Subgroups of G_2

$G_2 \subset SO(7)$ acts on $\mathbb{R}^7 \subset \mathbb{C}^7$ leaving invariant a 3-form φ in one of two open $GL(7,\mathbb{R})$ orbits of $\bigwedge^3 \mathbb{R}^7$.

subgroup	$\mathbb{C}^7$	3-forms
$SO(3)$	$\mathsf{S}^6\mathbb{C}^2$	1
$SO(4)$	$\mathbb{C}^4 \oplus \bigwedge^2_+ \mathbb{C}^4$	2
$SU(3)$	$\mathbb{C} \oplus (\mathbb{C}^3 \oplus \overline{\mathbb{C}^3})$	3

Starting from $SO(4)$ leads to explicit metrics with holonomy groups equal to G_2 on vector bundles over 3- and 4-manifolds.

$SU(3)$ structures

A reduction to $SU(3)$ on a 6-manifold is characterized by

- a 'Kähler' 2-form ω
- a pair of real 3-forms ψ^+, ψ^-

such that $\psi^+ + i\psi^-$ is a $(3,0)$-form for an almost complex structure J and $g(X,Y) = \omega(JX,Y)$ is positive definite.

Relative to an orthonormal basis of 1-forms,

$$\omega = e^{12} + e^{34} + e^{56}$$

$$Je^1 = e^2,\ Je^3 = e^4,\ Je^5 = e^6 \quad \text{with } J^2 = -1$$

$$\psi^+ + i\psi^- = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6),$$

giving compatibility equations:

$$\omega \wedge \psi^{\pm} = 0,$$

$$\psi^+ \wedge \psi^- = \tfrac{2}{3}\omega^3$$

Intrinsic torsion

The holonomy group $\mathrm{Hol}(g)$ is contained in $SU(3)$ iff all the forms are constant relative to the Levi-Civita connection:

$$\nabla\omega = 0, \quad \nabla\psi^{\pm} = 0.$$

The extent to which this fails is measured by a torsion tensor τ that takes values in a space

$$\mathfrak{su}(3)^{\perp} \otimes T^* \cong_c (\Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \mathbb{C}\omega) \otimes (\Lambda^{1,0} \oplus \Lambda^{0,1})$$

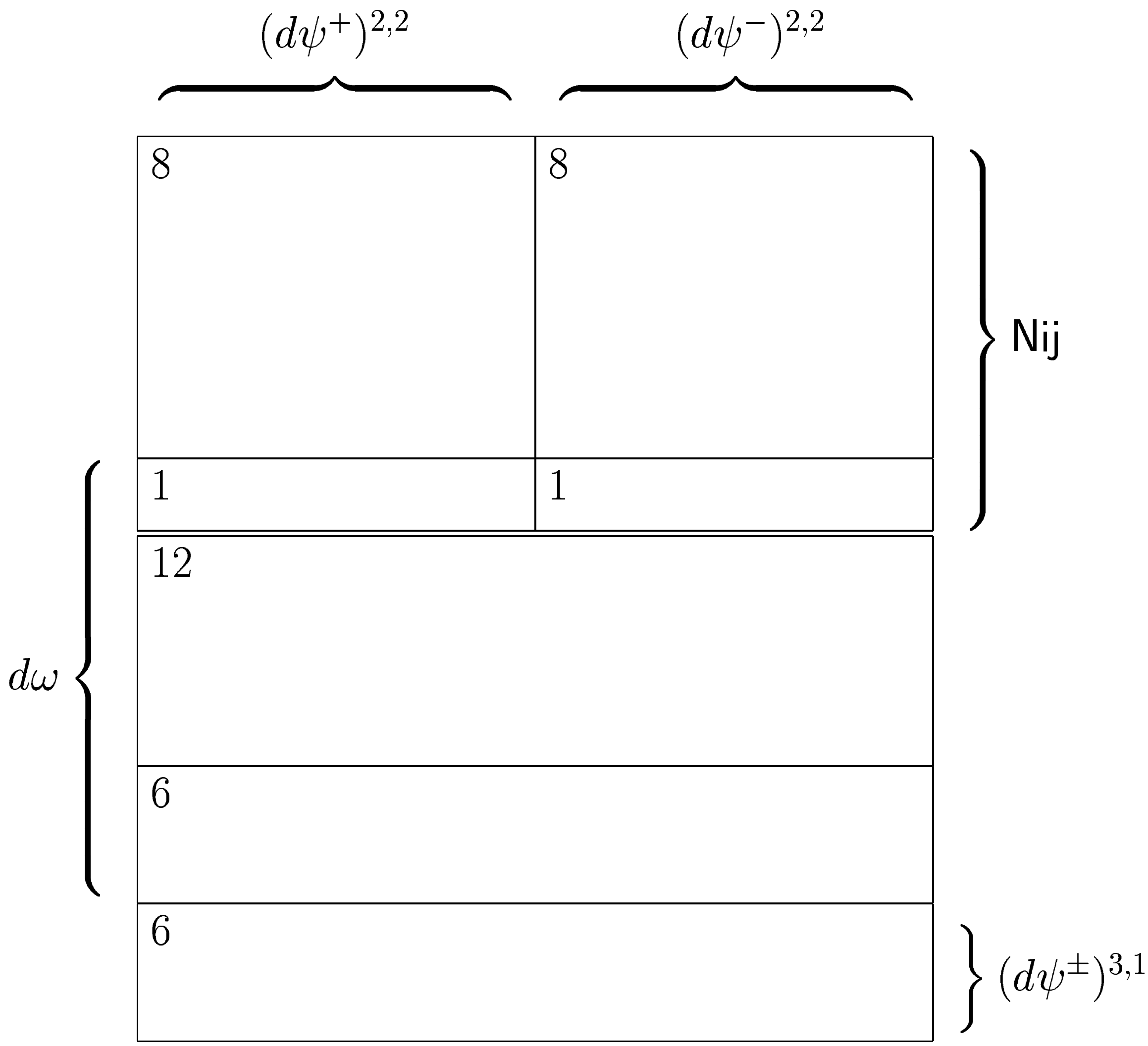
of 42 dimensions.

Corollary In fact, $\mathrm{Hol}(g) \subseteq SU(3)$ iff the forms are closed:

$$d\omega = 0, \quad d\psi^{+} = 0, \quad d\psi^{-} = 0.$$

In this case J is integrable, g is Kähler and the structure is 'Calabi-Yau'.

Components of τ:



Key features:

- The Nijenhuis tensor of J splits into two:

$$\mathsf{Nij} = \mathsf{Nij}_+ + \mathsf{Nij}_-$$

- 1-dimensional invariants common to $d\omega$ and $d\psi^\pm$:

$$d\omega \wedge \psi^\pm = \omega \wedge d\psi^\pm$$

- Two 6-dimensional components are determined by

$$d(\tfrac{1}{2}\omega^2) = \omega \wedge d\omega,$$
$$(d\psi^+)^{3,1} = i(d\psi^-)^{3,1}$$

  and a linear combination is conformally invariant.

G_2 structures

On a 7-manifold $\mathbb{M}$ with tangent space $T_x\mathbb{M} = \mathbb{R}^6 \oplus \mathbb{R}$ and $SU(3) \times \{e\}$ structure, define

$$\begin{aligned} \varphi &= \omega \wedge e^7 + \psi^+ \\ *\varphi &= \psi^- \wedge e^7 + \tfrac{1}{2}\omega^2 \end{aligned}$$

In terms of an orthonormal basis,

$$\begin{aligned} \varphi &= e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \\ *\varphi &= e^{1367} + e^{1457} + e^{2357} - e^{2467} + e^{3456} + e^{1256} + e^{1234} \end{aligned}$$

The associated holonomy group is contained in G_2 iff

$$d\varphi = 0 \quad \text{and} \quad d*\varphi = 0.$$

Instances of these equations fit into two cases:

- Nij$^- = 0$ ('self-dual'), realized when $\mathbb{M} \to M^6$
- Nij$^+ = 0$ ('ASD'), realized when $\mathbb{M} \to (a, b)$

If $\mathbb{M}$ is an S^1 bundle with curvature 2-form $\rho = de^7$,

$$0 \;=\; d\omega \wedge e^7 + (\omega \wedge \rho + d\psi^+)$$
$$0 \;=\; d\psi^- \wedge e^7 + (\psi^- \wedge \rho + \omega \wedge d\omega).$$

Thus $d\omega = 0$ and $d\psi^- = 0$. In general, if X generates the S^1 action,

$$\omega = X \lrcorner\, \varphi, \qquad \mathrm{e}^{-f}\psi^- = X \lrcorner\, (*\varphi)$$

are closed, where $|X| = \mathrm{e}^{2f}$. Thus M has a symplectic $SU(3)$ structure such that

- $(d\psi^-)^{2,2} = 0$, and $\psi^- \lrcorner\, d\psi^-$ is an exact 1-form;
- $\mathrm{e}^{3f}\big(\omega \lrcorner\, (d\psi^+)^{2,2} - 2df \lrcorner\, \psi^-\big)$ is a closed traceless 2-form.

Special case: M is Kähler

Let $\mathbb{M} = M \times (a,b)$ with $e^7 = dt$ and $\omega, \psi^\pm$ functions of t.

$$0 = \Big(d\omega - \frac{\partial \psi^+}{\partial t}\Big) \wedge dt + d\psi^+$$

$$0 = \Big(d\psi^- + \tfrac{1}{2}\frac{\partial\, \omega^2}{\partial t}\Big) \wedge dt + \tfrac{1}{2} d(\omega^2)$$

Definition An $SU(3)$ structure is half-flat if $d\psi^+ = 0$ and $d(\omega^2) = 0$ (21/42 of τ is zero)

Hitchin's theorem enables the equations to be solved on a compact 6-manifold with a half-flat $SU(3)$ structure.

Special case: $d\omega = a\psi^+$ and $d\psi^- = b\omega^2$ (like S^6)

A nilpotent example

Given the complex Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} : z^\alpha \in \mathbb{C} \right\},$$

define $M = \Gamma \backslash G$ where Γ is the subgroup with $z^\alpha \in \mathbb{Z}[i]$. Mapping to (z^1, z^2) realizes M as a T^2-bundle over T^4, and the real basis (e^i) of $T_e^*G \cong \mathfrak{g}^*$ with

$$dz^1 = e^1 + ie^2, \quad dz^2 = e^3 + ie^4, \quad -dz^3 + z^1 dz^2 = e^5 + ie^6$$

satisfies

$$de^i = \begin{cases} 0 & i = 1, 2, 3, 4 \\ e^{13} + e^{42}, & i = 5 \\ e^{14} + e^{23}, & i = 6. \end{cases}$$

Examples of $U(3)$ structures with (e^i) orthonormal:

$$\omega_0 = e^{12} + e^{34} + e^{56}, \qquad \omega_1 = e^{12} - e^{34} - e^{56},$$
$$\omega_2 = -e^{12} + e^{34} - e^{56}, \qquad \omega_3 = -e^{12} - e^{34} + e^{56}.$$

Whereas $\omega_0, \omega_1, \omega_2$ are complex structures, ω^3 defines an $SU(3)$ structure on M with $d\omega_3 = \psi^+$.

Given an invariant $SU(3)$ structure on M, ψ^+ determines J and $\psi^- = J\psi^+$. If $d\psi^+ = 0$, what can one say about $d\psi^-$? Certainly it belongs to $d(\bigwedge^3\mathfrak{g}^*)$, a space of dimension 5.

<u>Proposition</u> If $d\psi^+ = 0$ then $d\psi^- \in \langle e^{1234}\rangle$.

Roundabout proof: (i) $\omega = \omega_3$ is compatible with an $SU(3)$ structure with $d\psi^+ = 0$ and $d\psi^- = 4e^{1234}$;

(ii) $K = \ker(d : \bigwedge^3\mathfrak{g}^* \to \bigwedge^4\mathfrak{g}^*)$ has dimension $20 - 5 = 15$;

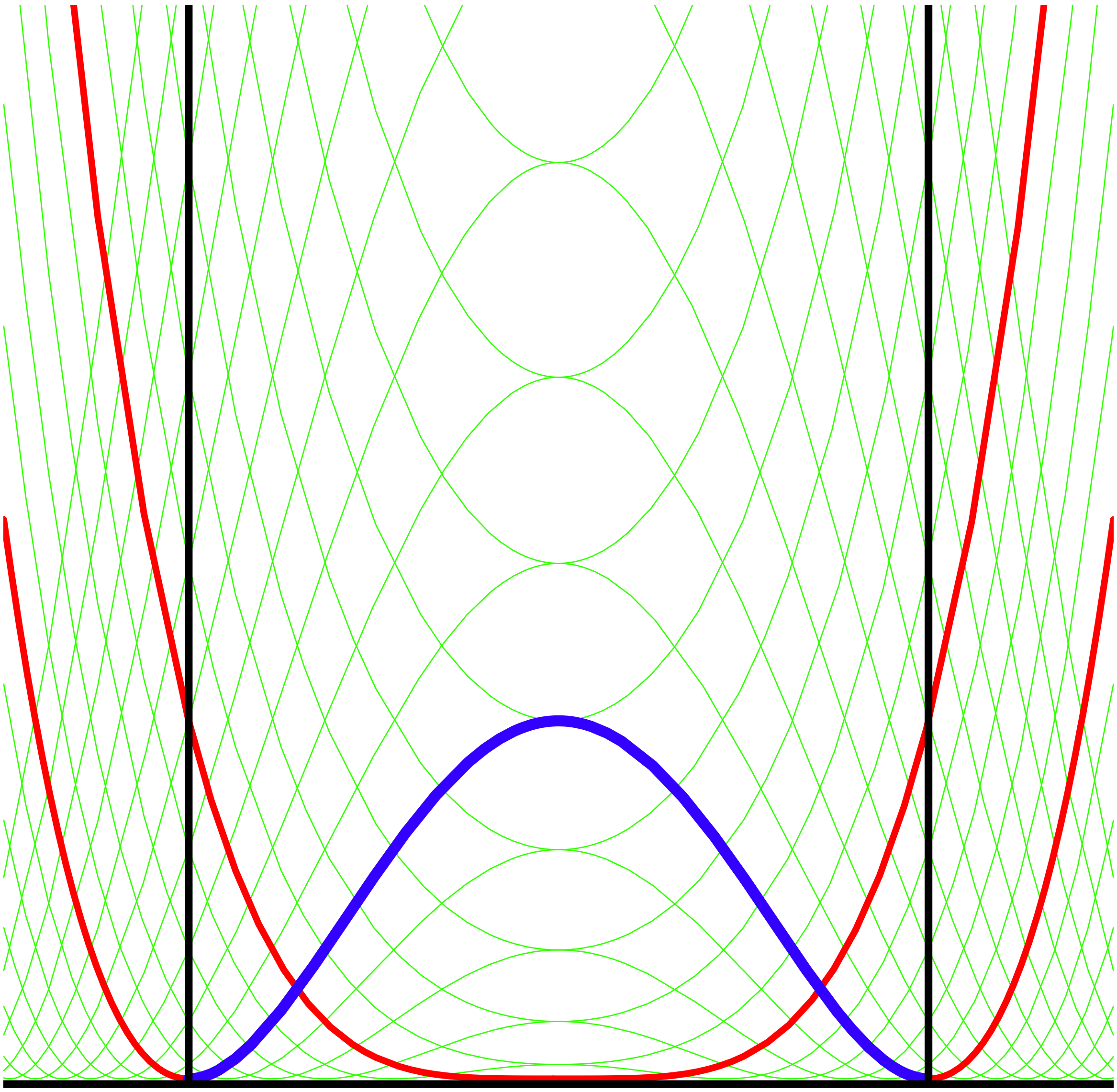
(iii) The space $\mathcal{C}$ of invariant complex structures on M can be identified with an open set of $\{\psi^+ \in K : d\psi^- = 0\}/\mathbb{C}^*$;

(iv) $\mathcal{C}$ has complex dimension 6 ($= h^{2,1} - h^{2,0} + 3$).

Similar considerations apply to $d\omega \in \bigwedge^3(\mathfrak{g}^*)$ and yield the

<u>Theorem</u> A metric with holonomy G_2 is obtained by deforming the standard half-flat $SU(3)$ structure on M as follows:

$$\begin{aligned} \psi^+(t) &= \psi_0^+ + x(t)\, d(e^{56}) \\ \tfrac{1}{2}\omega(t)^2 &= \tfrac{1}{2}\omega_0^2 + y(t)\, e^{1234} \end{aligned} \quad \text{with} \quad \begin{cases} x' = \frac{1}{\sqrt{y+1}}, \\ y' = -4x \end{cases}$$



$$H = \sqrt{y+1} + x^2$$

$$y = (H - x^2)^2 - 1$$

Explicit solutions

Taking $H=1$ gives $x^3+3x+3t=0$ and:

$$x(t)=s^{1/3}-s^{-1/3},\quad 2s=-3t+\sqrt{4+9t^2}$$

$$\omega(t)=(1-x^2)(e^{12}+e^{34})+(1-x^2)^{-1}e^{56}$$

$J(t)$ has $(1,0)$-forms $e^1+ie^2,\ e^3+ie^4,\ e^5+ie^6+xi(e^5-ie^6)$, and is only integrable for $t=0$.

In local coordinates $(p,q,\ldots,t)$ with $r=p^2+q^2$, the metric (g_{ij}) on $\mathbb{M}^7$ is given by

$$\begin{pmatrix} 1-x^2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1-x^2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1-x^2+r & 0 & -p & -q & 0\\ 0 & 0 & 0 & 1-x^2+r & q & -p & 0\\ 0 & 0 & -p & q & (1+x)^{-2} & 0 & 0\\ 0 & 0 & -q & -p & 0 & (1-x)^{-2} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Taking $H=0$ is possible with initial condition $\omega(0)=\omega_3$:

$x(t)=(3t)^{1/3}$;

$\omega(t)=-x^2(e^{12}+e^{34})+x^{-2}e^{56}$;

$J(t)$ is constant with $(1,0)$-forms $e^1-ie^2,\ e^3-ie^4,\ e^5+ie^6$.

As a T^2-bundle over T^4, M has an orthonormal basis of 1-forms $xe^1, \ldots, xe^4, x^{-1}e^5, x^{-1}e^6$. The Ricci-flat metric on $\mathbb{M}=M\times(0,\infty)$ is

$$t^{2/3}(\text{flat metric on base})+t^{-2/3}(\text{vertical metric})+dt^2$$

$\mathbb{M}$ has an S^1 quotient $(\Gamma'\backslash G')\times(0,\infty)$ with $(1,0)$-forms e^1+ie^3, e^4+ie^2, x^3dx+ie^5 and Kähler form

$$\omega=x(e^{13}+e^{42})+dx\wedge e^5=d(xe^5)=ic\,\partial\overline{\partial}(t^{5/3})$$

built from T^4.

References

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