- = (e^1{+}ie^2)\wedge(e^3{+}ie^4)\wedge(e^5{+}ie^6)$ then

$$U = \langle \alpha, \beta\rangle \oplus \langle \psi^+, \psi^-\rangle.$$

(ii) If $\gamma = \gamma(0) = e^{135}$ then

$$\langle \alpha, \beta, \psi^+\rangle = \langle \alpha, \beta, \gamma\rangle = \langle \gamma(0), \gamma(\tfrac{2\pi}{3}), \gamma(-\tfrac{2\pi}{3})\rangle$$

since $\varepsilon = \exp(\frac{2\pi i}{3})$ acts trivially on ψ^+.



Examples

...of symplectic 6-manifolds with α, β, γ all closed.

Let $N^6 \ (\stackrel{T^3}{\to} T^3)$ be a principal torus bundle with base 1-forms e^1, e^3, e^5 and connection 1-forms e^2, e^4, e^6 with

$$de^2 = ce^{35}, \quad de^4 = c'e^{51}, \quad de^6 = c''e^{13}.$$

Suppose $c + c' + c'' = 0$, so that $\omega = e^{12} + e^{34} + e^{56}$ is closed and N^6 is symplectic.

<u>Theorem</u> N^6 admits a triple of mutually transverse Lagrangian submanifolds through each point.

Follows from the existence of closed simple 3-forms

$$\begin{gathered} e^1 \wedge (ae^3 + be^4) \wedge (ae^5 - be^6) \\ (ae^1 - be^2) \wedge e^3 \wedge (ae^5 + be^6) \\ (ae^1 + be^2) \wedge (ae^3 - be^4) \wedge e^5. \end{gathered}$$

Take $a = -\frac{1}{2}$, $b = \frac{\sqrt{3}}{2}$, so that $\theta = a + bJ = \mathrm{rot}_{2\pi/3}$. Relative to the basis

$$e^1, \quad e^2, \quad \theta(e^3), \quad \theta(e^4), \quad \theta^2(e^5), \quad \theta^2(e^6),$$

the closed 3-forms above are $\gamma(0), \gamma(\frac{2\pi}{3}), \gamma(\frac{-2\pi}{3})$.

Corollary The 8-manifold $N^6 \times T^2$ has a closed 4-form Ω with stabilizer $Sp(2)Sp(1)$, but no quaternion-Kähler structure.

Remarks (i) There are essentially two examples:

(A) if c, c', c'' are non-zero then $b_1(N^6) = 3$;

(B) if $c = 0$ then $b_1(N^6) = 4$ and $N^6 = N^5 \times S^1$.

In (A), one can also take $c = c' = c''$ and define

$$\varphi = e^{12} + \varepsilon e^{34} + \varepsilon^2 e^{56} = B + i\omega.$$

(ii) It seems to be easier to $d\alpha = 0 = d\beta$ by postulating the existence of a simple closed 3-form. Is this integrability property predicted by a property of $SO(3)$ torsion?

(iii) The symplectic group $Sp(n, \mathbb{R})$ acts almost transitively on triples of transverse Lagrangian subspaces of $\mathbb{R}^{2n}$, with stabilizer $O(n)$.

A non-principal fibration

Let H denote the real matrix group

$$\left\{ X = \begin{pmatrix} 1 & x^1 & x^6+ix^4 \\ 0 & 1 & x^3+ix^5 \\ 0 & 0 & 1 \end{pmatrix} : x^i \in \mathbb{R} \right\}$$

and Γ the corresponding integral lattice. Then

$$N^5 = \Gamma \backslash H = \{\Gamma h \ : \ h \in H\}.$$

Consider the mapping

$$\begin{array}{ccc} N^5 \times S^1 & \ni & X \\ \big\downarrow \pi & & \downarrow \\ T^3 & \ni & [x^2, x^3, x^5] \end{array}$$

The Lagrangian fibres of π are necessarily T^3's, and there is a global section $x^1 = x^4 = x^6 = 0$. This is one leaf of the foliation determined by e^{146}; others will be multi-valued sections.

3-symmetric spaces

The homogeneous spaces

$$S^6, \quad S^3 \times S^3, \quad \mathbb{CP}^3, \quad \mathbb{F} = SU(3)/T^2$$

all have an almost complex structure J for which

$$\theta = -\tfrac{1}{2}1 + \tfrac{\sqrt{3}}{2}J$$

is the derivative of a 3-fold isometry around each point. It arises from a Lie algebra automorphism

$$\hat{\theta} : \mathfrak{g} \to \mathfrak{g} \quad \text{with} \quad \hat{\theta}^3 = 1.$$

Moreover, G/H has a nearly-Kähler metric g_{NK} characterized by $(\nabla_X J)X = 0$ (and $\nabla J \neq 0$).

<u>Examples</u> fall into the following types:

(i) isotropy irreducible spaces,

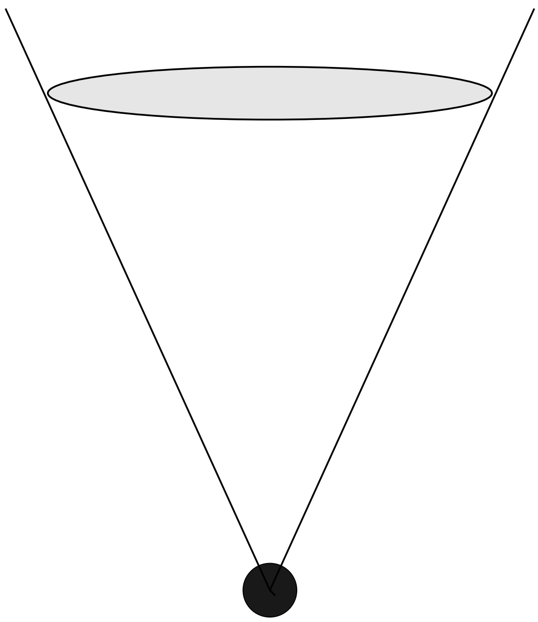
(ii) twistor spaces,

(iii) $(G\times G\times G)/G \cong G\times G$.

Conical singularities

$dt^2 + t^2 g_{\mathrm{NK}}$

has hol G_2



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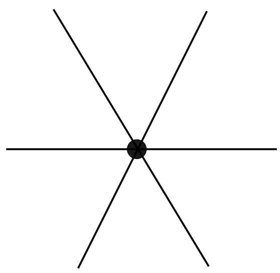
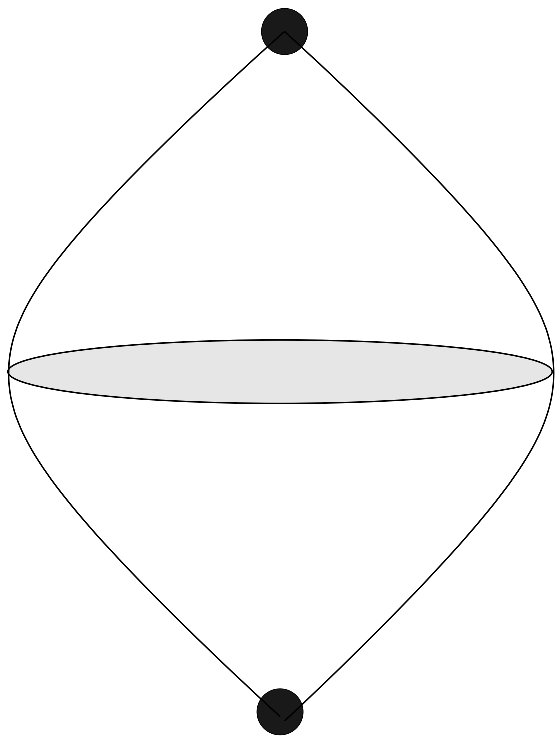


$\mathbb{R}^6$

$dt^2 + (\sin^2 t) g_{\mathrm{NK}}$

has weak hol G_2



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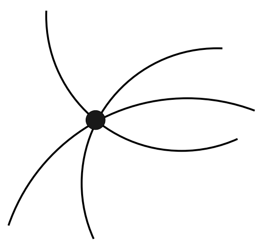

S^6 or $S^3 \times S^3$

Metrics with reduced holonomy

An $SU(3)$ structure on a 6-manifold is defined by a 2-form ω and a 3-form ψ^+. The latter determines ψ^- and an almost complex structure J so that $\psi^+ + i\psi^-$ is a $(3,0)$-form. Since $\psi^+ = 4\gamma - \alpha$, each of the two nilmanifolds N^6 has a natural half-flat $\mathrm{SU}(3)$ structure:

$$\begin{aligned} \psi^+(0) &= e^{135} - e^{146} - e^{236} - e^{245} \\ \tfrac{1}{2}\omega(0) \wedge \omega(0) &= e^{3456} + e^{5612} + e^{1234} \end{aligned}$$

are both closed.

<u>Problem</u> Extend to $\omega = \omega(t)$ and $\psi = \psi^+(t)$ so that

$$\begin{aligned} \varphi &= \omega \wedge dt + \psi^+ \\ *\varphi &= \psi^- \wedge dt + \tfrac{1}{2}\omega \wedge \omega. \end{aligned}$$

are closed, giving a metric h on $(a,b) \times N^6$ for which hol$(h) \subseteq G_2$.

Solution (A): the irreducible case

Proposition $(-1, \frac{3\sqrt{3}}{2} - 1) \times N^6$ has a metric h with $\mathrm{hol}(h) = G_2$.

Define s by setting $x = \frac{1}{\sqrt{3}} \cosh s$, $y = \frac{1}{\sqrt{2}} \sinh s$ and

$$4(t+1)x^5 - 5x^2 + 1 = 0.$$

Then h has an orthonormal basis

$$dt, \quad \frac{y^2}{x^2} e^1, \quad \frac{y}{x^2} e^3, \quad \frac{y}{x^2} e^5, \quad x e^2, \quad \frac{y}{x} e^4, \quad \frac{y}{x} e^6.$$

The original structure corresponds to the hypersurface $t=0$ and $x=1=y$.

Submanifold properties?

Solution (B): the reducible case

Proposition $(-\pi/2, \pi/2) \times N^5$ has a metric k with hol$(k) = SU(3)$.

Let $x = \cos u$, $y = \sin u$. Then k has an orthonormal basis

$$\begin{array}{ccc} xe^1, & (1+y)^{1/2}e^3, & (1-y)^{1/2}e^5, \\ x^2du, & (1-y)^{-1/2}e^4, & (1+y)^{-1/2}e^6. \end{array}$$

The Kähler form is $\omega = x^3e^1du + e^{45} - e^{63}$, and there are closed 3-forms

$$\begin{gathered} e^{146} \\ x(1+y)e^{34}du + x(1-y)e^{56}du + x^2e^{135} \\ xe^{46}du + (1-y)e^{156} + (1+y)e^{134} \\ x^3e^{35}du. \end{gathered}$$

There is a pencil of Lagrangian submanifolds through each point.

A twisted model

The Hermitian symmetric space

$$Q=\mathbb{G}\mathrm{r}_2(\mathbb{R}^5)=\frac{SO(5)}{SO(3)\times SO(2)}\subset\mathbb{CP}^4$$

has structure group

$$\left\{\begin{pmatrix}aP & -bP\\ bP & aP\end{pmatrix}\ :\ P\in SO(3),\ a^2+b^2=1\right\}$$

and $T_qQ\cong\mathbb{R}^3\otimes\mathbb{R}^2$. In addition to its Kähler form ω, there is a 2-dim subspace $\langle\alpha,\beta\rangle\subset\bigwedge_0^3 T^*Q$ that defines a tautological vector bundle

$$\begin{array}{ccc} & W & \qquad e^7, e^8\\ \mathbb{R}^2\!\! & \downarrow & \\ & Q & \end{array}$$

and a well-defined 4-form

$$\tfrac12\Omega=-\omega\wedge e^{78}+\alpha\wedge e^7+\beta\wedge e^8-\tfrac12\omega^2.$$

<u>Poposition</u> In fact, $\nabla\Omega=0$ and (W,Ω) is locally isometric to $\mathbb{G}\mathrm{r}_4(\mathbb{R}^6)\cong\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$.

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